Complexity of necessary efficiency in interval LP and MOLP

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Abstract

We present some complexity results on testing necessary efficiency in interval multiobjective linear programming. Supposing that objective function coefficients perturb within prescribed intervals, a feasible point x^* is called necessarily efficient if it is efficient for all instances of interval data. We show that the problem of testing necessary efficiency is co-NP-complete even for only one objective. Provided that x^* is non-degenerate basic solution, the problem is polynomially solvable for one objective, but remains NP-hard in the general case. Some open problems are mentioned at the end of the paper.

Keywords: *Multiobjective linear programming Interval matrix Efficient solution NP-completeness*

1 Introduction

Interval analysis is an approach for tackling uncertainties, which are common in real-life problems. It assumes that the problem entries perturb within some known lower and upper bounds. Modelling uncertainty in this way results in problems that are often either polynomial or NP-hard [3].

Linear programming with interval data was studied may many authors; cf. [3]. Even in this discipline there is a number of intractable problems. For instance, computing the exact range of optimal values [3], or checking strong solvability (whether each problem has an optimal solution) of an interval linear programming problem [3], [17]. Also testing whether each problem is unbounded turned out to be NP-hard [11].

Multiobjective linear programming (MOLP) problems with interval data were investigated, too [14]. However, usually authors considered uncertainties in objective functions coefficients only [13]. An important concept of efficiency in interval MOLP is necessary efficiency; it ensures that a feasible point is efficient for all realizations of interval data. It was dealt with in e.g. [1], [4], [5], [6], [7], [8], [10], [13], [20]. Basic properties and theoretical foundations for necessary efficiency were discussed in [1], [10], [13]. An implicit enumeration algorithm for testing necessary efficiency of a non-degenerate basic solution was proposed by Bitran [1], and later improved by Ida [6].

Throughout the paper, $A_{i\bullet}$ denotes the i-th row of a matrix A, and e a vector of ones (with convenient dimension). A diagonal matrix with entries $z_1, \dots, z_n$ is written as $\operatorname{diag}(z)$.

2 Preliminaries

A multiobjective linear programming (MOLP) problem reads

$$\max_{x \in \mathcal{M}} \ Cx, \tag{1}$$

where the feasible set $\mathcal{M} := \{x \in \mathbb{R}^n \mid Ax \leq b\}$, $C \in \mathbb{R}^{s\times n}$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. A feasible solution x^* to (1) is called *efficient* if there is no $x \in \mathcal{M}$ such that $Cx \geq Cx^*$ with at least one strict inequality; we denote it briefly $Cx \gneqq Cx^*$.

Efficiency of points may be characterized by tangent and normal cones [2], [12], [16]. The tangent cone of $\mathcal{M}$ at the point x^* is defined

$$\mathcal{T}(x^*) := \{x \in \mathbb{R}^n \mid A_P x \leq 0\},$$

where $P := \{i \mid A_{i\bullet}x^* = b_i\}$ denotes the set of active constraints, and A_P denotes the submatrix of A consisting of the rows indexed by P. The normal cone [12], [16] of $\mathcal{M}$ at the point x is defined as

$$\begin{aligned}\mathcal{N}(x^*) : &= \{x \in \mathbb{R}^n \mid x^T y \leq 0 \ \forall y \in \mathcal{T}(x^*)\} \\ &= \{A_P^T u \in \mathbb{R}^n \mid u \in \mathbb{R}^{|P|}, \ u \geq 0\}.\end{aligned}$$

The extremal directions of the cone $\mathcal{N}(x^*)$ are constituted by the rows of the matrix A_P. Since $\mathcal{N}(x^*)$ is a convex polyhedral cone, it can by described by means of linear inequalities

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n \mid Dx \leq 0\},$$

where $D \in \mathbb{R}^{r\times n}$ is an appropriate matrix. To determine such a description has exponential complexity in the worst case [15]. One way is to compute all extremal directions h_i, $i \in I$, of $\mathcal{T}(x^*)$ to obtain the desired description

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n \mid h_i^T x \leq 0 \ \forall i \in I\}.$$

Nevertheless, as long as x^* is a non-degenerate basic solution corresponding to a basis $B \subseteq \{1, \ldots, m\}$ then the normal cone reads

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n \mid (A_B^T)^{-1} x \geq 0\},$$

that is, D is effectively computable and $D = -(A_B^T)^{-1}$.

Normal and tangent cones relate to efficiency in the following way. A point $x^* \in \mathcal{M}$ is efficient if and only if there is some positive combination of objectives lying inside $\mathcal{N}(x^*)$. In other words, if and only if $DC^T\lambda \leq 0$ for some $\lambda \in \mathbb{R}^s$, $\lambda > 0$. A point $x^* \in \mathcal{M}$ is not efficient if and only if there is $y \in \mathcal{T}(x^*)$ such that $Cy \gneqq 0$.

In this paper, we suppose that the objective functions coefficients are not known precisely. We are given only some lower and upper bounds as follows $\underline{c}_{ij} \leq c_{ij} \leq \overline{c}_{ij}$, $i = 1, \ldots, s,\ j = 1, \ldots, n$. Define an interval matrix

$$\boldsymbol{C} := [\underline{C}, \overline{C}] = \{C \in \mathbb{R}^{s\times n} \mid \underline{c}_{ij} \leq c_{ij} \leq \overline{c}_{ij},\ i = 1, \ldots, s,\ j = 1, \ldots, n\}.$$

The corresponding midpoint matrix and radius matrix are denoted respectively by $C^c := \frac{1}{2}.(\overline{C} + \underline{C})$ and $C^\Delta := \frac{1}{2}.(\overline{C} - \underline{C})$. By an interval MOLP problem we understood a family of problems

$$\max_{x\in\mathcal{M}} \ Cx, \quad \text{where} \quad C \in \boldsymbol{C}. \tag{2}$$

A feasible solution x^* is called *necessarily efficient* if it is efficient to (2) for every $C \in \boldsymbol{C}$.

The following theorem gives a characterization of necessary efficiency developed in [5]. Contrary to original version, here we use a normalization $e^T|x| = 2$ instead of $e^T|x| = 1$. We will employ it in proofs of Theorem 2 and 3.

Theorem 1. *The vector x^* is necessarily efficient if and only if the system*

$$C^c x + C^\Delta |x| \gneqq 0,\ A_P x \leq 0,\ e^T|x| = 2 \tag{3}$$

has no solution.

3 Complexity

In this section we discuss complexity of testing necessary efficiency. First, we consider one-objective interval MOLP, that is, $\boldsymbol{C} = \boldsymbol{c}$ is an interval $1 \times n$ matrix. Even though the task belongs to the interval linear programming rather then interval MOLP, this issue—to the best of our knowledge—has been studied in neither discipline.

Lemma 1. *Let $M \in \mathbb{Q}^{n \times n}$ be a non-negative positive definite matrix. Checking the solvability of the system*

$$|Mx| \leq e, \; e^T|x| > 1 \tag{4}$$

is an NP-complete problem.

Proof. First we show that testing $\|M\|_{\infty,1} > 1$ is NP-hard on a class of symmetric rational M-matrices; it is a modification of a result by Rohn [18].

Let A be an MC-matrix and l a positive integer. In [18] there was proven that testing whether $z^TAz \geq l$ for some $z \in \{\pm 1\}^n$ is NP-hard. This is equivalent to testing $z^TAz > l-1$ for some $z \in \{\pm 1\}^n$. Hence checking whether $\|A\|_{\infty,1} > l-1$, or, equivalently, $\|\frac{1}{l-1}A\|_{\infty,1} > 1$ is also NP-hard. Here, $M := \frac{1}{l-1}A$ is a symmetric rational M-matrix. Note that the case $l = 1$ is pointless since A is an MC-matrix and therefore $\|A\|_{\infty,1} \geq e^TAe \geq n$.

Now, we can directly adapt the proof of Rohn's Theorem 2.3. in [3] with the inequality $\|M\|_{\infty,1} \geq 1$ replaced by $\|M\|_{\infty,1} > 1$. $\square$

Theorem 2. *Testing necessary efficiency is a co-NP-complete problem on a class of problems* (2) *with one objective function and rational inputs.*

Proof. We have to show that the problem belong to co-NP class and that it is NP-hard. The former is easily seen as any certificate x that x^* is not necessarily efficient must satisfy (3) (up to the normalization).

To prove NP-hardness we first show that (4) is solvable iff

$$|My| \leq ez, \; e^T|y| > z, \; e^T|y| + |z| = 2 \tag{5}$$

is solvable. Let $x \in \mathbb{Q}^n$ be a solution to (4). Put $y := \frac{1+\varepsilon}{e^T|x|}x$ and $z := 1 - \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Then $e^T|y| + |z| = 1 + \varepsilon + 1 - \varepsilon = 2$ and $e^T|y| > z$. Eventually,

$$|My| = \frac{1+\varepsilon}{e^T|x|}|Mx| \leq \frac{1+\varepsilon}{e^T|x|}e \leq (1-\varepsilon)e = ez$$

where we used the fact that $1 < e^T|x|$ implies $\frac{1+\varepsilon}{1-\varepsilon} \leq e^T|x|$ for $\varepsilon > 0$ small enough. Conversely, let $y \in \mathbb{Q}^n$ and $z \in \mathbb{Q}$ be a solution to (5). Put $x := y$. From the second and third condition in (5) it follows that $e^T|x| > 1$ and $z < 1$. Then also $|Mx| \leq ez < e$.

Now, we rewrite (5) in the form of (3). Note that $|My| \leq ez$ is equivalent to $-ez \leq My \leq ez$. Substitute $x^T := (y^T, z)^T$ and put

$$A_P := \begin{pmatrix} M & -e \\ -M & -e \end{pmatrix}, \quad C^c := \begin{pmatrix} 0^T & -1 \end{pmatrix}, \quad C^\Delta := \begin{pmatrix} e^T & 0 \end{pmatrix}.$$

Since C^c and C^Δ consist of just one row, the inequality $C^c x + C^\Delta|x| \gneqq 0$ is equivalent to $C^c x + C^\Delta|x| > 0$. Therefore we reduced the problem of checking solvability of (4) to the problem of testing necessary efficiency. □

Theorem 2 says that checking necessary efficiency is computationally expensive in general; compare e.g. an exponential time algorithm by Inuiguchi & Sakawa [9]. However, when we restrict the problem to non-degenerate basic solutions then it becomes polynomially solvable.

Let x^* be a non-degenerate basic solution corresponding to a basis $B \subseteq \{1, \dots, m\}$. We know that the normal cone to $\mathcal{M}$ at x^* reads

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n \mid Dx \leq 0\},$$

where $D = -(A_B^T)^{-1}$. The vector x^* is necessarily efficient iff each objective function vector $c \in \boldsymbol{c}$ lies within $\mathcal{N}(x^*)$. In other words, every solution to the system $\underline{c} \leq x \leq \overline{c}$ must satisfy $Dx \leq 0$. That is, for each $i = 1, \dots, r$ an optimal value to

$$\max\, D_{i,\bullet} x \quad \text{subject to} \quad \underline{c} \leq x \leq \overline{c}$$

should be non-positive. The maximum value is attained for the vector $c \in \boldsymbol{c}$ defined as $c_j := \overline{c}_j$ if $d_{ij} \geq 0$ and $c_j := \underline{c}_j$ otherwise. Hence the polynomial algorithm just checks positivity $D_{i,\bullet} c$; see Algorithm 1.

The readers familiar with interval computations know that this testing can be equivalently written by using interval arithmetic as $\overline{D\boldsymbol{c}^T} \leq 0$.

When we admit more then one criteria then the problem becomes NP-hard even when restricted to non-degenerate solutions.

Theorem 3. *Testing necessary efficiency of a non-degenerate basic solution is co-NP-complete problem on a class of problems* (2) *with rational inputs.*

Algorithm 1 (Necessary efficiency for one criterion)

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for i = 1, ..., r do
    put c_j := c̄_j if d_ij ≥ 0 and c_j := c_j (underlined) otherwise;
    if Σ_{j=1}^n d_ij c_j > 0 then
        return "x* is not necessarily efficient";
    end if
end for
return "x* is necessarily efficient".
```

for $i = 1, \dots, r$ **do**
  put $c_j := \overline{c}_j$ if $d_{ij} \geq 0$ and $c_j := \underline{c}_j$ otherwise;
  if $\sum_{j=1}^{n} d_{ij} c_j > 0$ **then**
    return "x^* is not necessarily efficient";
  end if
end for
return "x^* is necessarily efficient".

Proof. We adapt the proof of Theorem 2 rewriting the system (5) as follows

$$My - ez \leq 0,\ M_{1,\bullet} y + z \geq 0,\ M_{2:n,\bullet} y + ez \geq 0,\ e^T|y| > z,\ e^T|y| + |z| = 2 \tag{6}$$

Herein, $M_{2:n,\bullet}$ denotes the matrix M after removing the first row. The constraint $e^T|y| > z$ may be replaced by $e^T|y| \geq (1+\varepsilon)z$, where $\varepsilon > 0$ is sufficiently small. According to [19] this value has a polynomially large size which can be calculated from the system coefficients. Substituting $x^T := (y^T, z)^T$, the system now takes the form of (3) with

$$A_P := \begin{pmatrix} M & -e \\ -M_{1,\bullet} & -1 \end{pmatrix}, \quad C^c := \begin{pmatrix} M_{2:n,\bullet} & e \\ 0^T & -1-\varepsilon \end{pmatrix}, \quad C^\Delta := \begin{pmatrix} 0 & 0 \\ e^T & 0 \end{pmatrix}.$$

If (6) is solvable then (3) has a solution y, z such that $e^T|y| > (1+\varepsilon)z$, hence $C^c x + C^\Delta |x| \gneqq 0$ holds. Conversely, if $x^T = (y^T, z)^T$ is a solution to (3) then $e^T|x| \geq (1+\varepsilon)z > z$. Note that z cannot be zero since otherwise $0 \leq My \leq 0$ and due to regularity of M also y is zero, which contradicts $e^T|y| + |z| = 2$.

Notice that A_P is nonsingular, so the proof is completed. □

Theorem 3 states that testing necessary efficiency is NP-hard for a non-degenerate basic solution. From the proof we see that it remains true when we restrict considerations on n objective functions and only one of them is affected by intervals.

Corollary 1. *Testing whether there exists a necessarily efficient point is an NP-hard problem on a class of problems* (2) *with rational inputs.*

Proof. By Theorem 3, testing necessarily efficiency of a non-degenerate basic solution x^* is NP-hard. Its tangent cone

$$\mathcal{T}(x^*) = \{x \in \mathbb{R}^n \mid A_P x \leq 0\}$$

has just one vertex x^*. If there is some necessarily efficient solution on $\mathcal{T}(x^*)$ then it must be x^* [1], and vice versa. Therefore we reduced the problem of testing necessarily efficiency of x^* to the desired existence problem. $\square$

There are still some open questions remaining. For example:

- What is complexity of the problem of testing necessary efficiency of a non-degenerate basic solution when we have just two objective functions (or any fixed number of them)?
- What is complexity of the problem of testing existence of a necessarily efficient solution on a bounded convex polyhedron?

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