Diophantine equation $ax^n - by^n = c$. I

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Abstract

This exposition is a first (cf. [8]) draft of a fragment of a future book on number theory. We explain on the Diophantine equation $ax^n - by^n = c$ ($a, b, n \in \mathbb{N}$, $n \geq 3$, $c \in \mathbb{Z}$ and $c \neq 0$) the hypergeometric method as developed by Thue, Siegel and Baker. We show that (i) the equation has only finitely many solutions $x, y \in \mathbb{Z}$ (Thue's non-effective result), (ii) for ab sufficiently large depending on n and c there is at most one solution $x, y \in \mathbb{N}$ (Siegel's result) and (iii) for a, b, n in a certain range there is an explicit bound on the sizes of solutions and thus an algorithm for determining them (Baker's effective result). We also discuss Thue's effective result from 1918.

Contents

1 Introduction

We use notation: $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, \dots\}$, $\mathbb{Z}$ for the integers and (a, b) for the greatest common factor.

1.1 Overview of results

Consider solutions $x, y \in \mathbb{Z}$ of the generalized Pell equation

$$x^2 - by^2 = c$$

where $b \in \mathbb{N}$ is not a square and $c \in \mathbb{Z}$; the cases with $b < 0$ or square b are trivial. In 1770 Lagrange [9] proved that for $c = 1$ the solution set is always infinite. From this one derives that for any $c \in \mathbb{Z}$ there is either no solution or infinitely many. For example, the infinite series of positive integral solutions of

$$x^2 - 2y^2 = 1$$

begins with $(3, 2)$, $(17, 12)$, $(99, 70)$, $\dots$ (it follows the recurrence $x_{r+1} = 3x_r + 4y_r$, $y_{r+1} = 2x_r + 3y_r$) and the solutions $(1, 1)$, $(0, 1)$, $(1, 2)$ and $(5, 2)$ show, respectively, that for $c = -1, -2, -7$ and 17 we also have infinitely many solutions. On the other hand, by divisibility reasons e.g. for $c = -3, 0, 3$ and 5 there is no solution.

What about the similar cubic equation

$$x^3 - 2y^3 = c\ ?$$

Now the situation is different—one can prove that for $c = 1$ the only integral solutions are $(1, 0)$ and $(-1, -1)$ (Delone [5], see also Delone and Faddeev [6], and Nagell [12]). In 1908 Thue proved a fundamental result [16] that for any $c \in \mathbb{Z}$ this equation has only finitely many integral solutions. Can one find them? For some values of c ad hoc methods determine solution sets, for example for $c = 1$ (the mentioned two solutions) or $c = 0$ or $c = -4$ (no solution). Is there an algorithm determining for any $c \in \mathbb{Z}$ the solution set of $x^3 - 2y^3 = c$? Such algorithm would result from an explicit bound on sizes of solutions in terms of c. Unfortunately, Thue's proof does not provide such bound. (It does bound the number of solutions.) This is frustrating—the solution set is finite but we do not have even theoretical way of finding its elements. Can one give a finiteness proof for the solution set of $x^3 - 2y^3 = c$ that would provide bound $\max(|x|, |y|) < f(c)$ with an explicit function f? Despite all progress of number theory in the first half of the 20th century, we had to wait until 1964 for such proof when it was found by Baker [1], [2][1]. This text gives exposition of these two remarkable results of Thue and Baker.

We consider solutions $x, y \in \mathbb{Z}$ of the more general equation

$$ax^n - by^n = c$$

with parameters $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $abc = 0$, it is easily solved. We may assume that $a > 0$. If $b < 0$, then for even n the solution set is trivially finite and for odd n by replacing y with $-y$ we achieve $b > 0$. We exclude $n \leq 2$ as then generalized Pell equations have infinite solution sets. Thus we will assume that $a, b > 0$, $n \geq 3$ and $c \neq 0$. We shall present three results on solutions of this equation.

- The theorem of Thue [16]. For any $a, b, n \in \mathbb{N}$ and $c \in \mathbb{Z}$ with $n \geq 3$ and $c \neq 0$, the equation $ax^n - by^n = c$ has finitely many integral solutions.
- The theorem of Siegel [14]. If $a, b, n \in \mathbb{N}$, $n \geq 3$ and ab is sufficiently large depending on n and $c \in \mathbb{Z}$, $c \neq 0$ (an explicit bound will be given), then at most one pair $x, y \in \mathbb{N}$ may satisfy $ax^n - by^n = c$.
- The theorem of Baker [2]. If $x, y, c \in \mathbb{Z}$ are such that $x^3 - 2y^3 = c$ then $\max(|x|, |y|) \leq (263000|c|)^{22}$. This is an instance of a more general theorem.

Thue's result is general but non-effective and Baker's result is effective but not practical (the bound for sizes of solutions is already for $c = 1$ larger than 10^{110}). Siegel's result is still non-effective but despite it practical in many cases and determines many solution sets. For example, it implies that $(1, 1)$ is the only integral solution of $40x^{11} - 41y^{11} = -1$.

In the presentation of the last two results we closely follow [14] and [2]. We have not seen [16] and prove Thue's result by means of later versions of his method, as given in [14], [1], [2] or [15]. The proofs follow in the next section, modulo the construction of Padé approximations to $(1-x)^{1/n}$ that is postponed to Section 3.

Another rendering of these results will be given in [8]. More precisely, there we will give shorter derivation of the required properties of hypergeometric polynomials from Section 3. In the present version, although the

[1] I wrote this before I familiarized with the results of [17]. In Section 1.2 we will see that Thue himself did this feat already in 1918, not with $x^3 - 2y^3 = c$ but with the similar equation $x^7 - 17y^7 = c$.

derivation of the hypergeometric identitites from scratch is of some interest, it is tedious and too long. Also, we will omit Siegel's result and will replace Baker's result by Thue's effective result from 1918, which we discuss next.

1.2 Thue's 1918 effective result

Here we wanted to survey results on the binomial equation $ax^n - by^n = c$ but we move this to [8]. Instead we discuss Thue's article [17][2]. In it Thue, remarkably, demonstrated that his method, although non-effective in general, does provide effective resolution for certain binomial equations. In the next theorem we work out details of the final Examples 1 and 2 in [17].

Theorem 1.1 (Thue, 1918) *1. If the numbers $x, y, k, c \in \mathbb{Z}$ are such that $k \geq 55$ and*

$$(k+1)x^3 - ky^3 = c$$

then

$$\max(|x|, |y|) \leq 3k^7 |c|^{1+4/(\log k - 4)}.$$

2. If the numbers $x, y, c \in \mathbb{Z}$ satisfy

$$x^7 - 17y^7 = c$$

then

$$\max(|x|, |y|) \leq 693|c|^4.$$

(Chudnovsky [3, p. 357] gives in the case 2 bound $|x| \leq |c|^{0.50845}$ for $|c|$ larger than an effective bound.) These are corollaries of the next general theorem of Thue [17].

Theorem 1.2 (Thue, 1918) *Let $a, b, \alpha, \beta, \gamma, n \in \mathbb{N}$, $n \geq 3$ is a prime number, satisfy $a\alpha^n - b\beta^n = \gamma$ and*

$$(4a\alpha^n)^{n-2} > \gamma^{2n-2} n^{n^2/(n-1)} (a\alpha^n / b\beta^n)^{2(n-1)^2/n}.$$

Suppose that $c > 0$ and $r \in \mathbb{N}_0$ are such that $r \geq (\log c - \log K)/\log L$, where K and L are given by

$$K = \frac{n^{(n^2-4n+1)/(n-1)} b^{(n^2-n+1)/n} \beta^{(n-1)^2}}{2^{n+4} \gamma^{n-1} \alpha^{2n+1} a^{(n+1)/n}}$$

$$L = \frac{(4a\alpha^n)^{n-2}}{\gamma^{2n-2} n^{n^2/(n-1)} (a\alpha^n/b\beta^n)^{2(n-1)^2/n}} \ (> 1).$$

[2]We had access to [17] only via [18, pp. 565–571]. Unfortunately, it appears that the copy of [17] in [18] is incomplete.

Then every solution $p, q \in \mathbb{N}$ *of the inequality* $|ap^n - bq^n| \leq c$ *satisfies*

$$p \leq \frac{nb\beta^{n-1}}{2\gamma}\left(\frac{4a\alpha^n}{\gamma^2 n^{n/(n-1)}(a\alpha^n/b\beta^n)^{(2n-2)/n}}\right)^r.$$

We reproduce Thue's result with two changes: we switch the letters n and r and correct an error in the formula for K where [17] has an incorrect power α^{2n} (i.e., α^{2r}). This error arose in the transformation of formulae (24), (25) and (26) in [17] and although it makes very little difference in general bounds, it invalidates numerical Example 2 in [17] and an application of Thue's result by Lubelski [10].

We derive Theorem 1.1 from Theorem 1.2 (we will prove Theorem 1.2 or its variant in [8]). For the bound 1 (Example 1 in [17]) set $a = k + 1, b = k, \alpha = \beta = \gamma = 1, n = 3$. Then certainly $a\alpha^n - b\beta^n = \gamma$ and it is required that $4(k+1) > 3^{9/2}(1 + 1/k)^{8/3}$, which is satisfied for every $k \geq 37$. Strangely, at this moment Example 1 in [17] abruptly ends but we will continue. (Perhaps [17] is incompletely reproduced in [18]; references always state [17] has 9 pages but in [18] it takes 7 pages.) We have

$$K = \frac{k(1 - 1/(k+1))^{4/3}}{384} \quad \text{and} \quad L = \frac{4(k+1)}{3^{9/2}(1+1/k)^{8/3}}.$$

If $r \in \mathbb{N}_0$, $c \geq 1$ and $r \geq (\log c - \log K)/\log L$ then

$$p \leq \frac{3k}{2}\left(3^{-3/2}4(k+1)^{-1/3}k^{4/3}\right)^r < \frac{3k}{2}k^r$$

whenever $p, q \in \mathbb{N}$ satisfy $|(k+1)p^3 - kq^3| \leq c$. Thus

$$p \leq \frac{3k^\delta}{2} \cdot c^{\log k/\log L}$$

where $\delta = 2 + \max(0, -\log K/\log L)$. Since $L > 3^{-7/2}k$, $\log k/\log L < 1 + 4/(\log k - 4)$ for $k \geq 55 > e^4$. Since $K = K(k) > 0$ and $L = L(k) > 1$ are increasing functions of k and $K(55) < 1$, for $k \geq 55$ we have $\log K/\log L \geq \log K(55)/\log L(55) = -4.685\ldots$ and thus $\delta \leq 7$. For $k \geq 55$ we therefore get the bound

$$p \leq \frac{3k^7}{2} \cdot c^{1+4/(\log k - 4)}.$$

Clearly, $0 < q < 2p$. If $x, y, c \in \mathbb{Z}$ and $(k+1)x^3 - ky^3 = c$, we may assume that $x, y \geq 0$ and get for $k \geq 55$ the stated bound on $\max(|x|, |y|)$. A similar bound holds in the whole range $k \geq 37$ but with the exponent 7 replaced by 266.

For the bound 2 (Example 2 in [17]) set $a = 1, b = 17, \alpha = 3, \beta = 2, \gamma = 11, n = 7$; we use the "anchor" solution $3^7 - 17 \cdot 2^7 = 11$. The key condition holds as

$$L = 8748^5 \cdot 11^{-12} \cdot 7^{-49/6} \cdot (2187/2176)^{-72/7} = 1.943\ldots > 1.$$

If $p, q \in \mathbb{N}$ satisfy that $|p^7 - 17q^7| \leq c$ then

$$p \leq \frac{7 \cdot 17 \cdot 2^6}{2 \cdot 11} \left(\frac{4 \cdot 3^7}{11^2 7^{7/6} (3^7/(17 \cdot 2^7))^{12/7}} \right)^r < 347(15/2)^r$$

for any $r \in \mathbb{N}_0$ satisfying $r \geq (\log c - \log K)/\log L$ where

$$K = \frac{7^{11/3} 17^{43/7} 2^{36}}{2^{11} 11^6 3^{15}} = 59943.171\ldots .$$

It is required that $r \geq (\log c - \log K)/\log L = (1.505\ldots)\log c - 16.562\ldots$ and $r \geq 0$. Thus $r \leq 1.51 \log c$ works for $c \geq 1$ and we get that

$$p \leq 347c^{1.51\log(15/2)} < 347c^{3.1},\ c \geq 1.$$

If $x^7 - 17y^7 = c$ for $x, y, c \in \mathbb{Z}$, the bound on $\max(|x|, |y|)$ is twice this bound.

Thue in Example 2 of [17] calculates the bound on p if $|p^7 - 17q^7| \leq 10^6$ where he uses $r = 3$ based on the incorrect value $(\log 10^6 - \log K)/\log L = 2.58\ldots$. The correct value is larger by $\log \alpha/\log L = \log 3/\log(1.943\ldots) = 1.652\ldots$ and one has to set $r = 5$. (We use natural logarithms but in [17] all logarithms are decadic.)

Thue's article [17] was and is not rarely cited but unfortunately very often its results, if at all mentioned, are misinterpreted (are portrayed as dealing only with bounding the number of solutions) and their effective nature is ignored (see, e.g., Siegel [14] or Mordell [11, p. 273]). This was rectified by Bombieri ([4, p. 196]) who pointed out effectivity of the results in [17].

An interesting early exception is Lubelski [10] (and Nagell [13]). This note is in Germain, has subtitle and summary in Polish and quotes in French a passage from Nagell's memoir [13] describing in detail Thue's result, i.e., Theorem 1.2. Lubelski then uses it to prove that the only integral solutions to $x^4 - 15y^4 = 1$ are $\pm 1, 0$ and $\pm 2, \pm 1$. There are discrepancies in the quotation in [10] compared to Theorem 1.2: (i) slightly more general result is stated, allowing $n \geq 3$ be a prime power, (ii) the incorrect power α^{2n} in K is used and (iii) L contains an incorrect (more precisely, "weaker") power γ^{2n+2} (but this makes no difference as $\gamma = 1$ in the application). Lubelski sets $a = 1, b = 15, \alpha = 2, \beta = \gamma = 1, n = 4$ and $c = 1$, calculates that $r > 2.881\ldots$ and sets $r = 3$. But, as we know, the correct lower bound on r is larger by $\log\alpha/\log L = \log 2/\log(15^{27}/2^{100}) = 0.182\ldots$ and one has to set $r = 4$. This yields the bound $p < 210300$, which is larger than Lubelski's bound $p < 24000$. Hence his proof is incomplete (but could be easily completed, if anybody cared).

2 Three theorems on $ax^n - by^n = c$

The next three propositions on rational approximations $\frac{P_r(x)}{Q_r(x)}$ to $(1-x)^{1/n}$ will be proved in Section 3. To simplify notation we do not mark explicitly the dependence of several quantities like $P_r(x)$ or $Q_r(x)$ on n. We obtain Thue's theorem by means of the next proposition.

Proposition 2.1 *For every $n \in \mathbb{N}$ with $n \geq 2$ there is a constant $c > 1$ and a sequence of pairs of polynomials $P_r(x), Q_r(x)$ such that for every $r = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$ the following holds.*

1. *Both polynomials are nonzero and have degree at most r.*
2. *Their coefficients are integers, in absolute values smaller than $(c/2)^r$. Hence if $|x| < 1$ then $|P_r(x)|, |Q_r(x)| < (r+1)(c/2)^r \leq c^r$.*
3. *If $x \neq 0$ then $P_r(x)Q_{r+1}(x) - P_{r+1}(x)Q_r(x) \neq 0$.*
4. *If $|x| < 1$ then $\left|P_r(x) - (1-x)^{1/n}Q_r(x)\right| < c^r|x|^{2r+1}$.*

Siegel's result needs explicit estimates in parts 2 and 4. We define, for fixed $n \in \mathbb{N}$ and $r = 1, 2, \ldots$,

$$\sigma_r = (n^r, r!),\ s_r = \sigma_r n^r \binom{2r}{r} \text{ and } t_r = \sigma_r n^r \prod_{m=1}^{r}\left(1 - \frac{1}{mn}\right).$$

The number σ_r is obtained from the prime factorization of $r!$ by keeping only primes dividing n. Thus $\sigma_1 = 1$, $\sigma_r \geq 1$ and

$$\sigma_r = \prod_{p\,|\,n} p^{\lfloor r/p\rfloor + \lfloor r/p^2\rfloor + \cdots} \leq \prod_{p\,|\,n} p^{\lfloor r/(p-1)\rfloor} \leq \left(\prod_{p\,|\,n} p^{1/(p-1)}\right)^r.$$

Proposition 2.2 *The polynomials of Proposition 2.1 satisfy that for* $n = 3$ *is* $P_1(x) = 6-4x$, $Q_1(x) = 6-2x$ *and for* $n \geq 2$, $r = 1, 2, \ldots$ *and* $0 < x < 1$ *is*

$$0 < P_r(x) < s_r \quad \text{and} \quad 0 < P_r(x) - (1-x)^{1/n}Q_r(x) < t_r x^{2r+1}.$$

Baker's result is based on the following estimates.

Proposition 2.3 *The polynomials of Proposition 2.1 satisfy for* $r = 1, 2, \ldots$ *that* $n^{-r}P_r(nx) \in \mathbb{Z}[x]$, $n^{-r}Q_r(nx) \in \mathbb{Z}[x]$, *if* $0 < x < 1$ *then*

$$0 < P_r(x) < 2\sigma_r(2n)^r(2-x)^r,$$

and if $0 < x \leq \frac{1}{8}$ *then*

$$0 < P_r(x) - (1-x)^{1/n}Q_r(x) < (3\sigma_r/4)(n/4)^r x^{2r+1}(1-x)^{1/n-r-1}.$$

2.1 The theorem of Thue

We use the following gap principle.

Proposition 2.4 *Let* $t > 1$, $\theta, \alpha, \beta > 0$ *be real constants and* p_r, q_r *be pairs of integers such that for every* $r = 1, 2, \ldots$, $p_r q_{r+1} - p_{r+1}q_r \neq 0$ *(i.e., pairs* p_r, q_r *and* p_{r+1}, q_{r+1} *are not multiple of one another),*

$$|q_r| < t^{\beta r} \quad \text{and} \quad |p_r - q_r\theta| < \frac{1}{t^{\alpha r}}.$$

If $\gamma, c > 0$ *are such that* $\gamma > \beta/\alpha$ *then*

$$|p - q\theta| < \frac{c}{q^\gamma}$$

holds for only finitely many pairs of integers p, q, $q > 0$.

Proof. Let us suppose that pairs p, q and p_r, q_r satisfy the stated inequalities and are not multiple of one another. Eliminating θ we get

$$1 \leq |q_r p - q p_r| \leq |q_r| \cdot |p - q\theta| + q \cdot |q_r\theta - p_r| < \frac{ct^{\beta r}}{q^\gamma} + \frac{q}{t^{\alpha r}}$$

which we rewrite, denoting $\delta = \alpha\gamma - \beta > 0$, as the inequality

$$1 < \frac{c}{t^{\delta r}}\left(\frac{t^{\alpha r}}{q}\right)^\gamma + \frac{q}{t^{\alpha r}}.$$

For every $q \in \mathbb{N}$ there is a unique $r_0 = r_0(q) \in \mathbb{N}$ with

$$\frac{1}{2t^\alpha} \le \frac{q}{t^{\alpha r_0}} < \frac{1}{2}$$

and $r_0(q) \to \infty$ for $q \to \infty$. Thus there is a bound Q such that if $q > Q$ then $r_0 = r_0(q)$ satisfies

$$\frac{c}{t^{\delta r_0}}(2t^{2\alpha})^\gamma < \frac{1}{2}.$$

It follows that for any considered pair p, q with $q > Q$, one of the values $r = r_0(q)$ or $r = r_0(q) + 1$ (as one of the corresponding pairs p_r, q_r is not a multiple of p, q) turns the inequality into the contradiction $1 < \frac{1}{2} + \frac{1}{2}$. Hence $q \le Q$ for every pair p, q (and for any q there is at most one p forming pair with q). □

To save some notation, we divide $ax^n - by^n = c$ by a and write it as $x^n - \beta y^n = \gamma$, with $\beta = b/a$ and $\gamma = c/a$.

Theorem 2.5 (Thue, 1908) *Let $n \in \mathbb{N}$, $n \ge 3$, and $\beta, \gamma \in \mathbb{Q}$, $\beta > 0, \gamma \neq 0$. The equation*

$$x^n - \beta y^n = \gamma$$

has only finitely many solutions $x, y \in \mathbb{Z}$.

Proof. It suffices to prove finiteness of positive solutions. We assume that $p^n - \beta q^n = \gamma$ for infinitely many pairs $p, q \in \mathbb{N}$ and derive a contradiction. Clearly, p in these pairs may be arbitrary large. We write $p^n - \beta q^n = \gamma$ as

$$\left(1 - \frac{\gamma}{p^n}\right)^{1/n} = \frac{\theta q}{p} \quad \text{where} \quad \theta = \beta^{1/n}$$

and substitute γ/p^n for x in the polynomials of Proposition 2.1. We assume that p is so large that $|\gamma/p^n| < 1$. For $r = 0, 1, 2, \ldots$ we set (recall that a is the denominator of γ)

$$u_r = p(ap^n)^r P_r(\gamma/p^n), \; v_r = q(ap^n)^r Q_r(\gamma/p^n)$$

and

$$\begin{aligned} w_r &= p(ap^n)^r \Big(P_r(\gamma/p^n) - (1 - \gamma/p^n)^{1/n} Q_r(\gamma/p^n)\Big) \\ &= p(ap^n)^r \left(P_r(\gamma/p^n) - (\theta q/p) Q_r(\gamma/p^n)\right) \\ &= u_r - \theta v_r. \end{aligned}$$

Note that $q = ((p^n - \gamma)/\beta)^{1/n} \leq p(1 + |\gamma|)(1 + \beta^{-1})$. The properties in Proposition 2.1 give that $u_r, v_r \in \mathbb{Z}$, $u_r v_{r+1} - u_{r+1} v_r \neq 0$,

$$|v_r| = |q(ap^n)^r Q_r(\gamma/p^n)| < c_1^r p^{rn+1}$$

where $c_1 = ac(1 + |\gamma|)(1 + \beta^{-1})$ and c is the constant of Proposition 2.1, and

$$|u_r - \theta v_r| = |w_r| < p(ap^n)^r c^r |\gamma/p^n|^{2r+1} < \frac{c_2^r}{p^{rn+n-1}}$$

where $c_2 = ac|\gamma|^3$. The constants c_1 and c_2 depend only on β, γ and n. We fix p (in a pair $p, q \in \mathbb{N}$ satisfying $p^n - \beta q^n = \gamma$) so that

$$p > \max(|\gamma|, c_1^2, c_2^2).$$

Then for every $r = 1, 2 \ldots$,

$$|v_r| < p^{rn+r/2+1} \leq p^{r(n+3/2)} \text{ and } |u_r - \theta v_r| < \frac{1}{p^{rn-r/2+n-1}} < \frac{1}{p^{r(n-1/2)}}.$$

Since $n \geq 3$, $n - 1 > \frac{n+3/2}{n-1/2}$ and Proposition 2.4, used with the pairs u_r, v_r and with $t = p, \alpha = n - \frac{1}{2}, \beta = n + \frac{3}{2}, \gamma = n - 1$, shows that for any $d > 0$ only finitely many pairs $u, v \in \mathbb{N}$ satisfy $|u - \theta v| < d/v^{n-1}$. However, every positive solution of our equation satisfies such inequality because $u^n - \beta v^n = u^n - (\theta v)^n = \gamma$ with $u, v \in \mathbb{N}$ implies

$$|u - \theta v| = \frac{|\gamma|}{\sum_{i=0}^{n-1} u^i (\theta v)^{n-1-i}} < \frac{|\gamma|\theta^{1-n}}{v^{n-1}}.$$

Therefore $x^n - \beta y^n = \gamma$ has only finitely many solutions $x, y \in \mathbb{N}$, which indeed contradicts the initial assumption. □

We add a few remarks to this proof. The impression may arise that it belongs to the not so well organized proofs by contradiction where one proves a claim C by assuming first that C does not hold and derives a contradiction that C holds after all. This usually means that the proof can be redone to derive C directly. This is the case here where a modified direct argument would provide an explicit upper bound on the *number* of solutions. However, the present argument does not give easily an explicit upper bound on the *size* of solutions; this is the topic of the third, Baker's, result. The reader also notes that there is still quite a room for improvement in the argument. Indeed, it could be easily strengthened to prove Thue's inequality for $\theta = \beta^{1/n}$, namely, that for every $\varepsilon > 0$ only finitely many fractions satisfy the inequality $|\theta - p/q| < q^{-1-\varepsilon-n/2}$.

2.2 The theorem of Siegel

Theorem 2.6 (Siegel, 1937) *Let $a, b, c, n \in \mathbb{N}$ with $n \geq 3$. Define*

$$\lambda_n = 4n^n \prod_{p \mid n} p^{n/(p-1)}.$$

If

$$(ab)^{n/2-1} \geq \lambda_n c^{2n-2}$$

then the equation

$$ax^n - by^n = c$$

has at most one positive solution $x, y \in \mathbb{N}$.

Corollary 2.7 *Let $b, n \in \mathbb{N}$ be such that $n \geq 3$ and*

$$b \geq \kappa_n \quad \text{where} \quad \kappa_n = \lambda_n^{1/(n-2)} = 4^{1/(n-2)} \left(n \prod_{p \mid n} p^{1/(p-1)} \right)^{\frac{n}{n-2}}.$$

Then the equation

$$(b+1)x^n - by^n = 1$$

has in $\mathbb{N}$ only the trivial solution $x = y = 1$. For example, the only integral solution of either of the equations

$$33x^7 - 32y^7 = 1, \; 33x^{11} - 32y^{11} = 1 \quad \text{and} \quad 33x^{13} - 32y^{13} = 1$$

is $x = y = 1$.

Proof. Now $c = 1$, $a = b+1$ and $(ab)^{n/2-1} > b^{n-2}$. Thus $b \geq \kappa_n$ implies the inequality in Theorem 2.6 and by its conclusion there is besides the trivial solution no other positive solution. Since $\kappa_7 = 4^{1/5}(7^{7/6})^{7/5} = 31.67\ldots$, $\kappa_{11} = 29.30\ldots$ and $\kappa_{13} = 30.26\ldots$, for $b = 32$ and $n = 7, 11, 13$ the inequality holds and we get all integral solutions of the three equations (as n is odd, the cases $xy \leq 0$ and $x, y < 0$ contribute no solutions). □

For the proof of Theorem 2.6 we need two lemmas. The first relates quantities s_r, t_r and λ_n and the second proves the theorem in a particular case for $n = 3$.

Lemma 2.8 *Let $n \in \mathbb{N}$ with $n \geq 3$ and σ_r, s_r, t_r and λ_n be as defined before. For every $r \in \mathbb{N}$ one has*

$$ns_{r+2}\left(\frac{t_r}{1-1/n}\right)^{n-1} < \lambda_n^{r+1}.$$

Proof. For $r = 1$ this holds as the left side equals $ns_3(t_1/(1-1/n))^{n-1} = 20(n^3, 6)n^{n+3} \leq 120n^{n+3}$ and the right side is $\lambda_n^2 = 16n^{n+3}n^{n-3}\prod_{p\,|\,n} p^{2n/(p-1)} \geq 432n^{n+3}$.

Let $r \geq 2$. The left side equals

$$\binom{2r+4}{r+2}\sigma_{r+2}\sigma_r^{n-1}n^{(r+1)n-(n-3)}\prod_{m=2}^{r}\left(1-\frac{1}{mn}\right)^{n-1}$$

which is at most

$$\binom{2r+4}{r+2}\left(1-\frac{1}{2n}\right)^{n-1}\sigma_{r+2}\sigma_r^{n-1}n^{(r+1)n}.$$

We have

$$\sigma_{r+2}\sigma_r^{n-1} < \left(\prod_{p\,|\,n} p^{1/(p-1)}\right)^{n(r+1)-(n-2)} < \left(\prod_{p\,|\,n} p^{1/(p-1)}\right)^{n(r+1)}.$$

If we show that for every $r \geq 2$ is

$$\binom{2r+4}{r+2}\left(1-\frac{1}{2n}\right)^{n-1} < 4^{r+1},$$

we will be done by the definition of λ_n. It holds for $r \geq 3$ as for $k = r+2 \geq 5$ one has even $\binom{2k}{k}4^{-k} = \prod_{h=1}^{k}(1-1/2h) \leq \prod_{h=1}^{5}(1-1/2h) = \frac{63}{64}\cdot\frac{1}{4} < 4^{-1}$. For the case $r = 2$ we use that $(1-1/2n)^{n-1} \leq (1-1/6)^2 = \frac{25}{36}$, which follows from the monotonicity of $(n-1)\log(1-1/2n)$. Thus

$$\binom{8}{4}4^{-4}\left(1-\frac{1}{2n}\right)^{n-1} \leq \frac{35}{128}\cdot\frac{25}{36} < \frac{1}{4}.$$

□

Lemma 2.9 *Let $a, b, c \in \mathbb{N}$, $P_1(x) = 6-4x$, $Q_1(x) = 6-2x$ and p, q and r, s be two (necessarily distinct) pairs of positive integers such that $(r, s) = 1$,*

$$0 < ap^3 - bq^3 \leq c, \; |ar^3 - bs^3| \leq c \;\; \textit{and} \;\; \frac{s}{r} P_1(1 - bq^3/ap^3) - \frac{q}{p} Q_1(1 - bq^3/ap^3) = 0.$$

Then $ab < c^4$.

Proof. With $h = ap^3 - bq^3$ and $w = ap^3$ the last equality reads $\frac{s}{r} = \frac{q(3w-h)}{p(3w-2h)}$. Thus $dr = p(3w - 2h)$ and $ds = q(3w - h)$ for some $d \in \mathbb{N}$. We have

$$\begin{aligned} d^3 c &\geq |ad^3 r^3 - bd^3 s^3| = |ap^3(3w - 2h)^3 - bq^3(3w - h)^3| \\ &= |w(3w - 2h)^3 - (w - h)(3w - h)^3| \\ &= |h^3(2w - h)| \\ &> h^3 w \end{aligned}$$

and $(d/h)^3 c > w$. From this and

$$2w = (ap^3 - bq^3) + (ap^3 + bq^3) > ap^3 + bq^3 > a + b \geq 2\sqrt{ab}$$

we get the estimate

$$ab < (d/h)^6 c^2.$$

It remains to bound d in terms of h.

We write $u = (w, h)$, $w = uw_0$, $h = uh_0$ and $v = (d, h_0)$, $d = vd_0$, $h_0 = vh_1$. Thus $(w_0, h_0) = (d_0, h_1) = 1$. Since d^3 divides $ap^3(3w - 2h)^3$, $bq^3(3w - h)^3$ and also $h^3(2w - h) = w(3w - 2h)^3 - (w - h)(3w - h)^3$, it follows that

$$d_0^3 \text{ divides } u^4 w_0 (3w_0 - 2h_0)^3 \text{ and } u^4 h_1^3 (2w_0 - h_0).$$

But $(d_0, h_1) = 1$,

$$(2w_0 - h_0, w_0(3w_0 - 2h_0)^3) = (2w_0 - h_0, w_0^4) = (h_0, w_0^4) = 1$$

and therefore d_0^3 divides u^4. It follows that $d = vd_0 \leq h_0 u^{4/3} \leq h^{4/3}$, which with the above estimate of ab and $h \leq c$ gives that $ab < c^4$. □

Proof of Theorem 2.6. Let $a, b, c, n \in \mathbb{N}$ with $n \geq 3$. Let also $(ab)^{(n-2)/2} \geq \lambda_n c^{2n-2}$. We assume for contradiction that

$$ap^n - bq^n = at^n - bu^n = c$$

for two distinct pairs $p, q \in \mathbb{N}$ and $t, u \in \mathbb{N}$. Cancelling common factors and interchanging p, q with t, u, we may assume that $(p, q) = (t, u) = 1$,

$$0 < ap^n - bq^n \leq c,\ 0 < at^n - bu^n \leq c \ \text{ and } \ ap^n + bq^n \leq at^n + bu^n.$$

Note that $p/q \neq t/u$. We will eventually derive a contradiction.

We denote $v = ap^n$ and $w = at^n$. In the first step we prove the lower bounds

$$v > \sqrt{ab} \ \text{ and } \ w > abv^{n-1}/(2c)^n.$$

The first was derived in the proof of Lemma 2.9 and now we obtain the other. Dividing $0 < ap^n - bq^n \leq c$ by v and factoring, and similarly for $0 < at^n - bu^n \leq c$, we get

$$0 < 1 - (b/a)^{1/n}(q/p) < c/v \ \text{ and } \ 0 < 1 - (b/a)^{1/n}(u/t) < c/w.$$

From this and from $p/q \neq t/u$ we have

$$\frac{1}{pt} \leq \left|\frac{q}{p} - \frac{u}{t}\right| < \left(\frac{a}{b}\right)^{1/n}\left(\frac{c}{v} + \frac{c}{w}\right) \ \text{ and so } \ \left(\frac{ab}{vw}\right)^{1/n} < 2c\max(1/v, 1/w).$$

Now if $v \leq w$, the lower bound on w follows. We assume that $v > w$ and derive a contradiction. The last displayed inequality then would give the upper bound

$$w < \left((2c)^n v/ab\right)^{1/(n-1)}.$$

By the initial inequalities for ap^n, bq^n, at^n and bu^n,

$$v - w - c = ap^n - at^n - c \leq \frac{ap^n + bq^n - (at^n + bu^n)}{2} \leq 0.$$

On the other hand, $(ab)^{(n-2)/2} \geq \lambda_n c^{2n-2}$ implies that $\sqrt{ab} > 3c$ and with $v > \sqrt{ab}$ also that $v^{(n-2)/(n-1)} - ((2c)^n/ab)^{1/(n-1)} > 0$ because

$$abv^{n-2} > (ab)^{n/2} \geq \left(\lambda_n c^{2n-2}\right)^{n/(n-2)} > (n^n c^n)^{n/(n-2)} > (2c)^n.$$

From this, $v > \sqrt{ab}$ and the upper bound on w we get the contradiction

$$\begin{aligned} v - w - c &> v^{1/(n-1)}\left(v^{(n-2)/(n-1)} - ((2c)^n/ab)^{1/(n-1)}\right) - c \\ &> (ab)^{1/2} - \left((ab)^{-1/2}(2c)^n\right)^{1/(n-1)} - c \\ &> 3c - ((2c)^n/3c)^{1/(n-1)} - c = 3c - (2/3)^{1/(n-1)}2c - c \\ &> 0. \end{aligned}$$

Thus $v \le w$ and we obtain the lower bound $w > abv^{n-1}/(2c)^n$.

Recall that $s_r = \binom{2r}{r}\sigma_r n^r$. In the second step we prove that there is a unique $l \in \mathbb{N}$ such that

$$s_l v^l \le \frac{(ab)^{1/n} w^{1-1/n}}{ncv^{1/n}} < s_{l+1} v^{l+1}$$

and that $l \ge 2$ if $n \ge 4$. As $s_k v^k$, $k = 1, 2, \ldots$, strictly increase to $+\infty$, it suffices to show that

$$s_k v^k \le \frac{(ab)^{1/n} w^{1-1/n}}{ncv^{1/n}}$$

holds with $k = 1$ if $n = 3$ and with $k = 2$ if $n \ge 4$. For $k \le \min(n-2, 2)$ we have, due to the lower bounds on w and v from the first step and the lower bound on ab in terms of λ_n and c, that

$$\begin{aligned} \frac{(ab)^{1/n} w^{1-1/n}}{ncs_k v^{k+1/n}} &> \frac{ab}{ncs_k}(2c)^{1-n} v^{n-2-k} \ge \frac{(2c)^{1-n}}{ncs_k}(ab)^{(n-k)/2} \\ &\ge \frac{2^{1-n}\lambda_n c^{n-2}}{ns_k} \ge \frac{2^{1-n}\lambda_n}{ns_k}. \end{aligned}$$

Since $s_1 = 2n$, $s_2 = 6(n,2)n^2 \le 12n^2$ and $\lambda_n > 4n^n$ we have, respectively for $n \ge 3$ and $n \ge 4$, that

$$\frac{2^{1-n}\lambda_n}{ns_1} > (n/2)^{n-2} > 1 \quad \text{and} \quad \frac{2^{1-n}\lambda_n}{ns_2} \ge \frac{1}{12}(n/2)^{n-3} \prod_{p \mid n} p^{n/(p-1)} > 1.$$

This proves the required upper bound on $s_k v^k$.

In the third step we derive the final contradiction by substituting the value

$$z = 1 - (b/a)(q/p)^n = 1 - bq^n/v$$

for x in the polynomials of Proposition 2.1. Clearly,

$$0 < z \le c/v < 1.$$

We select r so that $r \in \{l-1, l\}$, where l is the number from the second step, and

$$\frac{u}{t} P_r(z) - \frac{q}{p} Q_r(z) \ne 0.$$

This is possible due to property 3 in Proposition 2.1. We may assume that $r \ge 1$ because $r = 0$ is forced only when $n = 3$ and $(u/t)P_1(z) - (q/p)Q_1(z) =$

0 (and $l = 1$) but then Proposition 2.2 and Lemma 2.9 show that $ab < c^4$, which contradicts the assumption as $ab \geq \sqrt{ab} = (ab)^{n/2-1} > \lambda_n c^{2n-2} > c^4$.

By Proposition 2.2 we have

$$0 < P_r(z) - \left(\frac{b}{a}\right)^{1/n} \frac{q}{p} Q_r(z) < t_r \left(\frac{c}{v}\right)^{2r+1}$$

and, using that $0 < 1 - (b/a)^{1/n}(u/t) < c/w$, also

$$0 < P_r(z) - \left(\frac{b}{a}\right)^{1/n} \frac{u}{t} P_r(z) < s_r \frac{c}{w}.$$

Together this implies that

$$\left|\frac{u}{t} P_r(z) - \frac{q}{p} Q_r(z)\right| < \left(\frac{a}{b}\right)^{1/n} \left(s_r \frac{c}{w} + t_r \left(\frac{c}{v}\right)^{2r+1}\right).$$

The rational number on the left is positive (by the selection of r) and has denominator at most ptv^r. Multiplying by ptv^r we therefore get

$$\left(\frac{vw}{ab}\right)^{1/n} v^r \left(s_r \frac{c}{w} + t_r \left(\frac{c}{v}\right)^{2r+1}\right) > 1.$$

We show that the left side is in fact smaller than 1. By the selection of r and by the first inequality defining l in the 2nd step we have the bound

$$\left(\frac{vw}{ab}\right)^{1/n} v^r s_r \frac{c}{w} \leq \left(\frac{vw}{ab}\right)^{1/n} v^l s_l \frac{c}{w} \leq \frac{1}{n}.$$

Since $l \leq r+1$, the second inequality defining l yields

$$(vw)^{1-1/n} < ncs_{r+2} v^{r+3}/(ab)^{1/n}.$$

Using this, Lemma 2.8, $v > \sqrt{ab}$, $r \geq 1$ (this implies that $\frac{2}{n-1} - (r+1)\frac{n-2}{n-1} \leq 0$), and finally the bound on ab in terms of λ_n and c, we obtain another bound

$$\begin{aligned}
\left(\frac{vw}{ab}\right)^{1/n} v^r t_r \left(\frac{c}{v}\right)^{2r+1} &< (ab)^{-1/n} c^{2r+1} t_r v^{-r-1} \left((ab)^{-1/n} ncs_{r+2} v^{r+3}\right)^{\frac{1}{n-1}} \\
&= t_r (ns_{r+2})^{\frac{1}{n-1}} (ab)^{-\frac{1}{n-1}} c^{2r+1+\frac{1}{n-1}} v^{\frac{2}{n-1} - (r+1)\frac{n-2}{n-1}} \\
&\leq (1 - 1/n) \left((ab)^{1-n/2} \lambda_n c^{2n-2}\right)^{(r+1)/(n-1)} \\
&\leq 1 - 1/n.
\end{aligned}$$

Both bounds together give a contradiction that proves the theorem. □

2.3 The theorem of Baker

Theorem 2.10 (Baker, 1964) *If the numbers $x, y, c \in \mathbb{Z}$ satisfy*

$$x^3 - 2y^3 = c$$

then $\max(|x|, |y|) \leq (263000|c|)^{22}$.

The bound stated in [2] is in fact $(300000|c|)^{23}$. This will follow from the next more general theorem, which is in [2] formulated with $(a/b)^{m/n}$, $1 \leq m < n$, but here we set $m = 1$.

Theorem 2.11 (Baker, 1964) *Let $a, b, n \in \mathbb{N}$ be such that $n \geq 3$, $7a/8 \leq b < a$ and n divides $a - b$. Let $\mu_n = \prod_{p \mid n} p^{1/(p-1)}$ and suppose that*

$$\lambda = \frac{4b}{\mu_n (a-b)^2} > 1.$$

Define positive constants κ and c by the relations

$$\lambda^\kappa = 2\mu_n(a+b) \quad \text{and} \quad c = \frac{1}{2^{\kappa+3}(a+b)}.$$

Then

$$\left| p - (a/b)^{1/n} q \right| > \frac{c}{q^\kappa}$$

for every pair $p, q \in \mathbb{N}$.

Proof of Theorem 2.10. We apply the last theorem for $a = 128 = 2^7, b = 125 = 5^3$ and $n = 3$. Then $\mu_3 = \sqrt{3}$, $\lambda = 500/9\sqrt{3} > 1$ and also other assumptions are met. We get

$$\kappa = \frac{\log(506\sqrt{3})}{\log(500/9\sqrt{3})} = 1.95377\ldots \quad \text{and} \quad c = \frac{1}{2^\kappa \cdot 2024} = 0.000127\ldots .$$

Then

$$\left| 4p - (a/b)^{1/3} 5q \right| = \left| 4p - 4q\sqrt[3]{2} \right| > \frac{c}{(5q)^\kappa}$$

and

$$\left| p - q\sqrt[3]{2} \right| > \frac{c}{4(5q)^\kappa} > \frac{10^{-2}c}{q^\kappa} > \frac{10^{-6}}{q^{1.954}}$$

for every $p, q \in \mathbb{N}$.

Now let $x^3 - 2y^3 = c$ for $x, y, c \in \mathbb{Z}$. If $xy \le 0$ then $|c| = |x^3 - 2y^3| = |x|^3 + 2|y|^3$ and

$$\max(|x|, |y|) \le |c|.$$

If $xy > 0$, we may assume that $x, y > 0$. From $x^3 - 2y^3 = c$ we get that $x = (c + 2y^3)^{1/3} \le 2^{1/3}y + |c|$ and

$$x^2 + 2^{1/3}xy + 2^{2/3}y^2 = \frac{|c|}{|x - y2^{1/3}|} \le 10^6|c|y^{1.954}.$$

For $0 < y \le x$ it implies that $y^{0.046} \le 10^6|c|/(1 + 2^{1/3} + 2^{2/3}) = (2^{1/3} - 1)10^6|c|$, or $y \le (259930|c|)^{22}$ and

$$\max(|x|, |y|) = x \le 2^{1/3}y + |c| \le (263000|c|)^{22}.$$

For $0 < x < y$ we have $|x - y2^{1/3}| > 1$ and therefore $2^{2/3}y^2 \le |c|$, which gives

$$\max(|x|, |y|) = y \le |c|.$$

So in all cases is $\max(|x|, |y|) \le (263000|c|)^{22}$. $\square$

Proof of Theorem 2.11. Let $a, b, n \in \mathbb{N}$ satisfy the assumptions and $\mu_n, \lambda, \kappa, c$ be the mentioned constants. For $r = 1, 2, \ldots$ we set

$$p_r = (a/n)^r Q_r(1 - b/a) \quad \text{and} \quad q_r = (a/n)^r P_r(1 - b/a)$$

where $P_r(x)$ and $Q_r(x)$ are the polynomials of Proposition 2.1. Propositions 2.1 and 2.3 tell us that $p_r, q_r \in \mathbb{Z}$, $q_r > 0$, and $p_{r+1}q_r - p_r q_{r+1} \ne 0$. Setting in Proposition 2.3 $x = 1 - b/a$, multiplying the first bound in it by $(a/n)^r$, the second by $(a/n)^r(a/b)^{1/n}$, using that $\sigma_r \le \mu_n^r$ and the definition of λ, we get the bounds

$$q_r < 2\left(2\mu_n(a + b)\right)^r \quad \text{and} \quad 0 < (a/b)^{1/n}q_r - p_r < 3(a - b)/4b\lambda^r.$$

Let $p, q \in \mathbb{N}$ be arbitrary pair. We first assume that $q \ge \frac{1}{2}\lambda\mu_n$. Then (as $\lambda > 1$) there is a unique $l \in \mathbb{N}$ such that

$$\lambda^l \le 2q/\mu_n < \lambda^{l+1}.$$

We select $r \in \{l, l+1\}$ so that $pq_r - p_r q \ne 0$, which is possible due to $p_{l+1}q_l - p_l q_{l+1} \ne 0$. By the upper bound on q_r, the definition of κ, the first inequality defining l, and the definition of c we have

$$q_r < 2\lambda^{\kappa(l+1)} \le 2(2\lambda q/\mu_n)^\kappa = \mu_n^{1-\kappa}q^\kappa/2c.$$

From the upper bound on $(a/b)^{1/n}q_r - p_r$, the second inequality defining l, the definition of λ and $a-b \geq n \geq 3$ we have

$$|(a/b)^{1/n}q_r - p_r| < \frac{3(a-b)\lambda\mu_n}{8bq} = \frac{3}{2(a-b)q} \leq \frac{1}{2q}.$$

From this, $pq_r - p_r q \neq 0$, the second upper bound on q_r and $\kappa > 1$ we get the desired bound

$$\begin{aligned} |p-(a/b)^{1/n}q| &= q_r^{-1}|pq_r - (a/b)^{1/n}qq_r| \\ &\geq q_r^{-1}\left(|pq_r - p_r q| - q|p_r - (a/b)^{1/n}q_r|\right) \\ &> q_r^{-1}(1-1/2) \\ &> \frac{2c\mu_n^{\kappa-1}}{2q^\kappa} > \frac{c}{q^\kappa}. \end{aligned}$$

In the case $0 < q < \frac{1}{2}\lambda\mu_n$ we use the expansion

$$(a/b)^{1/n} = (1+(a-b)/b)^{1/n} = 1 + \frac{a-b}{nb} + S,\ S = \sum_{j\geq 2}\binom{1/n}{j}(a/b-1)^j.$$

We will prove the upper bound

$$|S| < \frac{1}{3aq}.$$

Using it, the inequality $p \neq q + q(a-b)/nb$ (otherwise nb would divide $q(a-b)$, which is impossible as $q(a-b) < \frac{1}{2}(a-b)\lambda\mu_n = 2b/(a-b) < b$) and the fact that n divides $a-b$, we get again the desired bound

$$\left|p-(a/b)^{1/n}q\right| \geq \frac{1}{b} - q|S| > \frac{2}{3a} > \frac{1}{a+b} > 16c > \frac{c}{q^\kappa}.$$

It remains to prove the upper bound on S. For $j \geq 2$ is

$$\left|\binom{1/n}{j}\right| = \frac{(1/n)(1-1/n)}{j!}\prod_{i=2}^{j-1}(i-1/n) \leq \frac{(1/4)(j-1)!}{j!} \leq \frac{1}{8}.$$

Thus

$$|S| \leq \frac{1}{8}\sum_{j\geq 2}(a/b-1)^j = \frac{(a-b)^2}{8b(2b-a)} \leq \frac{(a-b)^2}{6ab}.$$

Since $q < \frac{1}{2}\lambda\mu_n = 2b/(a-b)^2$, it follows that $|S| < 1/3aq$. $\square$

3 The polynomials $P_r(x)$ and $Q_r(x)$

Following [14] and [2] we define by means of the hypergeometric series the polynomials $P_r(x)$ and $Q_r(x)$ of Propositions 2.1–2.3. The formulas in [14], taken literally, are ill defined because of zero denominators. We justify them by rigorous evaluations of indeterminate expressions of the type $\frac{0}{0}$. The next two subsections consist therefore mainly of filling in details missing in [14] and [2] (many of them are standard results on the hypergeometric series). We take this lengthy route because of our love of power series and because we are interested in how to cope with the expressions $\frac{0}{0}$. Short and self-contained proof of Proposition 2.2, based on hypergeometric formulas with nonzero denominators, was given by Evertse [7, Lemma 8]. Simple inductive definition of $P_r(x)$ and $Q_r(x)$ avoiding hypergeometric formulas altogether is in Skolem [15].

The hypergeometric series is the formal power series

$$\begin{aligned} F(a,b,c,x) &= \sum_{j=0}^{\infty}\left(\prod_{k=0}^{j-1}\frac{(a+k)(b+k)}{(1+k)(c+k)}\right)x^j \\ &= 1+\frac{ab}{c}\cdot\frac{x}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\cdot\frac{x^2}{2!}+\cdots\in\mathbb{C}(a,b,c)[[x]] \end{aligned}$$

in variable x, where a,b,c are formal parameters. We denote the coefficient of x^j by $k_j = k_j(a,b,c)$, thus $k_0(a,b,c)=1$ and, for $j=1,2,\ldots$,

$$k_j(a,b,c)=\frac{a(a+1)\ldots(a+j-1)\cdot b(b+1)\ldots(b+j-1)}{j!\cdot c(c+1)\ldots(c+j-1)}\in\mathbb{C}(a,b,c).$$

We need to specialize $F(a,b,c,x)$ by assigning to some or all of a,b,c values in $\mathbb{C}$. For example, $P_r(x)$ is in [14] defined, up to a normalizing factor, as $F(-\frac{1}{n}-r,-r,-2r,x)$. Unfortunately, for $j>2r$ the coefficients are indeterminate, $k_j=\frac{0}{0}$. How to make sense of them? They are redefined as 0s so that $P_r(x)$ is a polynomial. Now we discuss this in detail and greater generality.

3.1 Limit specializations in $\mathbb{C}(a,b,c)[[x]]$

In view of the applications to $F(a,b,c,x)$ we work in the field of rational functions $\mathbb{C}(a,b,c)$ and in the ring of formal power series with coefficients in $\mathbb{C}(a,b,c)$.

A *specialization* is a triple

$$(\alpha, \beta, \gamma) \in (\mathbb{C} \cup \{a\}) \times (\mathbb{C} \cup \{b\}) \times (\mathbb{C} \cup \{c\})$$

where a, b, c are formal variables. For any polynomial $p \in \mathbb{C}[a, b, c]$, $p(\alpha, \beta, \gamma)$ is a well defined element of $\mathbb{C}[a, b, c]$ and (α, β, γ) determines a ring homomorphism from $\mathbb{C}[a, b, c]$ to itself. If $p_k, p \in \mathbb{C}[a, b, c]$, $k = 1, 2, \ldots$, we write

$$\lim_{k\to\infty} p_k = p$$

if for every $j \geq 0$ and $k \to \infty$ the coefficient of x^j in p_k goes in the euclidean metric to the coefficient of x^j in p. This limit commutes with addition and multiplication in $\mathbb{C}[a, b, c]$. A *limit specialization*

$$(\alpha_k, \beta_k, \gamma_k) \to (\alpha, \beta, \gamma),\ k \to \infty,$$

is a specialization (α, β, γ) and a sequence of specializations $(\alpha_k, \beta_k, \gamma_k)$, $k = 1, 2, \ldots$, such that, for every k, either $\alpha_k, \alpha \in \mathbb{C}$ and $\alpha_k \to \alpha$ in the euclidean metric or $\alpha_k = \alpha = a$, and similarly for the other two coordinates. Every specialization can be viewed as a limit one via the constant sequence $(\alpha_k, \beta_k, \gamma_k) = (\alpha, \beta, \gamma)$.

Recall that a rational function $f \in \mathbb{C}(a, b, c)$ is an equivalence class defined on the set of fractions of polynomials $\frac{p}{q}$, $p, q \in \mathbb{C}[a, b, c], q \neq 0$, by the equivalence relation $\frac{p}{q} \sim \frac{p_0}{q_0} \iff pq_0 - p_0q = 0$. Thus

$$f = \{\tfrac{p}{q} \mid pq_0 - p_0q = 0\}$$

where $\frac{p_0}{q_0}$ is a representative of f. Instead of $\frac{p}{q} \in f$ and $\frac{p}{q} \sim \frac{p_0}{q_0}$ we write less precisely $f = \frac{p}{q}$ and $\frac{p}{q} = \frac{p_0}{q_0}$. Let (α, β, γ) be a specialization, $f \in \mathbb{C}(a, b, c)$ and $\frac{p}{q}$ run through the representatives of f. We say that the specialization of f at (α, β, γ) is *regular* if $q(\alpha, \beta, \gamma) \neq 0$ for some $\frac{p}{q}$, *irregular* if $p(\alpha, \beta, \gamma) = q(\alpha, \beta, \gamma) = 0$ for every $\frac{p}{q}$, and *singular* if $p(\alpha, \beta, \gamma) \neq 0$ but $q(\alpha, \beta, \gamma) = 0$ for some $\frac{p}{q}$. Equivalently, $\frac{p}{q}$ in these definitions can be taken in lowest terms, i.e., with coprime polynomials p and q. In the regular case $f(\alpha, \beta, \gamma) := \frac{p(\alpha,\beta,\gamma)}{q(\alpha,\beta,\gamma)}$ is a well defined element of $\mathbb{C}(a, b, c)$ and regular specialization commutes with addition and multiplication in $\mathbb{C}(a, b, c)$. In the irregular case we write $f(\alpha, \beta, \gamma) = \frac{0}{0}$. We assign no value to singular specialization but will do so for some irregular specializations. Note that in one variable, in $\mathbb{C}(a)$, irregular specializations do not occur.

Let $f, f_0 \in \mathbb{C}(a,b,c)$ and L be a limit specialization $(\alpha_k, \beta_k, \gamma_k) \to (\alpha, \beta, \gamma)$. We say that f is *regular* at L and has at L *value* f_0, written $f(L) = f_0$, or that L *redefines* $f(\alpha, \beta, \gamma)$ *as* f_0, if every specialization $f(\alpha_k, \beta_k, \gamma_k)$ is regular and there exist fractions $\frac{p_k}{q_k}$ and $\frac{p_0}{q_0}$, $f(\alpha_k, \beta_k, \gamma_k) = \frac{p_k}{q_k}$ and $f_0 = \frac{p_0}{q_0}$, such that

$$\lim_{k \to \infty} \frac{p_k}{q_k} = \frac{p_0}{q_0},$$

that is, $\lim_{k\to\infty} p_k = p_0$ and $\lim_{k\to\infty} q_k = q_0$. We collect properties of evaluations at limit specializations.

Proposition 3.1 *Let L be a limit specialization $(\alpha_k, \beta_k, \gamma_k) \to (\alpha, \beta, \gamma)$ and $f, g \in \mathbb{C}(a,b,c)$.*

1. *If f is regular at L then the value $f(L)$ is unique and does not depend on the choice of representatives.*
2. *If f is regular at L then $f(\alpha, \beta, \gamma)$ is non-singular.*
3. *If $f(\alpha, \beta, \gamma)$ is regular then f is regular at L and $f(L) = f(\alpha, \beta, \gamma)$.*
4. *If f and g are regular at L then so are $f + g$ and fg and, moreover, $f(L) + g(L) = (f+g)(L)$ and $f(L)g(L) = (fg)(L)$.*

Proof. 1. If $\frac{p_k}{q_k} = \frac{r_k}{s_k}$ are two representatives of $f(\alpha_k, \beta_k, \gamma_k)$ and the fractions $\frac{p_0}{q_0}$, $\frac{r_0}{s_0}$ satisfy that $\lim_{k\to\infty} p_k = p_0$, $\lim_{k\to\infty} q_k = q_0$, $\lim_{k\to\infty} r_k = r_0$ and $\lim_{k\to\infty} s_k = s_0$, then $p_k s_k - r_k q_k = 0$ in limit turns in $p_0 s_0 - r_0 q_0 = 0$, which means that $\frac{p_0}{q_0} = \frac{r_0}{s_0}$.

2. Suppose for contrary that $f(\alpha, \beta, \gamma)$ is singular but that in limit some representatives of $f(\alpha_k, \beta_k, \gamma_k)$ give a value $f_0 = \frac{p_0}{q_0}$ of f at L. So we have a representative $f = \frac{p}{q}$ such that $p(\alpha, \beta, \gamma) \neq 0$ but $q(\alpha, \beta, \gamma) = 0$, and representatives $f = \frac{p_k}{q_k}$ such that $q_k(\alpha_k, \beta_k, \gamma_k) \neq 0$ for every k and $p_k(\alpha_k, \beta_k, \gamma_k)$, $q_k(\alpha_k, \beta_k, \gamma_k)$ go in limit to p_0, q_0, respectively. But then the relations $pq_k - p_k q = 0$, specialized at $\alpha_k, \beta_k, \gamma_k$, go in limit to $p(\alpha, \beta, \gamma)q_0 - q(\alpha, \beta, \gamma)p_0 = 0$, giving $q_0 = 0$, which is impossible.

3. We take $f = \frac{p}{q}$ with $q(\alpha, \beta, \gamma) \neq 0$ and consider the representatives $f(\alpha_k, \beta_k, \gamma_k) = \frac{p(\alpha_k, \beta_k, \gamma_k)}{q(\alpha_k, \beta_k, \gamma_k)}$ (omitting the finitely many terms with zero denominators). In limit we get that f has at L the value $f_0 = \frac{p(\alpha, \beta, \gamma)}{q(\alpha, \beta, \gamma)}$, which is $f(\alpha, \beta, \gamma)$.

4. We show this only for product; the case of sum is similar. We have fractions $\frac{p_k}{q_k}$ and $\frac{r_k}{s_k}$, $f(\alpha_k,\beta_k,\gamma_k)=\frac{p_k}{q_k}$ and $g(\alpha_k,\beta_k,\gamma_k)=\frac{r_k}{s_k}$, such that $\lim_{k\to\infty}p_k=p_0$, $\lim_{k\to\infty}q_k=q_0$, $\lim_{k\to\infty}r_k=r_0$ and $\lim_{k\to\infty}s_k=s_0$ where the polynomials q_0 and s_0 are nonzero. For $k\to\infty$ the fractions $\frac{p_kr_k}{q_ks_k}$ representing $(fg)(\alpha_k,\beta_k,\gamma_k)$ go to $\frac{p_0r_0}{q_0s_0}$, which means that fg is regular at L and that $(fg)(L)=\frac{p_0r_0}{q_0s_0}=\frac{p_0}{r_0}\cdot\frac{q_0}{s_0}=f(L)g(L)$. □

Thus the value at a limit specialization is unambiguous and for regular specialization $f(\alpha,\beta,\gamma)$ the way $(\alpha_k,\beta_k,\gamma_k)$ approach (α,β,γ) is irrelevant. However, for irregular specialization $f(\alpha,\beta,\gamma)$ its redefinition typically depends on the limit transition to (α,β,γ). For example, if $f=\frac{ab}{c}$ then $(u/k,b,1/k)\to(0,b,0)$, $k\to\infty$, redefines $f(0,b,0)=\frac{0}{0}$ as ub where $u\in\mathbb{C}$ is arbitrary. For irregular specialization the existence of redefinitions of $f(\alpha,\beta,\gamma)$ and $g(\alpha,\beta,\gamma)$ in general does not imply the existence of redefinitions of $(f+g)(\alpha,\beta,\gamma)$ and $(fg)(\alpha,\beta,\gamma)$. For example, if $f=\frac{a}{bc}$ and $g=\frac{b}{a}$ then it is easy to find redefinitions of $f(0,0,0)$ and $g(0,0,0)$ (e.g., $(1/k^2,1/k,1/k)\to(0,0,0)$ and $(1/k,1/k,0)\to(0,0,0)$ redefine, respectively, both $\frac{0}{0}$ as 1) but no redefinition of $(fg)(0,0,0)$ exists because $fg=\frac{1}{c}$ is singular at $(0,0,0)$.

We proceed to the ring of formal power series $\mathbb{C}(a,b,c)[[x]]$. For

$$f=f(a,b,c,x)\in\mathbb{C}(a,b,c)[[x]],\ f=\sum_{j\geq 0}k_jx^j,$$

and a specialization (α,β,γ) we call the specialization $f(\alpha,\beta,\gamma,x)$ of f *regular* if every $k_j(\alpha,\beta,\gamma)$ is regular, *irregular* if one or more $k_j(\alpha,\beta,\gamma)$ are irregular but the other $k_j(\alpha,\beta,\gamma)$ are regular, and *singular* if at least one $k_j(\alpha,\beta,\gamma)$ is singular. If L is a limit specialization $(\alpha_k,\beta_k,\gamma_k)\to(\alpha,\beta,\gamma)$ and every coefficient k_j is regular at L, we say that f is *regular at* L and write

$$f(L,x)=\sum_{j\geq 0}k_j(L)x^j$$

for the formal power series whose coefficients are the values of the coefficients of $f(a,b,c,x)$ at L; we then say that f *has value* $f(L,x)$ *at* L or that L *redefines* $f(\alpha,\beta,\gamma,x)$ *as* $f(L,x)$.

By a *differential equation* we shall mean here a homogeneous linear differential equation with polynomial coefficients for elements of $\mathbb{C}(a,b,c)[[x]]$, that is, an equation

$$D=D(a,b,c,x,X)=p_0X+p_1X'+p_2X''+\cdots+p_uX^{(u)}=0$$

where $p_r \in \mathbb{C}[a,b,c,x]$, X is a variable with domain $\mathbb{C}(a,b,c)[[x]]$ and differentiation is with respect to x. If L is a limit specialization $(\alpha_k,\beta_k,\gamma_k) \to (\alpha,\beta,\gamma)$ then $D(L,x,X) = D(\alpha,\beta,\gamma,x,X)$ is

$$\begin{aligned} & p_0(L,x)X + p_1(L,x)X' + \cdots + p_u(L,x)X^{(u)} \\ = \quad & p_0(\alpha,\beta,\gamma,x)X + p_1(\alpha,\beta,\gamma,x)X' + \cdots + p_u(\alpha,\beta,\gamma,x)X^{(u)}. \end{aligned}$$

Proposition 3.2 *Let $f, g \in \mathbb{C}(a,b,c)[[x]]$ be formal power series, L be a limit specialization $(\alpha_k,\beta_k,\gamma_k) \to (\alpha,\beta,\gamma)$ and $D = 0$ be a differential equation.*

1. *If f and g are regular at L then so are $f+g$ and fg and, denoting $f_0 = f(L,x)$, $g_0 = g(L,x)$, $h_0 = (f+g)(L,x)$ and $h_1 = (fg)(L,x)$, we have that $f_0 + g_0 = h_0$ and $f_0 g_0 = h_1$.*

2. *If $D(a,b,c,x,f) = 0$ and f is regular at L, then $f(L,x)$ is a solution to the specialized equation, $D(\alpha,\beta,\gamma,x,f(L,x)) = 0$.*

Proof. 1. This follows by part 4 of Proposition 3.1 and by the definition of addition and multiplication in $\mathbb{C}(a,b,c)[[x]]$

2. The fact that $f = \sum_{j\ge 0} k_j x^j$ solves $D = \sum_{r=0}^{u} p_r X^{(r)} = 0$ is equivalent, by equating the coefficient of x^j to 0, to the fact that in $\mathbb{C}(a,b,c)$ one has the equalities (given by finite sums)

$$\sum_{t-r+s=j} t(t-1)\dots(t-r+1)p_{r,s}(a,b,c)k_t(a,b,c) = 0, \ j = 0,1,2,\dots,$$

where $p_r = \sum_{s=0}^{d_r} p_{r,s}x^s$ with $p_{r,s} \in \mathbb{C}[a,b,c]$ and summation indices have ranges $t = 0,1,2,\dots,$ $r = 0,1,\dots,u$ and $s = 0,1,\dots,d_r$. By part 4 of Proposition 3.1, all these equalities remain preserved when "L" is written in place of "a,b,c". Because $p_{r,s}(L) = p_{r,s}(\alpha,\beta,\gamma)$, we see that $f(L,x)$ solves the specialized differential equation. □

We will often apply the result in part 2. Note that for fixed (α,β,γ), for every limit specialization $(\alpha_k,\beta_k,\gamma_k) \to (\alpha,\beta,\gamma)$ the diff. equation $D(a,b,c,x,X) = 0$ specializes to the same equation $D(\alpha,\beta,\gamma,x,X) = 0$ but irregular specialization $f(\alpha,\beta,\gamma,x)$ of $f \in \mathbb{C}(a,b,c)[[x]]$ may get redefined in different ways, which means that we obtain several solutions to the same differential equation.

3.2 Properties of the hypergeometric series

We apply the above theory to the hypergeometric series

$$F(a,b,c,x) = \sum_{j\geq 0} k_j x^j = 1 + \frac{ab}{c}\cdot\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\cdot\frac{x^2}{2!} + \cdots \in \mathbb{C}(a,b,c)[[x]].$$

The specialization $F(\alpha,\beta,\gamma,x)$ is

- regular if $\gamma \neq 0,-1,-2,\ldots$;
- irregular if $\gamma \in \mathbb{Z}$, $\gamma \leq 0$ but α or β belongs to $\{\gamma,\gamma+1,\ldots,0\}$, then $k_j(\alpha,\beta,\gamma) = \frac{0}{0}$ exactly if $j > -\gamma$;
- singular if $\gamma \in \mathbb{Z}$, $\gamma \leq 0$ and neither α nor β belongs to $\{\gamma,\gamma+1,\ldots,0\}$.

If it is not said else, we redefine any irregular specialization $F(\alpha,\beta,\gamma,x)$ by the standard limit specialization

$$(\alpha,\beta,\gamma+\tfrac{1}{k}) \to (\alpha,\beta,\gamma),\ k = 2,3,\ldots,$$

which redefines every coeficient $k_j = \frac{0}{0}$ in $F(\alpha,\beta,\gamma,x)$ as 0. Any irregular specialization $F(\alpha,\beta,\gamma,x)$ is thus redefined as a nonzero polynomial in $\mathbb{C}(a,b)[x]$ with degree at most $-\gamma$ (recall that always $k_0 = 1$). This will be the case for the polynomials $P_r(x)$ and $Q_r(x)$. An exception is the redefinition $f_r(x) = F(L,x)$ in Proposition 3.11 that is not a polynomial.

Proposition 3.3 *The hypergeometric series* $F(a,b,c,x) = \sum_{j\geq 0} k_j(a,b,c)x^j$ *satisfies the differential equation*

$$x(x-1)X'' + ((1+a+b)x-c)X' + abX = 0.$$

Proof. We work in the ring $\mathbb{C}(a,b,c)[[x]]$. The coefficient of x^j, $j = 0,1,\ldots$, on the left side equals

$$\begin{aligned} & j(j-1)k_j - j(j+1)k_{j+1} + j(1+a+b)k_j - (j+1)ck_{j+1} + abk_j \\ &= (j^2 + j(a+b) + ab)k_j - (j+1)(c+j)k_{j+1} \\ &= 0 \end{aligned}$$

because $k_{j+1}/k_j = (a+j)(b+j)/(1+j)(c+j)$. □

We call this differential equation the *hypergeometric differential equation with parameters* a,b,c or, for short, the hde(a,b,c). The transformations of it that we need usually affect only a,b,c but one involves also x.

Proposition 3.4 *Suppose that $G \in \mathbb{C}(a,b,c)[x]$ is a polynomial in x that solves the* hde(a,b,c). *Then $G_0(x) = G(1-x)$ solves the* hde$(a,b,a+b+1-c)$.

Proof. This follows by the substitution $x := 1-x$, which for polynomials is admissible. Note that $G_0(x) = G(1-x)$ and $G_0''(x) = G''(1-x)$ but $G_0'(x) = -G'(1-x)$. □

Proposition 3.5 *We denote $F_{-1} = F(a-1,b,c,x)$, $F = F(a,b,c,x)$ and $F_1 = F(a+1,b,c,x)$. In $\mathbb{C}(a,b,c)[[x]]$ we have the formal identity*

$$(c-a)F_{-1} + (2a-c-x(a-b))F + a(x-1)F_1 = 0.$$

It holds also with $F_{-1} = F(a,b-1,c,x)$ and $F_1 = F(a,b+1,c,x)$ when a and b are interchanged.

Proof. The ratios of $k_j(a-1,b,c)$, $k_j(a,b,c)$, $k_{j-1}(a,b,c)$, $k_{j-1}(a+1,b,c)$ and $k_j(a+1,b,c)$ to $k_j(a,b,c)$ show that the coefficient of x^j, $j = 0,1,\ldots$, on the left side is $k_j(a,b,c)$ multiplied by

$$(c-a)\frac{a-1}{a+j-1} + 2a - c - \frac{(a-b)j(c+j-1)}{(a+j-1)(b+j-1)} + a\frac{j(c+j-1)}{(b+j-1)a} - a\frac{a+j}{a},$$

which can be checked to be 0 identically in j. The second identity follows by the symmetry of $F(a,b,c,x)$ in a and b. □

Proposition 3.6 *We denote $F_{-1} = F(a,b,c-1,x)$, $F = F(a,b,c,x)$ and $F_1 = F(a,b,c+1,x)$. In $\mathbb{C}(a,b,c)[[x]]$ we have the formal identity*

$$c(c-1)(x-1)F_{-1} + c(c-1-x(2c-a-b-1))F + x(c-a)(c-b)F_1 = 0.$$

Proof. The ratios of $k_{j-1}(a,b,c-1)$, $k_j(a,b,c-1)$, $k_j(a,b,c)$, $k_{j-1}(a,b,c)$, and $k_{j-1}(a,b,c+1)$ to $k_j(a,b,c)$ show that the coefficient of x^j, $j = 0,1,\ldots$, on the left side is $k_j(a,b,c)$ multiplied by

$$\frac{c(c-1)j(c+j-2)(c+j-1)}{(c-1)(a+j-1)(b+j-1)} - c(c-1)\frac{c+j-1}{c-1} + c(c-1)$$
$$- \frac{c(2c-a-b-1)j(c+j-1)}{(a+j-1)(b+j-1)} + \frac{(c-a)(c-b)jc}{(a+j-1)(b+j-1)},$$

which can be checked to be 0 identically in j. □

We will use the formal binomial series

$$(1+x)^a = \sum_{j\geq 0} \binom{a}{j} x^j = \sum_{j\geq 0} \frac{a(a-1)\dots(a-j+1)}{j!} x^j \in \mathbb{C}(a)[[x]].$$

Every specialization $(1+x)^a$ with $a \in \mathbb{C}$ is regular. It is easy to verify the formal identities

$$((1+x)^a)' = a(1+x)^{a-1} \ \text{ and } \ (1+x)^a \cdot (1+x)^b = (1+x)^{a+b}.$$

Proposition 3.7 *No specialization $a \in \mathbb{C}\backslash\mathbb{Z}$ of $(1+x)^a$ is a rational power series, that is, of the form $\frac{P(x)}{Q(x)}$ where $P, Q \in \mathbb{C}[x]$ and Q is a unit in $\mathbb{C}[[x]]$.*

Proof. Suppose the contrary that for some $a \in \mathbb{C}\backslash\mathbb{Z}$ and $P, Q \in \mathbb{C}[x]$, $Q(0) \neq 0$, we have in $\mathbb{C}[[x]]$ the equality

$$Q(x) \cdot (1+x)^a = P(x).$$

Neither P nor Q is a zero polynomial and we may assume that $Q(x)$ is not divisible by $1+x$ (else we move a power of $1+x$ from $Q(x)$ to $(1+x)^a$ and replace a with another nonintegral value). Let $d = \deg P$. Differentiating the equality $d+1$ times we get the equality

$$S(x)\cdot(1+x)^{a-d-1} = (a(a-1)\dots(a-d)Q(x)+(1+x)R(x))\cdot(1+x)^{a-d-1} = 0$$

where $S, R \in \mathbb{C}[x]$. Since a is not an integer and $Q(x)$ is not divisible by $(1+x)$, we see that $S(x)$ is not a zero polynomial. Also, $(1+x)^{a-d-1} \neq 0$. But then the equality $S(x) \cdot (1+x)^{a-d-1} = 0$ is impossible because $\mathbb{C}[[x]]$ is an integral domain. We have a contradiction. □

Proposition 3.8 *In $\mathbb{C}(a,b,c)[[x]]$ (parts 1 and 2), or $\mathbb{C}(a,b)[[x]]$ (part 3), the following identities hold.*

1. *The power series*

$$(1-x)^{c-a-b} F(c-a, c-b, c, x)$$

satisfies the $\mathrm{hde}(a,b,c)$.

2. *In fact,*

$$F(a,b,c,x) = (1-x)^{c-a-b}F(c-a,c-b,c,x).$$

3. *For every* $m = 0, 1, 2, \ldots,$ *the power series*

$$x^{m+1}F(a+m+1, b+m+1, m+2, x)$$

satisfies the $\mathrm{hde}(a, b, -m)$.

Proof. 1. We denote $H = (1-x)^{c-a-b}$, $F = F(a,b,c,x)$ and $F_0 = F(c-a, c-b, c, x)$. We verify that

$$x(x-1)(HF_0)'' + ((1+a+b)x-c)(HF_0)' + abHF_0 = 0.$$

Using $H' = (a+b-c)H(1-x)^{-1}$ and $H'' = (a+b-c)(a+b-c+1)H(1-x)^{-2}$, dividing by H and collecting like terms we transform the left side into

$$x(x-1)F_0'' + ((2c-a-b+1)x-c)F_0' + (c^2 - c(a+b) + ab)F_0.$$

This equals 0 as an instance of the $\mathrm{hde}(c-a, c-b, c)$ for F_0.

2. We have $HF_0(0) = H(0)F_0(0) = 1 = F(0)$ and $(HF_0)'(0) = H'(0)F_0(0) + H(0)F_0'(0) = a+b-c+(c-a)(c-b)/c = ab/c = F'(0)$. Thus HF_0 and F not only satisfy the same hde but actually coincide.

3. For $m \in \mathbb{Z}$, $m \geq 0$, we denote $H = x^{m+1}$ and $F_0 = F(a+m+1, b+m+1, m+2, x)$. We verify that

$$x(x-1)(HF_0)'' + ((1+a+b)x+m)(HF_0)' + abHF_0 = 0.$$

Using $H' = (m+1)Hx^{-1}$ and $H'' = (m+1)mHx^{-2}$, dividing by H and collecting like terms we transform the left side into

$$x(x-1)F_0'' + ((2m+3+a+b)x-m-2)F_0' + ((m+1)(m+a+b+1)+ab)F_0.$$

This is 0 as an instance of the $\mathrm{hde}(a+m+1, b+m+1, m+2)$ for F_0. □

The identity in part 2 demonstrates usefulness of limit specializations because in them it specializes correctly, while the convention "set $k_j(\alpha,\beta,\gamma) = 0$ whenever the numerator of $k_j(\alpha,\beta,\gamma)$ is zero" leads to confusion. For example, the specialization $(\alpha,\beta,\gamma) = (-1,0,0)$ turns the identity in $F(-1,0,0,x) = (1-x)F(1,0,0,x)$ and the convention that $F(-1,0,0) =$

$F(1,0,0)=1$ is not very helpful as it gives the invalid equality $1=(1-x)\cdot 1$. Correct intepretation is via limit specializations. For example, if K is $(-1,0,\frac{1}{k})\to(-1,0,0)$ then the corresponding L is $(\frac{1}{k}+1,\frac{1}{k},\frac{1}{k})\to(1,0,0)$ and the identity specializes to $F(K,x)=(1-x)F(L,x)$, which is the valid equality $1=(1-x)(1+x+x^2+\dots)$.

We will need to specialize the formal power series $F(a,b,c,x)=\sum_{j\geq 0}k_jx^j$ and $(1-x)^a=\sum_{j\geq 0}\binom{a}{j}(-x)^j$ at $x=1$.

Proposition 3.9 *Let α,β,γ be real numbers, $\gamma\notin\{0,-1,\dots\}$. The following results on convergence hold.*

1. *If $\gamma>\alpha+\beta-1$ then $k_j(\alpha,\beta,\gamma)\to 0$ as $j\to\infty$. If $\gamma>\alpha+\beta$ then $F(\alpha,\beta,\gamma,x)$ absolutely converges at $x=1$.*

2. *If $\alpha>-1$ then $\binom{\alpha}{j}\to 0$ as $j\to\infty$. If $\alpha>0$ then $(1-x)^\alpha$ absolutely converges at $x=1$.*

Proof. 1. We may assume that neither α nor β is a non-positive integer (hence always $k_j\neq 0$) because else both claims hold trivially. We have

$$\frac{k_{j+1}}{k_j}=\frac{(1+\alpha/j)(1+\beta/j)}{(1+\gamma/j)(1+1/j)}=1+\frac{\alpha+\beta-\gamma-1}{j}+O(j^{-2}),\ j\to\infty,$$

where the implicit constant in O depends only on α,β,γ. Suppose that $j>j_0$ is large enough so that $|(\alpha-\beta-\gamma-1)/j|<\frac{1}{2}$. Then

$$\begin{aligned}\log|k_j| &= \log|k_{j_0}|+\sum_{i=j_0}^{j-1}\log|k_{i+1}/k_i|=(\alpha+\beta-\gamma-1)\sum_{i=1}^{j-1}i^{-1}+O(1)\\ &= (\alpha+\beta-\gamma-1)\log j+O(1)\end{aligned}$$

because $\log|1+x|=x+O(x^2)$ for $x\in(-\frac{1}{2},\frac{1}{2})$, the series $\sum_{j\geq 1}j^{-2}$ converges and $\sum_{i=1}^{j-1}i^{-1}=\log j+O(1)$ by the integral estimate. Hence

$$|k_j|<cj^{\alpha+\beta-\gamma-1},\ j=1,2,\dots,$$

with a constant $c>0$. From this estimate both claims follow at once.

2. As $(1-x)^a=F(-a,1,1,x)$, this is a particular case of 1. □

Proposition 3.10 *Let α,β,γ be real numbers, $\gamma\notin\{0,-1,\dots\}$.*

1. *If $\gamma > \alpha + \beta$, then*

$$F(\alpha, \beta, \gamma, 1) = \frac{\gamma - \alpha}{\gamma - \alpha - \beta} F(\alpha - 1, \beta, \gamma, 1)$$

$$F(\alpha, \beta, \gamma, 1) = \frac{\gamma - \beta}{\gamma - \alpha - \beta} F(\alpha, \beta - 1, \gamma, 1).$$

   If $\gamma > \alpha + \beta + 1$ and $\gamma \neq \alpha + 1, \beta + 1$, then

$$F(\alpha, \beta, \gamma, 1) = \frac{(\gamma - 1)(\gamma - 1 - \alpha - \beta)}{(\gamma - 1 - \alpha)(\gamma - 1 - \beta)} F(\alpha, \beta, \gamma - 1, 1).$$

2. *If $\alpha > 0$ then $(1-x)^\alpha$ converges at $x = 1$ to 0.*

Proof. For both results we need this simple fact: If $A(x) = a_0 + a_1 x + \dots$ and $B(x) = b_0 + b_1 x + \dots$ in $\mathbb{C}[[x]]$ are bound by the identity $A(x) = (1-x)B(x)$, $A(1)$ converges and $b_n \to 0$ as $n \to \infty$, then in fact $A(1) = 0$. Indeed, the polynomial identity $a_0 + a_1 x + \cdots + a_n x^n = (1-x)(b_0 + b_1 x + \cdots + b_n x^n) + b_n x^{n+1}$ for $x = 1$ gives $a_0 + a_1 + \cdots + a_n = b_n$ and the fact follows.

Thus the recurrences in 1 follow by specializing the formal identities of Propositions 3.5 and 3.6 at (α, β, γ), respectively at $(\alpha, \beta, \gamma - 1)$, and then using part 1 of Proposition 3.9 and the simple fact. In the same way, 2 follows by specializing $(1-x)^a = (1-x) \cdot (1-x)^{a-1}$ at α and then using part 2 of Proposition 3.9 and the simple fact. $\square$

The polynomials $P_r(x)$ and $Q_r(x)$ come from the following specializations of $F(a, b, c, x)$.

Proposition 3.11 *For $r, n \in \mathbb{N}$ with $n \geq 2$ we set, redefining every $\frac{0}{0}$ coefficient in F as 0,*

$$\begin{aligned} d_r(x) &= F(-n^{-1} - r, -r, 1 - n^{-1}, 1 - x) \\ e_r(x) &= F(-n^{-1} - r, -r, -2r, x) \\ f_r(x) &= (1-x)^{1/n} F(n^{-1} - r, -r, -2r, x) \\ g_r(x) &= x^{2r+1} F(-n^{-1} + r + 1, r + 1, 2r + 2, x). \end{aligned}$$

These four elements of $\mathbb{C}[[x]]$ ($d_r(x)$ and $e_r(x)$ are nonzero degree r polynomials) are all solutions to the $\mathrm{hde}(-\frac{1}{n} - r, -r, -2r)$*. Consequently, we have the identities*

$$e_r(x) = e_r(1) d_r(x) \quad \text{and} \quad e_r(x) - f_r(x) = \rho_r g_r(x)$$

where $0 < e_r(1) < 1$ and $\rho_r = e_r(1)/g_r(1) > 0$.

Proof. The polynomial $d_r(x)$ satisfies the hde$(-\frac{1}{n}-r,-r,-\frac{1}{n}-r-r+1-(1-\frac{1}{n}))=$ hde$(-\frac{1}{n}-r,-r,-2r)$ by Proposition 3.4. For $e_r(x)$ consider the standard limit specialization

$$K:\ (-\tfrac{1}{n}-r,-r,-2r+\tfrac{1}{k})\to(-\tfrac{1}{n}-r,-r,-2r).$$

By Proposition 3.3 and part 2 of Proposition 3.2, $e_r(x)=F(K,x)$ satisfies the hde$(K)=$ hde$(-\frac{1}{n}-r,-r,-2r)$. The power series $g_r(x)$ satisfies the same hde by part 3 of Proposition 3.8 for $m=2r$, $a=-\frac{1}{n}-r$ and $b=-r$ and by Proposition 3.2; the corresponding specialization is regular.

To deal with the power series $f_r(x)$ we set $\alpha=-\frac{1}{n}-r,\beta=-r,\gamma=-2r$ and consider specializations (both are non-standard)

$$\begin{array}{rcl} L:\ (\alpha,\beta+\frac{1}{k},\gamma+\frac{1}{k}) & \to & (\alpha,\beta,\gamma) \\ K':\ (\gamma-\alpha+\frac{1}{k},\gamma-\beta,\gamma+\frac{1}{k}) & \to & (\gamma-\alpha,\gamma-\beta,\gamma). \end{array}$$

Then $F_0(a,b,c,x)=F(c-a,c-b,c,x)$ is regular at L and $F_0(L,x)=F(K',x)$. By the formal identity $F(a,b,c,x)=(1-x)^{c-a-b}F(c-a,c-b,c,x)$ in part 2 of Proposition 3.8, Proposition 3.3 and Proposition 3.2 we see that

$$F(L,x)=(1-x)^{\gamma-\alpha-\beta}F_0(L,x)=(1-x)^{\frac{1}{n}}F(K',x)=f_r(x)$$

satisfies the hde$(L)=$ hde$(-\frac{1}{n}-r,-r,-2r)$.

Since $d_r(x)$, $e_r(x)$ and $f_r(x)$ solve the same second order differential equation, they are linearly dependent over $\mathbb{C}$. In fact, already the polynomials $d_r(x)$ and $e_r(x)$ are linearly dependent because otherwise we could express $f_r(x)$ as their linear combination, which contradicts Proposition 3.7. Setting $x=1$ and using $d_r(1)=1$ we get the first identity. As a polynomial in $1-x$, $d_r(x)$ has positive coefficients and for $0\le x\le 1$ is positive and decreasing. Thus $e_r(1)=e_r(0)/d_r(0)=1/d_r(0)>0$, $e_r(x)=e_r(1)d_r(x)$ is decreasing for $0\le x\le 1$ and $e_r(1)<e_r(0)=1$.

Similarly, $e_r(x)$, $f_r(x)$ and $g_r(x)$ are linearly dependent,

$$ue_r(x)+vf_r(x)+wg_r(x)=0$$

with $u,v,w\in\mathbb{C}$ not all zero. From $e_r(0)=f_r(0)=1$ and $g_r(0)=0$ we get $u+v=0$. If $u=v=0$ then $w\neq 0$ and $g_r(x)=0$, which is not possible. Thus $u=-v\neq 0$ and we may assume that $u=1$ and $v=-1$. We set $x=1$, which we may by Proposition 3.9, and get, since $f_r(1)=0$ and $g_r(1)>0$, that $\rho_r=-w=e_r(1)/g_r(1)>0$. Also Proposition 3.7 shows that $w\neq 0$. □

Note that $e_r(x) = F(K, x)$ and $f_r(x) = F(L, x)$ are two distinct redefinitions of the irregular specialization $F(-\frac{1}{n} - r, -r, -2r, x)$.

Proposition 3.12 *Let $n, r \in \mathbb{N}$ with $n \geq 2$ and $e_r(x), g_r(x)$ be as in Proposition 3.11. Then*

$$\begin{aligned} e_r(1) &= \frac{r!}{(2r)!}\prod_{k=1}^{r}(k - \frac{1}{n}) = \binom{2r}{r}^{-1}\prod_{k=1}^{r}(1 - 1/kn) \\ g_r(1) &= n\frac{(2r+1)!}{r!}\prod_{k=1}^{r}(k + \frac{1}{n})^{-1} = n(2r+1)\binom{2r}{r}\prod_{k=1}^{r}(1 + 1/kn)^{-1}. \end{aligned}$$

Consequently,

$$\rho_r = e_r(1)/g_r(1) = \frac{1}{n(2r+1)}\binom{2r}{r}^{-2}\prod_{k=1}^{r}(1 - 1/(kn)^2).$$

Proof. By the first identity of the previous proposition, $e_r(1) = 1/d_r(0)$. It suffices to check the formula

$$d_r(0) = F(-\frac{1}{n} - r, -r, 1 - \frac{1}{n}, 1) = \frac{(2r)!}{r! \cdot (r - \frac{1}{n})(r - 1 - \frac{1}{n}) \dots (1 - \frac{1}{n})}.$$

Let $\gamma > \alpha + \beta$. By part 1 of Proposition 3.10,

$$F(\alpha - 1, \beta - 1, \gamma, 1) = \frac{(\alpha + \beta - \gamma)(\alpha + \beta - \gamma - 1)}{(\gamma - \alpha)(\gamma - \beta)}F(\alpha, \beta, \gamma, 1).$$

For $\alpha = -\frac{1}{n} - r + 1, \beta = -r + 1, \gamma = 1 - \frac{1}{n}$ this gives the recurrence $d_r(0) = \frac{2r(2r-1)}{r(r-\frac{1}{n})}d_{r-1}(0)$ which with $d_0(0) = 1$ proves the first formula. But part 1 of Proposition 3.10 also gives

$$\begin{aligned} F(\alpha, \beta, \gamma, 1) &= \frac{(\gamma - \alpha)(\gamma - \beta)}{(\gamma - \alpha - \beta)(\gamma - \alpha - \beta + 1)}F(\alpha - 1, \beta - 1, \gamma, 1) \\ &= \frac{(\gamma - 1)(\gamma - 2)}{(\gamma - \alpha - 1)(\gamma - \beta - 1)}F(\alpha - 1, \beta - 1, \gamma - 2, 1). \end{aligned}$$

For $\alpha = -\frac{1}{n} + r + 1, \beta = r + 1, \gamma = 2r + 2$ we get the recurrence $g_r(1) = \frac{(2r+1)2r}{r(r+\frac{1}{n})}g_{r-1}(1)$. Since $g_0(1) = F(-\frac{1}{n}+1, 1, 2, 1) = n \cdot F(-\frac{1}{n}+1, 0, 2, 1) = n$, the second formula follows. Together they yield the formula for ρ_r. □

In [14] and [1] these values are obtained by means of the Gauss formula

$$F(\alpha,\beta,\gamma,1)=\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

where Γ is the gamma function. In [7] a simple binomial identity is employed to get $d_r(0)$.

3.3 Proofs of Propositions 2.1–2.3

After this special functions detour we finally get to the polynomials $P_r(x)$ and $Q_r(x)$ and their properties. For given $n \in \mathbb{N}$ with $n \geq 2$ we set $P_0(x) = Q_0(x) = 1$ and, for $r = 1, 2, \ldots,$

$$\begin{aligned}
P_r(x) &= \sigma_r n^r \binom{2r}{r} \cdot F(-n^{-1}-r,-r,-2r,x) = \sigma_r n^r \binom{2r}{r} \cdot F(K,x) \\
&= s_r e_r(x) \\
Q_r(x) &= \sigma_r n^r \binom{2r}{r} \cdot F(n^{-1}-r,-r,-2r,x) = \sigma_r n^r \binom{2r}{r} \cdot F(K',x) \\
&= s_r f_r(x)(1-x)^{-1/n}
\end{aligned}$$

where F is the hypergeometric series, K and K' are standard limit specializations given in the proof of Proposition 3.11, σ_r and s_r were defined after Proposition 2.1 and $e_r(x)$ and $f_r(x)$ in Proposition 3.11.

The coefficient of x^j is here nonzero for $j = r$ and zero for $j > r$. Therefore $P_r(x)$ and $Q_r(x)$ are nonzero polynomials with degree r, which establishes part 1 of Proposition 2.1. In particular,

$$P_1(x) = 2n-(n+1)x \quad \text{and} \quad Q_1(x) = 2n-(n-1)x.$$

For $n = 3$ this gives the two polynomials of Proposition 2.2.

We prove part 2 of Proposition 2.1. $P_r(x)$ equals

$$\begin{aligned}
&\sigma_r n^r \binom{2r}{r} \sum_{j=0}^{r} \frac{(r+n^{-1})(r+n^{-1}-1)\ldots(r+n^{-1}-j+1)}{(2r)(2r-1)\ldots(2r-j+1)} \binom{r}{j} (-x)^j \\
&= \sigma_r n^r \sum_{j=0}^{r} \binom{r+n^{-1}}{j} \binom{2r-j}{r} (-x)^j
\end{aligned}$$

and similarly

$$Q_r(x) = \sigma_r n^r \sum_{j=0}^{r} \binom{r - n^{-1}}{j} \binom{2r - j}{r} (-x)^j.$$

To prove that $P_r(x), Q_r(x) \in \mathbb{Z}[x]$, we need to show that for both choices of signs and every $j = 0, 1, \ldots, r$, the fraction

$$\frac{\sigma_r \cdot (rn \pm 1) \cdot ((r-1)n \pm 1) \cdot \ldots \cdot ((r-j+1)n \pm 1)}{j!}$$

is an integer. We show that every prime p in the factorization of this fraction has nonnegative exponent. It is clearly so if p divides n because by definition is σ_r divisible by at least as large power of p as is $j!$. Suppose that p does not divide n. It suffices to show that for any $e \in \mathbb{N}$ the number of multiples of p^e in the j-term arithmetic progression $N - n, N - 2n, \ldots, N - jn$, where $N = (r+1)n \pm 1$, majorizes this number for the progression $1, 2, \ldots, j$. It is true because the latter number equals $\lfloor j/p^e \rfloor$ and the former number is the same as this number for the progression $mN+1, mN+2, \ldots, mN+j$, where $m \in \mathbb{N}$ is the modulo p^e inverse of $-n$ (which exists as p does not divide n), which equals $\lfloor j/p^e \rfloor$ or $\lfloor j/p^e \rfloor + 1$. Thus the coefficients of $P_r(x)$ and $Q_r(x)$ are integers. In fact, to make $\binom{r \pm \frac{1}{n}}{j}$ an integer it suffices to multiply it by $\sigma_r n^j$ instead of $\sigma_r n^r$, which proves the initial claim of Proposition 2.3.

It is easy to see that the coefficients of $P_r(x)$ and $Q_r(x)$ are only exponentially big in r. Indeed, all factors σ_r, n^r,

$$\binom{2r - j}{r} \leq \binom{2r}{r} \leq 2^{2r} \quad \text{and} \quad \binom{r \pm n^{-1}}{j} \leq \binom{r+1}{\lfloor (r+1)/2 \rfloor} \leq 2^{r+1}$$

are only exponentially big in r. We have proved part 2 of Proposition 2.1.

We proceed to parts 4 and 3. By Proposition 3.11 and definition of $P_r(x)$ and $Q_r(x)$,

$$P_r(x) - (1-x)^{\frac{1}{n}} Q_r(x) = s_r(e_r(x) - f_r(x)) = s_r \rho_r g_r(x)$$

where

$$\rho_r = \tfrac{F(-\frac{1}{n} - r, -r, -2r, 1)}{F(-\frac{1}{n} + r + 1, r+1, 2r+2, 1)} \quad \text{and}$$

$$g_r(x) = x^{2r+1} F(-\tfrac{1}{n} + r + 1, r + 1, 2r + 2, x).$$

We show that $|\rho_r| \le 1$ and that $g_r(x)/x^{2r+1}$ is for $|x| \le 1$ bounded. Let $(x)_j = x(x-1)\dots(x-j+1)$ and $(x)^{(j)} = x(x+1)\dots(x+j-1)$. Since

$$\frac{|k_j(-\frac{1}{n}-r,-r,-2r)|}{k_j(-\frac{1}{n}+r+1,r+1,2r+2)}$$

is 0 for $j > r$ and positive, and equals

$$\frac{(2r+2)^{(j)}/2^j}{(r+1)^{(j)}} \cdot \frac{(r)_j}{(2r)_j/2^j} \cdot \frac{(r+\frac{1}{n})_j}{(r-\frac{1}{n}+1)^{(j)}} \le 1 \cdot 1 \cdot 1 = 1$$

for $j = 0, 1, \dots, r$, we get that $|\rho_r| \le 1$. As for $g_r(x)/x^{2r+1}$, the bound on k_j in the proof of Proposition 3.9 shows that for $|x| \le 1$ we indeed have

$$\left|\frac{g_r(x)}{x^{2r+1}}\right| \le F(-n^{-1}+r+1, r+1, 2r+2, 1) = \sum_{j\ge 0} k_j < c\sum_{j\ge 1} j^{-1-\frac{1}{n}} < c'.$$

The factor $s_r = \sigma_r n^r \binom{2r}{r}$ is only exponentially big in r and therefore part 4 of Proposition 2.1 has been proved.

The nonidentity in part 3 follows by eliminating $(1-x)^{1/n}$ among $P_r(x) - (1-x)^{1/n}Q_r(x)$ and $P_{r+1}(x) - (1-x)^{1/n}Q_{r+1}(x)$, which gives the identity

$$P_r(x)Q_{r+1}(x) - P_{r+1}(x)Q_r(x) = Q_{r+1}(x)h_r(x) - Q_r(x)h_{r+1}(x)$$

where $h_r(x) = s_r\rho_r g_r(x)$. Since $Q_{r+1}(0) = s_{r+1} \neq 0$ and the power series $h_r(x)$ begins

$$h_r(x) = s_r\rho_r x^{2r+1} + \dots$$

with a nonzero coefficient at x^{2r+1} (the nonvanishing of ρ_r, proved in Proposition 3.11, is crucial), the right side of this identity is a power series $cx^{2r+1} + \dots$ with $c = s_{r+1}s_r\rho_r \neq 0$. On the other hand, the left side is by properties of $P_r(x)$ and $Q_r(x)$ a polynomial with degree at most $2r+1$. It follows that there are no other terms on the right side,

$$P_r(x)Q_{r+1}(x) - P_{r+1}(x)Q_r(x) = cx^{2r+1} = s_{r+1}s_r\rho_r x^{2r+1},\ c \neq 0.$$

This implies part 3 of Proposition 2.1 which is now completely proved.

We continue with Proposition 2.2 whose initial claim was already established. By Proposition 3.11,

$$P_r(x) = s_r e_r(x) = e_r(1)s_r d_r(x)$$

is decreasing and positive on $[0,1)$ because $d_r(x)$ is a polynomial in $1-x$ with positive coefficients. From this and $e_r(0) = 1$ we get that $0 < P_r(x) < s_r$ on $(0,1)$. Further we have

$$P_r(x) - (1-x)^{1/n} Q_r(x) = s_r \rho_r g_r(x).$$

The power series $g_r(x)$ has positive coefficients and is therefore positive and increasing on $(0,1]$. Also, by Propositions 3.11 and 3.12, $s_r \rho_r > 0$ and $s_r \rho_r g_r(1) = s_r e_r(1) = t_r$. Thus $0 < P_r(x) - (1-x)^{1/n} Q_r(x) < t_r$ on $(0,1)$ and Proposition 2.2 is proven.

We proceed to Proposition 2.3 whose initial claim was already established too. Let $0 < x < 1$. By Proposition 3.11 we have $P_r(x) = s_r e_r(1) d_r(x) = s_r e_r(1) F(-\frac{1}{n} - r, -r, 1 - \frac{1}{n}, 1-x)$ which we know is positive. Thus, using Proposition 3.12 and denoting $l_r^{(j)} = \prod_{k=r-j+1}^{r} (kn + 1)$, we express $P_r(x)$ as

$$P_r(x) = \sigma_r (r!)^{-1} \sum_{j=0}^{r} \prod_{k=j+1}^{r} (kn-1) \cdot l_r^{(j)} \binom{r}{j} (1-x)^j.$$

Due to

$$l_r^{(j)} \prod_{k=j+1}^{r} (kn-1) \le n^r \prod_{k=r-j+1}^{r} (k+1) \prod_{k=j+1}^{r} k = r! n^r \binom{r+1}{j} \le r! n^r 2^{r+1}$$

we get the bound

$$P_r(x) \le \sigma_r n^r 2^{r+1} \sum_{j=0}^{r} \binom{r}{j} (1-x)^j = 2\sigma_r (2n)^r (2-x)^r.$$

It remains to estimate, for $x \in (0, \frac{1}{8}]$, once again the positive quantity

$$P_r(x) - (1-x)^{1/n} Q_r(x) = s_r \rho_r g_r(x) = s_r \rho_r x^{2r+1} h_r(x)$$

where $h_r(x) = F(-\frac{1}{n} + r + 1, r+1, 2r+2, x)$. Plugging in $s_r = \sigma_r n^r \binom{2r}{r}$, the formula for ρ_r of Proposition 3.12 and using the bounds $n \ge 2$, $(2r+1)\binom{2r}{r} \ge \frac{3}{2} 4^r$ $(r = 1, 2, \dots)$ we get that

$$P_r(x) - (1-x)^{1/n} Q_r(x) < (\sigma_r/3)(n/4)^r x^{2r+1} h_r(x).$$

Setting $s = \lfloor r/2 \rfloor$ we have

$$h_r(x) \le U_r + V_r = \sum_{j=0}^{s} w_r^{(j)} x^j + \sum_{j=s+1}^{\infty} w_r^{(j)} x^j, \; w_r^{(j)} = \binom{r+j}{j}^2 \binom{2r+j+1}{j}^{-1},$$

because $w_r^{(j)}$ majorize the coefficients in $h_r(x)$. From $w_r^{(j)} \le \binom{r+j}{j} \le 2^{r+j}$ and $0 < x \le \frac{1}{8}$ we get

$$V_r \le \sum_{j=s+1}^{\infty} 2^{r+j} x^j \le \sum_{j=s+1}^{\infty} 2^{r-2j} \le 2 \sum_{j=1}^{\infty} 2^{-2j} = 2/3.$$

If $0 \le k \le s$ and $r \ge 5$ we have $(r+1+k)^2 \le r(2r+2+k)$, which implies that $w_r^{(j)} = \frac{1}{j!} \prod_{k=0}^{j-1} (r+1+k)^2/(2r+2+k) \le r^j/j!$. Then, for $r \ge 5$,

$$U_r \le \sum_{j=0}^{s} \frac{r^j}{j!} x^j < \exp(xr) < (1-x)^{-r}.$$

The bound $U_r < (1-x)^{-r}$ for $x > 0$ holds also for $r = 1, 2, 3, 4$ because $U_1 = 1$, $U_2 = 1 + 3x/2$, $U_3 = 1 + 2x$ and $U_4 = 1 + 5x/2 + 4x^2$. Together we get the bound

$$h_r(x) < (5/3)(1-x)^{-r}.$$

Thus for $0 < x \le \frac{1}{8}$ we have

$$0 < P_r(x) - (1-x)^{1/n} Q_r(x) < (5\sigma_r/9)(n/4)^r x^{2r+1} (1-x)^{-r},$$

which in view of $\frac{5}{9} < \frac{3}{4}$ and $1 < (1-x)^{1/n-1}$ gives the last bound of Proposition 2.3.

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