Enclosures for the solution set of parametric interval linear systems

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Abstract

We investigate parametric interval linear systems of equations. The main result is a generalization of the Bauer–Skeel and Hansen–Bliek–Rohn bounds for this case, comparing and refinement of both. We show that not only the latter bounds are not provably better, but they are sometimes too pessimistic. The presented form of both methods is suitable for combining both of them into one to get a more efficient algorithm. Some numerical experiments are carried out to illustrate performances of the methods.

Keywords: *Linear interval systems, solution set, interval matrix.*

1 Introduction

Solving systems of interval linear equations is a fundamental problem in interval computing [2, 10]. Therein, one assumes that the matrix entries and the right-hand side components perturb independently and simultaneously within given intervals. However, this assumption comes hardly true in practical problems. Very often various correlations between input quantities appear.

Linear dependences were investigated by several authors. Theoretical papers involve e.g. characterization of the boundary of the solution set [19],

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quality of the solution set [12], or an explicit characterization of a class of parametric interval systems [3, 17].

Kolev proposed a direct method [5, 8] and an iterative method [6] for computing an enclosure of the solution set. Parametrized Gauss-Seidel iteration was employed by Popova [13]. A direct method was given by Skalna in [28], and a monotonicity approach in [29]. Inner and outer approximations by a fixed-point method was developed by Rump [25] and implemented in [18]. A *Mathematica* package for solving parametric interval systems is introduced in [14]. More general systems with nonlinear dependences between interval quantities were handled e.g. by Kolev [7] or Popova [16].

Let

$$\boldsymbol{p} := [\underline{p}, \overline{p}] = \{p \in \mathbb{R}^n \mid \underline{p} \leq p \leq \overline{p}\}$$

be an interval vector. By $p^c := \frac{1}{2}(\overline{p} + \underline{p})$ and $p^\Delta := \frac{1}{2}(\overline{p} - \underline{p})$ we denote the corresponding center and radius vector. Analogous notation is used for interval matrices.

In this paper, we consider a general parametric system of interval linear equations in the form

$$A(p)x = b(p), \tag{1}$$

where

$$A(p) = \sum_{k=1}^{K} p_k A^k, \quad b(p) = \sum_{k=1}^{K} p_k b^k, \quad p \in \boldsymbol{p},$$

and $A^k \in \mathbb{R}^{n \times n}$, and $b^k \in \mathbb{R}^n$, $k = 1, \dots, K$. The solution set is defined as

$$\Sigma := \{x \in \mathbb{R}^n \mid A(p)x = b(p),\ p \in \boldsymbol{p}\}.$$

We use the following notations: $\rho(\cdot)$ stands for the spectral radius; $A_{i\bullet}$ for the ith row of A; I for the identity matrix and e_i for its ith column. The diagonal matrix with entries $z_1, \dots, z_n$ is denoted by $\operatorname{diag}(z)$.

2 Regularity of parametric interval matrices

In order to develop an enclosure for the parametric interval system we have to discuss regularity of the parametric interval matrix $A(p)$, $p \in \boldsymbol{p}$, first.

The parametric interval matrix is called *regular* if $A(p)$ is nonsingular for every $p \in \boldsymbol{p}$.

Preconditioning and relaxing the parametric interval matrix we obtain an interval matrix

$$\boldsymbol{A} = \sum_{k=1}^{K} \boldsymbol{p}_k (RA^k),$$

i.e.

$$\boldsymbol{A}_{ij} = \left[\sum_{k=1}^{K} \min\left(\underline{p}_k (RA^k)_{ij}, \overline{p}_k (RA^k)_{ij}\right),\ \sum_{k=1}^{K} \max\left(\underline{p}_k (RA^k)_{ij}, \overline{p}_k (RA^k)_{ij}\right) \right].$$

Clearly, if $\boldsymbol{A}$ is regular then $A(p)$, $p \in \boldsymbol{p}$, is regular. Thus we can employ the well-known Beeck–Rump sufficient condition for regularity of interval matrices [1, 24, 20].

Theorem 1. *Let $R \in \mathbb{R}^{n\times n}$ be such that*

$$\rho\left(|I - RA(p^c)| + \sum_{k=1}^{K} p_k^{\Delta} |RA^k| \right) < 1. \tag{2}$$

Then $A(p)$, $p \in \boldsymbol{p}$, is regular.

Usually, the best choice for the matrix R is numerically computed inverse of $A(p^c)$. In the following, we consider the case $R = A(p^c)^{-1}$. For this special case the sufficient condition was already stated by Popova [15].

How strong is the sufficient condition presented in Theorem 1? The following result shows a class of problems where the condition is not only sufficient, but also necessary. It is a generalization of the Rohn's result [21, Corollary 5.1.(ii)].

Proposition 1. *Suppose that $A(p^c)$ is nonsingular and there is $z \in \{\pm 1\}^n$ such that for every $k \in \{1, \dots, K\}$ we have $y_k \operatorname{diag}(z)\, A(p^c)^{-1} A^k \operatorname{diag}(z) \geq 0$ for some $y_k \in \{\pm 1\}$. Then $A(\boldsymbol{p})$ is regular iff and only if*

$$\rho\left(\sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k| \right) < 1.$$

Proof. One implication is obvious in view of Theorem 1. We prove the converse by contradiction. Denote

$$A^* := \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k| = \sum_{k=1}^{K} p_k^{\Delta} y_k \operatorname{diag}(z)\, A(p^c)^{-1} A^k \operatorname{diag}(z),$$

and suppose for contradiction that $\rho(A^*) \geq 1$. Since A^* is non-negative, according to Perron–Frobenius theorem [4, 9] there is some non-zero vector x such that

$$A^* x = \rho(A^*) x,$$

or, equivalently,

$$\left(I - \frac{1}{\rho(A^*)} A^*\right) x = 0.$$

Premultiplying by $A(p^c) \operatorname{diag}(z)$ we get

$$\left(A(p^c) \operatorname{diag}(z) - \frac{1}{\rho(A^*)} A(p^c) \operatorname{diag}(z)\, A^*\right) x = 0,$$

or

$$\left(\sum_{k=1}^{K} \left(p_k^c - \frac{y_k}{\rho(A^*)} p_k^{\Delta}\right) A^k\right) (\operatorname{diag}(z)\, x) = 0.$$

The vector $\operatorname{diag}(z)\, x$ is non-zero, and the constraint matrix belongs to $A(\boldsymbol{p})$ since

$$p_k^c - \frac{y_k}{\rho(A^*)} p_k^{\Delta} \in \boldsymbol{p}_k, \quad k = 1, \ldots, K.$$

Thus we found a singular matrix in $A(\boldsymbol{p})$, which is a contradiction. □

3 Enclosures for parametric interval linear systems

The main problem studied within this paper is to find a tight enclosure for the solution set Σ, where an enclosure is any interval vector containing Σ. A simple enclosure can be acquired by relaxing the system (1) to an interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, where

$$\boldsymbol{A} := \sum_{k=1}^{K} \boldsymbol{p}_k A^k, \quad \boldsymbol{b} := \sum_{k=1}^{K} \boldsymbol{p}_k b^k.$$

Since many efficient solvers of interval linear systems use preconditioning, we should note that instead of preconditioning the system $\boldsymbol{A}x = \boldsymbol{b}$ by a matrix R it is better to precondition the original data. That is, to consider $\boldsymbol{A}'x = \boldsymbol{b}'$, where

$$\boldsymbol{A}' := \sum_{k=1}^{K} \boldsymbol{p}_k (RA^k), \quad \boldsymbol{b}' := \sum_{k=1}^{K} \boldsymbol{p}_k (Rb^k). \tag{3}$$

Proposition 2. *We have $\boldsymbol{A}' \subseteq R\boldsymbol{A}$ and $\boldsymbol{b}' \subseteq R\boldsymbol{b}$.*

Proof. Let $i, j \in \{1, \ldots, n\}$. Due to the sub-distributivity of interval arithmetic we can write

$$\begin{aligned}
\boldsymbol{A}'_{ij} &= \sum_{k=1}^{K} \boldsymbol{p}_k (RA^k)_{ij} = \sum_{k=1}^{K} \boldsymbol{p}_k \left(\sum_{l=1}^{n} R_{il} A^k_{lj} \right) \\
&\subseteq \sum_{k=1}^{K} \sum_{l=1}^{n} \boldsymbol{p}_k R_{il} A^k_{lj} = \sum_{l=1}^{n} \sum_{k=1}^{K} R_{il} (\boldsymbol{p}_k A^k_{lj}) \\
&= \sum_{l=1}^{n} R_{il} \left(\sum_{k=1}^{K} \boldsymbol{p}_k A^k_{lj} \right) = \sum_{l=1}^{n} R_{il} \boldsymbol{A}_{lj} = (R\boldsymbol{A})_{ij}.
\end{aligned}$$

Similarly for $\boldsymbol{b}' \subseteq R\boldsymbol{b}$. □

To obtain tighter enclosures we have to inspect the parametric systems more carefully. Recently, Popova [17] proved that (4) is an explicit description of a parametric interval linear system of the so called zero or first class. We first show this is necessary (but generally not sufficient) characterization for any parametric interval linear system.

Theorem 2. *If $x \in \mathbb{R}^n$ solves (1) for some $p \in \boldsymbol{p}$ then it solves*

$$|A(p^c)x - b(p^c)| \leq \sum_{k=1}^{K} p^{\Delta}_k |A^k x - b^k|. \tag{4}$$

Proof. Let $x \in \mathbb{R}^n$ be a solution to $A(p)x = b(p)$ for some $p \in \boldsymbol{p}$. Then

$$\begin{aligned}
|A(p^c)x - b(p^c)| &= \left| \sum_{k=1}^{K} p_k^c (A^k x - b^k) \right| \\
&= \left| \sum_{k=1}^{K} p_k^c (A^k x - b^k) - \sum_{k=1}^{K} p_k (A^k x - b^k) \right| \\
&= \left| \sum_{k=1}^{K} (p_k^c - p_k)(A^k x - b^k) \right| \le \sum_{k=1}^{K} |p_k^c - p_k||A^k x - b^k| \\
&\le \sum_{k=1}^{K} p_k^\Delta |A^k x - b^k|.
\end{aligned}$$

□

A sufficient and necessary characterization of Σ is given below in terms of an infinite systems of inequalities. From another viewpoint, the system is composed of a union of systems (4) over all possible preconditionings of (1). There is an open question arising, whether or not particular extremal points of Σ can be achieved by an appropriate preconditioning of (1).

Theorem 3. *We have that $x \in \Sigma$ if and only if it solves*

$$y^T \left(A(p^c)x - b(p^c) \right) \le \sum_{k=1}^{K} p_k^\Delta |y^T (A^k x - b^k)| \tag{5}$$

for every $y \in \mathbb{R}^n$.

Proof. Let $x \in \mathbb{R}^n$. Then $x \in \Sigma$ if and only if there is a vector $q \in [-1, 1]^K$ such that

$$A(p^c)x - b(p^c) = \sum_{k=1}^{K} q_k p_k^\Delta (A^k x - b^k).$$

Denote $d := A(p^c)x - b(p^c)$, and let $D \in \mathbb{R}^{n \times K}$ be a matrix whose kth column is equal to $p_k^\Delta (A^k x - b^k)$, $k = 1, \ldots, K$. Then $x \in \Sigma$ iff there is an optimal solution to the linear system

$$Dq = d, \ -1 \le q \le 1,$$

or, in other words, iff the linear program

$$\max\ 0^T q \ \text{ subject to } \ Dq = d,\ -1 \le q \le 1$$

has an optimal solution. Consider the dual problem

$$\min\ d^T y + 1^T(u+v) \ \text{ subject to } \ D^T y + u - v = 0,\ u, v \ge 0,$$

which is always feasible. According to the theory of duality in linear programming [11, 27], existence of an optimal solution to one problem implies the same for the second one and the optimal values are equal.

For an optimal solution of the dual problem and every $i \in \{1, \dots, K\}$ either $u_i = 0$ or $v_i = 0$; otherwise we can subtract a small positive amount from both u_i and v_i and decrease the optimal value. If $u_i = 0$ then $(u+v)_i = v_i = (D^T y)_i \ge 0$. Similarly, $v_i = 0$ implies $(u+v)_i = u_i = -(D^T y)_i \ge 0$. Hence we can derive $u + v = |D^T y|$, and the dual problem takes the form

$$\min\ d^T y + 1^T |D^T y| \ \text{ subject to } \ y \in \mathbb{R}^n.$$

Since the objective function is positive homogeneous, the problem has an optimal solution (equal to zero) iff the objective function is non-negative, i.e.

$$d^T y + 1^T |D^T y| \ge 0 \quad \forall y \in \mathbb{R}^n,$$

or, substituting $y := -y$,

$$y^T d \le |y^T D| 1 \quad \forall y \in \mathbb{R}^n.$$

In the setting of D and d, we get (5). □

Based on Theorem 2 we develop a generalization of the Bauer–Skeel bounds [23, 30] to parametric interval systems. Note that the generalized Bauer–Skeel bounds yield the same enclosure as the direct method by Skalna [28], however, the following form is more convenient for combining it with the Hansen–Bliek–Rohn method and for refinements.

Theorem 4. *Suppose that $A(p^c)$ be nonsingular. Denote*

$$M := \sum_{k=1}^{K} p_k^\Delta |A(p^c)^{-1} A^k|,$$
$$x^* := A(p^c)^{-1} b(p^c).$$

If $\rho(M) < 1$ then

$$\left[x^* - (I-M)^{-1}\sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}(A^k x^* - b^k)|,\right.$$
$$\left. x^* + (I-M)^{-1}\sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}(A^k x^* - b^k)|\right]$$

is an interval enclosure to Σ.

Proof. Preconditioning the system $A(p)x = b(p)$ by the matrix $A(p^c)^{-1}$ we obtain an equivalent system $A(p^c)^{-1}A(p)x = A(p^c)^{-1}b(p)$, or

$$\sum_{k=1}^{K} p_k A(p^c)^{-1}A^k x = \sum_{k=1}^{K} p_k A(p^c)^{-1} b^k, \quad p \in \boldsymbol{p}.$$

According to Theorem 2 each solution to this system satisfies

$$|A(p^c)^{-1}A(p^c)x - A(p^c)^{-1}b(p^c)| \leq \sum_{k=1}^{K} |p_k^{\Delta} A(p^c)^{-1}(A^k x - b^k)|.$$

Rearranging the system we get

$$\begin{aligned} |x - x^*| &\leq \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}(A^k x - b^k)| \qquad (6) \\ &= \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}(A^k(x - x^* + x^*) - b^k)| \\ &\leq \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}A^k(x - x^*)| + \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}(A^k x^* - b^k)| \\ &\leq \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}A^k||x - x^*| + \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}(A^k x^* - b^k)|. \end{aligned}$$

Equivalently,

$$\left(I - \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}A^k|\right)|x - x^*| \leq \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}(A^k x^* - b^k)|$$

From $\rho(M) < 1$, it follows [2, Theorem 1.31] [9]

$$(I - M)^{-1} = \sum_{j=0}^{\infty} M^j.$$

Since the matrix M is non-negative, $(I - M)^{-1}$ is also non-negative. Thus we may multiply the system by $(I - M)^{-1}$ to obtain

$$|x - x^*| \leq (I - M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|.$$

It means,

$$x \geq x^* - (I - M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|,$$
$$x \leq x^* + (I - M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|.$$

□

The Hansen–Bliek–Rohn method [2, Theorem 2.39], [22] gives an enclosure for the solution set of an interval linear system. The following is a generalization to parametric interval linear systems, however, the result is the same as the Hansen–Bliek–Rohn bounds applied on the preconditioned system (3) by $R := A(p^c)^{-1}$. For readers convenience, we present the detailed proof, which will be followed up in the next section for a refinement.

Theorem 5. *Suppose that $A(p^c)$ be nonsingular. Denote*

$$\begin{aligned}
M &:= \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k|, \\
x^* &:= A(p^c)^{-1} b(p^c), \\
M^* &:= (I - M)^{-1}, \\
x^0 &:= M^* |x^*| + \sum_{k=1}^{K} p_k^{\Delta} M^* |A(p^c)^{-1} b^k|.
\end{aligned}$$

If $\rho(M) < 1$ then any solution x to (1) *satisfies*

$$x_i \leq \max\left\{ x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*,\ \frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*\big) \right\},$$

and

$$x_i \geq \min\left\{ -x_i^0 + (x_i^* + |x^*|_i)m_{ii}^*,\ \frac{1}{2m_{ii}^* - 1}\big(-x_i^0 + (x_i^* + |x^*|_i)m_{ii}^*\big) \right\}.$$

Proof. From the proof of Theorem 4 we know that each solution to (1) satisfies

$$\begin{aligned} |x - x^*| &\leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x - b^k)| \qquad (7)\\ &\leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}A^k||x| + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}b^k|. \end{aligned}$$

This inequality system implies

$$x - x^* \leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}A^k||x| + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}b^k|, \qquad (8)$$

and

$$|x| - |x^*| \leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}A^k||x| + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}b^k|. \qquad (9)$$

Let $i \in \{1, \ldots, n\}$. Consider the system (9) in which the i-th inequality is replaced by the i-th inequality from (8)

$$\begin{aligned} &|x| - |x^*| + (x_i - x_i^* - |x|_i + |x^*|_i)e_i \leq \\ &\leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}A^k||x| + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}b^k|. \end{aligned}$$

This can be rewritten to

$$\begin{aligned} &\left(I - \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}A^k|\right)|x| + (x_i - |x|_i)e_i \leq \\ &\leq |x^*| + (x_i^* - |x^*|_i)e_i + \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}b^k|. \end{aligned}$$

From $\rho(M) < 1$, it follows [2, Theorem 1.31] [9]

$$(I - M)^{-1} = \sum_{j=0}^{\infty} M^j.$$

Since the matrix M is non-negative, $M^* = (I - M)^{-1} \geq I$. Thus we may multiply the system by $M^* \geq 0$ to obtain

$$|x| + (x_i - |x|_i)M^* e_i \leq M^*|x^*| + (x_i^* - |x^*|_i)M^* e_i + \sum_{k=1}^{K} p_k^{\Delta} M^*|A(p^c)^{-1}b^k|.$$

Substituting

$$x^0 = M^*|x^*| + \sum_{k=1}^{K} p_k^{\Delta} M^*|A(p^c)^{-1}b^k|$$

the system reads

$$|x| + (x_i - |x|_i)M^* e_i \leq x^0 + (x_i^* - |x^*|_i)M^* e_i.$$

The i-th inequality becomes

$$|x_i| + (x_i - |x|_i)m_{ii}^* \leq x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*.$$

Distinguish two cases. If $x_i \geq 0$ then

$$x_i \leq x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*.$$

If $x_i < 0$ then

$$-x_i + 2x_i m_{ii}^* \leq x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*.$$

Using the fact that $M^* \geq I$ we get that $2m_{ii}^* \geq 2 > 1$ and

$$x_i \leq \frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*\big).$$

Summing up, we have an upper bound on x_i as follows

$$x_i \leq \max\left\{x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*,\ \frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*\big)\right\}.$$

To obtain a lower bound on x_i we realize that $Ax = b$ if and only if $A(-x) = -b$. Thus, we apply the previous result to the parametric interval system

$$A(p)(-x) = -b(p).$$

That is, the sign of b^c and x^* will be changed and

$$-x_i \le \max\left\{x_i^0 + (-x_i^* - |x^*|_i)m_{ii}^*,\ \frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (-x_i^* - |x^*|_i)m_{ii}^*\big)\right\},$$

or,

$$x_i \ge \min\left\{-x_i^0 + (x_i^* + |x^*|_i)m_{ii}^*,\ \frac{1}{2m_{ii}^* - 1}\big(-x_i^0 + (x_i^* + |x^*|_i)m_{ii}^*\big)\right\}.$$

□

Remark 1. As we will see in Section 5 the Bauer–Skeel method and the Hansen–Bliek–Rohn method are incomparable; sometimes is better the former and sometimes the latter. Thus to obtain as tight as possible enclosure we propose to compute both and take their intersection. The overall computational cost is low since we calculate inverses $A(p^c)^{-1}$, $M^* = (I - M)^{-1}$ and other intermediate expressions only once. Using notations of Theorems 4 and 5, we compute the upper endpoints of the resulting enclosure as minima of

$$x_i^* + M_{i\bullet}^* \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|,$$

and

$$\max\left\{x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*,\ \frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*\big)\right\},$$

$i = 1, \dots, n$. Similarly for the lower endpoints.

4 Refinement of enclosures

Now we show that the enclosures discussed in the previous section can be made tighter. The idea is to use that enclosures to check some sign invariancies, and if they holds true then the process of deriving the enclosures can be refined. Note that the proposed refinements run always in polynomial time.

4.1 Refinement of Bauer–Skeel bounds

First, we consider the Bauer–Skeel bounds. Let $\boldsymbol{x}$ be the enclosure obtained by Theorem 4 or by any other method, and let $k \in \{1, \ldots, K\}$. Denote $\boldsymbol{a}^k := A(p^c)^{-1}(A^k\boldsymbol{x} - b^k)$. If $\underline{a}^k \geq 0$ then for every $x \in \Sigma$ one has

$$|A(p^c)^{-1}(A^k x - b^k)| = A(p^c)^{-1}A^k(x - x^*) + A(p^c)^{-1}(A^k x^* - b^k), \quad (10)$$

else if $\overline{a}^k \leq 0$ then

$$|A(p^c)^{-1}(A^k x - b^k)| = -A(p^c)^{-1}A^k(x - x^*) - A(p^c)^{-1}(A^k x^* - b^k), \quad (11)$$

otherwise we estimate the term from above as in the proof

$$|A(p^c)^{-1}(A^k x - b^k)| \leq |A(p^c)^{-1}A^k||x - x^*| + |A(p^c)^{-1}(A^k x^* - b^k)|. \quad (12)$$

Anyway, the inequality (6) can be written as

$$\begin{aligned} |x - x^*| &\leq \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}(A^k x - b^k)| \\ &\leq Y(x - x^*) + y + Z|x - x^*| + z \end{aligned}$$

for certain $Y, Z \in \mathbb{R}^{n\times n}$, $y, z \in \mathbb{R}^n$, and $Z \geq 0$. Here, Y and y are summed up from (10) and (11), whereas Z and z come from (12). Now, we proceed as follows

$$\begin{aligned} |x - x^*| &\leq Y(x - x^*) + y + Z|x - x^*| + z \\ &\leq |Y||x - x^*| + y + Z|x - x^*| + z, \end{aligned}$$

whence

$$(I - |Y| - Z)|x - x^*| \leq y + z,$$

and

$$\begin{aligned} x &\leq x^* + (I - |Y| - Z)^{-1}(y + z), \\ x &\geq x^* - (I - |Y| - Z)^{-1}(y + z). \end{aligned}$$

Since $|Y| + Z$ is non-negative and $|Y| + Z \leq M$, the inverse matrix $(I - |Y| - Z)^{-1}$ exists and is non-negative.

Notice that even more tighter bounds can be calculated by splitting the terms of (6) componentwise. That is, we check $\underline{a}_i^k \geq 0$ for every $i = 1, \dots, n$, and use the ith estimate either in (10), (11) or (12) accordingly. The method is described in Algorithm 1.

In the following we claim that the resulting enclosure is always as good as the initial Bauer–Skeel bounds.

Proposition 3. *Let $\boldsymbol{x}$ be the enclosure obtained by Theorem 4, and $\boldsymbol{x}'$ the enclosure obtained by Algorithm 1. Then $\boldsymbol{x}' \subseteq \boldsymbol{x}$.*

Proof. Recall that

$$\begin{aligned} M &:= \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1} A^k|, \\ \overline{x} &= x^* + (I - M)^{-1} \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)| \\ &= x^* + \sum_{j=1}^{\infty} M^j \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|, \end{aligned}$$

and

$$\overline{x}' = x^* + (I - |Y| - Z)^{-1}(y + z) = x^* + \sum_{j=1}^{\infty} (|Y| + Z)^j (y + z).$$

From

$$y + z \leq \sum_{k=1}^{K} p_k^{\Delta} |A(p^c)^{-1}(A^k x^* - b^k)|,$$

and

$$0 \leq |Y| + Z \leq M$$

we obtain $\overline{x}' \leq \overline{x}$. Similarly for $\underline{x}' \geq \underline{x}$. □

4.2 Refinement of Hansen–Bliek–Rohn bounds

We will refine the Hansen–Bliek–Rohn bounds is the same manner as the Bauer–Skeel ones. Let $\boldsymbol{x}$ be the enclosure of Σ obtained e.g. by Theorem 5, and let $k \in \{1,\dots,K\}$. Denote $\boldsymbol{a}^k := A(p^c)^{-1}(A^k\boldsymbol{x} - b^k)$. If $\underline{a}^k \geq 0$ then

$$|A(p^c)^{-1}(A^k x - b^k)| = A(p^c)^{-1}A^k x - A(p^c)^{-1}b^k, \tag{13}$$

else if $\overline{a}^k \leq 0$ then

$$|A(p^c)^{-1}(A^k x - b^k)| = -A(p^c)^{-1}A^k x + A(p^c)^{-1}b^k, \tag{14}$$

otherwise we use the standard estimation for the Hansen–Bliek–Rohn method

$$|A(p^c)^{-1}(A^k x - b^k)| \leq |A(p^c)^{-1}A^k||x| + |A(p^c)^{-1}b^k|. \tag{15}$$

Thus (7) takes the form of

$$\begin{aligned}|x - x^*| &\leq \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}(A^k x - b^k)|\\ &\leq Yx - y + Z|x| + z\\ &\leq (|Y| + Z)|x| - y + z,\end{aligned}$$

where $Y, Z \in \mathbb{R}^{n\times n}$, $y, z \in \mathbb{R}^n$, and $Z \geq 0$. Next, we proceed as in the proof of Theorem 5. The method is summarized in Algorithm 2.

We show that the refinement of the Hansen–Bliek–Rohn method is in each component at least as tight as the original Hansen–Bliek–Rohn bounds.

Proposition 4. *Let $\boldsymbol{x}$ be the enclosure obtained by Theorem 5, and $\boldsymbol{x}'$ the enclosure obtained by Algorithm 2. Then $\boldsymbol{x}' \subseteq \boldsymbol{x}$.*

Proof. Let $i \in \{1,\dots,n\}$. We prove $\overline{x}'_i \leq \overline{x}_i$; the lower case is done accordingly. Denote

$$\begin{aligned}M^* &:= \left(I - \sum_{k=1}^{K} p_k^{\Delta}|A(p^c)^{-1}A^k|\right)^{-1},\\ x^0 &:= M^*|x^*| + \sum_{k=1}^{K} p_k^{\Delta} M'^*|A(p^c)^{-1}b^k|.\end{aligned}$$

and

$$M'^* := (I - |Y| - Z)^{-1}, \quad x'^0 := M^*(|x^*| - y + z);$$

Clearly, $M'^* \le M^*$ and $x'^0 \le x^0$. Thus

$$-x_i'^0 + (x_i^* + |x^*|_i)m_{ii}'^* \le -x_i^0 + (x_i^* + |x^*|_i)m_{ii}^*.$$

Since $m_{ii}'^* \ge 0$, we have $\frac{1}{2m_{ii}'^*-1} \le 1$ and the term

$$\frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*\big)$$

is the maximizer in the step 18 of Algorithm 2 if and only if it is non-positive. In this case,

$$\frac{1}{2m_{ii}'^* - 1}\big(x_i'^0 + (x_i^* - |x^*|_i)m_{ii}'^*\big) \le \frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x^*|_i)m_{ii}^*\big),$$

which completes the proof. □

5 Examples

In his paper, Rohn [23] claims that for the standard system of interval linear equations the Hansen–Bliek–Rohn bound are never worse then the Bauer–Skell ones. In the following examples we show that this is not the case for the (more general) parametric systems. Surprisingly, the Bauer–Skell bounds are sometimes notably better (Example 2).

Example 1. Consider the Okumura's problem of a linear resistive network [19, Example 5.3.]:

$$\begin{pmatrix} p_1+p_6 & -p_6 & 0 & 0 & 0 \\ -p_6 & p_2+p_6+p_7 & -p_7 & 0 & 0 \\ 0 & -p_7 & p_3+p_7+p_8 & -p_8 & 0 \\ 0 & 0 & -p_8 & p_4+p_8+p_9 & -p_9 \\ 0 & 0 & 0 & -p_9 & p_5+p_9 \end{pmatrix} x = \begin{pmatrix} 10 \\ 0 \\ 10 \\ 0 \\ 0 \end{pmatrix},$$

where $p_i \in [0.99, 1.01]$, $i = 1, \ldots, 9$. The Bauer–Skeel bounds computed according to Theorem 4 are

$$([7.0148, 7.1671], [4.1173, 4.2463], [5.3933, 5.5158],\\ [2.1377, 2.2260], [1.0601, 1.1217])^T,$$

and refinement by Algorithm 1 yields

$$([7.0151, 7.1667], [4.1180, 4.2456], [5.3938, 5.5153], \\ [2.1382, 2.2255], [1.0605, 1.1213])^T.$$

The Hansen–Bliek–Rohn method (Theorem 5) results in a not-as-tight enclosure

$$([6.9693, 7.2150], [4.0689, 4.2971], [5.3501, 5.5612], \\ [2.1083, 2.2568], [1.0397, 1.1431])^T.$$

The refinement by Algorithm 2 gives

$$([6.9925, 7.1913], [4.1134, 4.2504], [5.3799, 5.5307], \\ [2.1324, 2.2317], [1.0576, 1.1244])^T.$$

Notice that for this example the exact interval hull of the solution set Σ is known [19]

$$([7.0170, 7.1663], [4.1193, 4.2454], [5.3952, 5.5150], \\ [2.1392, 2.2253], [1.0614, 1.1211])^T.$$

Example 2. Example 3.4. from [19] reads

$$\begin{pmatrix} p_1 & p_2 - 1 \\ p_2 & p_1 \end{pmatrix} x = \begin{pmatrix} -p_2 + \frac{1}{3} \\ p_2 \end{pmatrix},$$

where $p_1 \in [-2, -1]$ and $p_2 \in [3, 5]$. Here, the Bauer–Skeel enclosure draws

$$([0.1282, 1.2052], [-1.4103, -0.3675])^T,$$

whereas the Hansen–Bliek–Rohn method yields

$$([-0.4359, 3.7693], [-4.8718, -0.0923])^T.$$

No refinement for this very low dimensional example was successful.

Example 3. The last example is devoted to numerical experiments with randomly generated data. Even though the real-life data are not random, such experiments reveal something on the performance of algorithms. The computations were carried out in MATLAB 7.7.0.471 (R2008b) on a machine with `AMD Athlon 64 X2 Dual Core Processor 4400+`, CPU 2.2 GHz, with 1004 MB RAM. Interval arithmetics and some basic interval functions were provided by the interval toolbox INTLAB v5.3 [26].

First, we consider systems with symmetric matrices that were generated in the following way. First, entries of A^c were chosen randomly and independently in $[-10, 10]$ with uniform distribution, and then we set $A^c := A^c + (A^c)^T + 10nI$. The entries of the radius matrix A^Δ are equal to R, where $R > 0$ is a parameter. The right-hand side interval vector was chosen to be crisp with entries randomly from $[-10, 10]$.

In diverse settings of a dimension n and radius R we carried out sequences of 10 runs. The results are displayed in Table 1. We compare the resulting enclosures by sums of radii, that is for an interval vector $\boldsymbol{v}$ we calculate $\sum v_i^\Delta$.

Second, we consider Toeplitz systems, i.e, systems with matrices A satisfying $a_{i,j} = a_{i+1,j+1}$, $i, j = 1, \ldots, n$. Herein, $A^c_{i,1}$ and $A^c_{1,i}$, $i = 2, \ldots, n$, were chosen randomly in $[-10, 10]$, whereas $A^c_{1,1}$ in $[-10 + 10n, 10 + 10n]$. The entries of A^Δ are equal to R. The right-hand side vector was again crisp with entries randomly in $[-10, 10]$. The results are displayed in Table 2.

6 Concluding remarks

Numerical experiments revealed that the generalization of the Bauer–Skeel method is a competitive alternative to Hansen–Bliek–Rohn method. The best is to use the combination of both to obtain the tight enclosure. Efficient refinements of both methods was proposed in order to compute more tighter enclosure.

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Algorithm 1 (Refinement of the Bauer–Skeel method)

$Y := 0$; $y := 0$; $Z := 0$; $z := 0$;
$x^* := A(p^c)^{-1} b(p^c)$;
Let $\boldsymbol{x}$ be an initial an enclosure to Σ;
for $k = 1, \ldots, K$ **do**
    $\boldsymbol{a}^k := A(p^c)^{-1}(A^k \boldsymbol{x} - b^k)$;
    for $j = 1, \ldots, n$ **do**
        if $\underline{a}_j^k \geq 0$ **then**
            $Y_{j\bullet} := Y_{j\bullet} + p_k^\Delta A(p^c)^{-1}_{j\bullet} A^k$; $y_j := y_j + p_k^\Delta A(p^c)^{-1}_{j\bullet}(A^k x^* - b^k)$;
        else if $\overline{a}_j^k \leq 0$ **then**
            $Y_{j\bullet} := Y_{j\bullet} - p_k^\Delta A(p^c)^{-1}_{j\bullet} A^k$; $y_j := y_j - p_k^\Delta A(p^c)^{-1}_{j\bullet}(A^k x^* - b^k)$;
        else
            $Z_{j\bullet} := Z_{j\bullet} + p_k^\Delta |A(p^c)^{-1}_{j\bullet} A^k|$; $z_j := z_j + p_k^\Delta |A(p^c)^{-1}_{j\bullet}(A^k x^* - b^k)|$;
        end if
    end for
end for
return $\left[x^* - (I - |Y| - Z)^{-1}(y + z),\, x^* + (I - |Y| - Z)^{-1}(y + z)\right]$, an enclosure to Σ.

Algorithm 2 (Refinement of the Hansen–Bliek–Rohn method)

$Y := 0$; $y := 0$; $Z := 0$; $z := 0$;
$x^* := A(p^c)^{-1} b(p^c)$;
Let $\boldsymbol{x}$ be an initial an enclosure to Σ;
for $k = 1, \ldots, K$ **do**
  $\boldsymbol{a}^k := A(p^c)^{-1}(A^k \boldsymbol{x} - b^k)$;
  for $j = 1, \ldots, n$ **do**
    if $\underline{a}_j^k \geq 0$ **then**
      $Y_{j\bullet} := Y_{j\bullet} + p_k^\Delta A(p^c)^{-1}_{j\bullet} A^k$; $y_j := y_j + p_k^\Delta A(p^c)^{-1}_{j\bullet} b^k$;
    else if $\overline{a}_j^k \leq 0$ **then**
      $Y_{j\bullet} := Y_{j\bullet} - p_k^\Delta A(p^c)^{-1}_{j\bullet} A^k$; $y_j := y_j - p_k^\Delta A(p^c)^{-1}_{j\bullet} b^k$;
    else
      $Z_{j\bullet} := Z_{j\bullet} + p_k^\Delta |A(p^c)^{-1}_{j\bullet} A^k|$; $z_j := z_j + p_k^\Delta |A(p^c)^{-1}_{j\bullet} b^k|$;
    end if
  end for
end for
$M^* := (I - |Y| - Z)^{-1}$; $x^0 := M^*(|x^*| - y + z)$;
for $i = 1, \ldots, n$ **do**
  $\overline{x}'_i := \max\left\{ x_i^0 + (x_i^* - |x^*|_i) m_{ii}^*,\ \frac{1}{2m_{ii}^* - 1}\big(x_i^0 + (x_i^* - |x^*|_i) m_{ii}^*\big) \right\}$;
  $\underline{x}'_i := \min\left\{ -x_i^0 + (x_i^* + |x^*|_i) m_{ii}^*,\ \frac{1}{2m_{ii}^* - 1}\big(-x_i^0 + (x_i^* + |x^*|_i) m_{ii}^*\big) \right\}$;
end for
return $\boldsymbol{x}'$, an enclosure to Σ.

n	R	sum of radii				average computing time (in sec.)			
		BS	refined BS	HBR	refined HBR	BS	refined BS	HBR	refined HBR
5	0.05	0.04411	0.04407	0.04803	0.04416	0.01537	0.0893	0.01421	0.0886
5	0.1	0.08454	0.08441	0.09157	0.08446	0.0155	0.09176	0.0148	0.08898
5	0.5	0.4293	0.4247	0.4755	0.4255	0.01553	0.09054	0.01441	0.0911
5	1	1.02	1.003	1.098	1.015	0.01412	0.08438	0.01366	0.08453
10	0.05	0.03669	0.03666	0.04041	0.03674	0.04731	0.5632	0.0456	0.5579
10	0.1	0.06011	0.06	0.06576	0.06013	0.04588	0.5559	0.04498	0.5494
10	0.5	0.3504	0.347	0.3835	0.3496	0.04839	0.5813	0.04604	0.5679
10	1	0.9912	0.9752	1.075	0.9948	0.04638	0.5461	0.04401	0.5449
15	0.05	0.03551	0.03548	0.0389	0.03553	0.1017	1.802	0.1	1.778
15	0.1	0.07195	0.07181	0.07872	0.07197	0.09783	1.719	0.09587	1.695
15	0.5	0.3912	0.3878	0.4269	0.3918	0.09836	1.759	0.09593	1.724
15	1	0.8311	0.8172	0.907	0.8338	0.09666	1.733	0.0956	1.731
20	0.05	0.03385	0.03381	0.03717	0.03385	0.1758	3.979	0.1695	3.956
20	0.1	0.07319	0.07305	0.08081	0.07326	0.1721	3.937	0.1697	3.928
20	0.5	0.4017	0.3979	0.4427	0.4033	0.1726	3.961	0.1671	3.976
20	1	0.793	0.7799	0.8648	0.7986	0.1699	4.01	0.169	3.996
25	0.05	0.03413	0.0341	0.03743	0.03414	0.2774	7.591	0.2647	7.524
25	0.1	0.067	0.06687	0.07359	0.06704	0.283	7.712	0.2775	7.644
25	0.5	0.3948	0.3913	0.4331	0.3966	0.2726	7.599	0.2669	7.493
25	1	0.8265	0.8126	0.9052	0.8345	0.2767	7.723	0.2704	7.77

Table 1: Symmetric systems with random data.

n	R	sum of radii				average computing time (in sec.)			
		BS	refined BS	HBR	refined HBR	BS	refined BS	HBR	refined HBR
5	0.05	0.02881	0.02876	0.03739	0.02892	0.008363	0.05246	0.008132	0.05139
5	0.1	0.04278	0.04268	0.05122	0.04312	0.0102	0.05672	0.009942	0.05684
5	0.5	0.2485	0.2453	0.3032	0.2514	0.01007	0.05786	0.009805	0.05706
5	1	0.8072	0.7862	0.9695	0.8153	0.01092	0.05661	0.009788	0.05831
10	0.05	0.02995	0.02989	0.03995	0.03016	0.01879	0.2069	0.01853	0.21
10	0.1	0.05936	0.05913	0.07881	0.06001	0.01832	0.2046	0.0172	0.2001
10	0.5	0.2848	0.2791	0.3847	0.2858	0.018	0.2023	0.01751	0.1963
10	1	0.6083	0.587	0.7969	0.6231	0.01894	0.2084	0.01806	0.2091
15	0.05	0.0291	0.02904	0.03915	0.02918	0.02552	0.4194	0.02465	0.4127
15	0.1	0.05711	0.05688	0.0768	0.05761	0.02636	0.4542	0.03089	0.455
15	0.5	0.3054	0.2988	0.4281	0.3124	0.02747	0.4478	0.02632	0.4443
15	1	0.7093	0.679	0.9737	0.7287	0.02684	0.4291	0.02627	0.4324
20	0.05	0.02799	0.02793	0.03814	0.02804	0.03518	0.7715	0.03548	0.7492
20	0.1	0.05503	0.0548	0.07594	0.05523	0.03628	0.7911	0.03554	0.7797
20	0.5	0.2742	0.2686	0.3756	0.2776	0.03488	0.757	0.03406	0.7627
20	1	0.6921	0.664	0.9403	0.7168	0.03498	0.769	0.03447	0.776
25	0.05	0.02589	0.02583	0.03582	0.02596	0.04478	1.157	0.04359	1.156
25	0.1	0.05519	0.05495	0.07707	0.05554	0.04647	1.195	0.04478	1.192
25	0.5	0.2802	0.2745	0.3846	0.2856	0.0467	1.198	0.04404	1.194
25	1	0.6381	0.613	0.8766	0.6614	0.04455	1.166	0.04308	1.165

Table 2: Toeplitz systems with random data.