Towards a theory of Ground state uniqueness

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Abstract

We study groundstates in a generalization of Edwards-Anderson Ising model. We develop abstract discreate mathematical theory of incongruency called XOR-system.

1 Introduction

A fundamental and extensively studied problem in spin glass physics is the multiplicity of infinite-volume groundstate in finite dimensinal short-ranged systems [14]. In the mean-field Sherrington-Kirkpatrick model [8], it is conjectured that finite dimensional short-ranged systems with frustration have infinitely many groundstate pairs [11, 3]. A different conjecture based on droplet-scaling theories predicts that there is only a single groundstate pair in all finite dimensions [10, 4, 6]. The later scenario has received support from recent simulations, some [1, 15] based on "chaotic size dependence" [12] and some [7] using other techniques.

We focus our attention on the nearest-neighbor Edwards-Anderson Ising model [5] on a graph $G = (V, E)$ with Hamiltonian

$$H_J(\sigma) = -\sum_{uv \in E} J_{uv} \sigma_u \sigma_v \qquad (1)$$

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where J is the set of couplings J_{uv}, $uv \in E$. We take the spins σ_u, $u \in V$, to be Ising, i.e. $\sigma_u = \pm 1$. The couplings J_{uv} are independent, identically distributed random variables and their common distribution is symmetric around zero. The most common examples are the Gaussian and $\pm J$ distribution. The Hamiltonian (1) has clear sense if the graph G is finite. If G is infinite, then we consider expression (1) as formal power series in variables J_{uv}.

Mathematically the problem remains open. Newman and Stein [13] ruled out the appearance of multiple domain walls between groundstates. Arguin et al. [2] considered the Edwards-Anderson Ising spin glass model on the half-plane $\mathbb{Z} \times \mathbb{Z}^+$. They took finite-volume measures corresponding to joint distributions of the couplings and groundstates and proved that these converge to a unique limit and the conditional distribution of the limiting measure is supported on a single groundstate pair. On the other hand, there are also papers supporting existence of different groundstate pairs [9].

2 Summary of main results

A groundstate is a state σ of locally minimal Hamiltonian. The Hamiltonian is essentially a sum over all edges of their weights multiplied by 1 or -1. But we cannot choose arbitrary set of edges whose weight we want to multiply by -1 in the Hamiltonian. We can choose only those sets of edges which form cuts in the graph. A groundstate is also a cut which leads us to study cut systems $(E, \mathcal{C}_G)$ where $G = (V, E)$ is a graph on countably many vertices with bounded degree and $\mathcal{C}_G$ is the family of all cuts of G.

For our convenience, we denote the coupling constants J by a weight function $\omega : E \to \mathbb{R}$. A sum $\sum_{e \in A} \omega(e)$ is denoted by $\omega(A)$, and ω_A is the weight function obtained from ω by flipping the sings on all edges of A, where $A \subseteq E$ (Definition 3.2). In this terminology and notation, a cut A is an ω-groundstate if $\omega_A(B) \geq 0$ for every finite cut B (Definition 3.1).

The essential property of a cut system is the closure on symmetric differences; i.e. the symmetric difference of every pair of cuts is a cut also. We focus our attention on this property and say that (M, S) is a XOR-system if M is a countable ground set and S is a family of subsets of M which is closed under finite symmetric differences (Definition 4.1). It is clear that every cut system forms a XOR-system.

It is quite unnatural to consider only finite symmetric differences. Therefore, we define a limit and a symmetric difference of a sequence of S by the classical way in Definitions 4.2 and 4.3. We restrict our attention on limits instead of symmetric differences of sequences because it is more convenient and Proposition 4.5 proves that a XOR-system contains limits of all converging sequences of S if and only if it contains symmetric differences of sequences of S.

We consider XOR-systems (M, S) that contain limits of converging sequences of S (Definition 4.4). This gives us a very useful tool called compactness (Theorem 5.1). Let S_k be the set of all finite sets of S. Let S_σ be the sets of all limits of converging sequences of S_k. We say that a XOR-system is a σ-XOR-system if it is closed under limits and $S = S_\sigma$ (Definition 4.6). Theorem 4.11 shows that every cut system is a σ-XOR-system which is one of the reasons why we are mainly interested in σ-XOR-systems.

Another interesting property of S_σ is that S_σ does not contain only all limits of converging sequences of S_k but also limits of converging sequences of S_σ by Lemma 4.9. Moreover, Proposition 4.10 states that for a XOR-system (M, S) which is closed under limits it holds that (M, S_σ) is a σ-XOR-system.

We can use S_σ for factorization of a σ-XOR-system (M, S). Let $A, B \in S$ be in a relation σ if $A \bigtriangleup B \in S_\sigma$. The relation σ is an equivalence which factorizes S into classes S/σ. The translation invariance of a σ-XOR-system (M, S) also shows that classes S_1 and S_2 of S/σ are the same because there is the bijection between $B \in S_1$ and $B \bigtriangleup A_1 \bigtriangleup A_2 \in S_2$ where $A_1 \in S_1$ and $A_2 \in S_2$. These classes are crucial for groundstates.

We say that $A \in S$ is an ω-groundstate in a XOR-system (M, S) where $\omega : M \to \mathbb{R}$ if $\omega_A(B) \geq 0$ for all $B \in S_k$ (Definition 6.1). Note that this definition only extends the definition of groundstates in cut systems into more general set systems. Theorem 6.5 proves that an ω-groundstate exists in every σ-XOR-system (M, S) and every $\omega : M \to \mathbb{R}$. Moreover, Corollary 6.6 shows that the translation invariance gives us an ω-groundstate in every class S/σ of a XOR-system (M, S) which is closed under limits. This leads us to study groundstates in σ-XOR-systems and ask whether groundstates are unique in σ-XOR-systems.

A XOR-system (M, S) is finite if M is finite; otherwise, (M, S) is infinite. We present dichotomy between finite and infinite σ-XOR-systems. Proposition 4.8 proves that a finite XOR-system (M, S) is a σ-XOR-system and $S = S_k = S_\sigma$ and they are finite. On the other hand, if (M, S) is an infinite σ-XOR-system, then S_k is infinite and S_σ is uncountable by Theorem 5.3.

We know that an ω-groundstate always exists in a σ-XOR-system (M, S)

and we are interested whether the ω-groundstate is unique almost surely. This question depends on the distribution function of $\omega : M \to \mathbb{R}$. The most common distributions are independent and identical Gaussian and $\pm J$ distributions. Generally, the distribution of ω is symmetric around zero (Definition 7.1).

The unnatural property of $\pm J$ distribution is that $\mathrm{P}\left(\omega(B) = 0\right)$ is positive for every $B \in S_k$. Moreover, Proposition 7.11 shows that an ω-groundstate is not unique in 2D lattice if ω is chosen from $\pm J$ distribution. On the other hand, there is almost surely no cut A such that $\omega_A(B) > 0$ for every finite and non-empty cut B. But for Gaussian distribution, every ω-groundstate $A \in S$ almost surely satisfies the stronger condition $\omega_A(B) > 0$ for every non-empty $B \in S$ in every σ-XOR-system (M, S). This leads us to Definition 7.2 of unique distribution which ensures that $\omega(B) \neq 0$ for all $B \in S_K$ almost surely.

Proposition 7.4 claims that if a weight function ω is chosen from a unique distribution, there are almost surely no pair of ω-groundstates $A, B \in S$ such that $A \triangle B$ contains a non-empty set of S_k as a subset.

We say that a XOR-system (M, S) is transitive if for every $m, n \in M$ there exists a bijection $f : M \to M$ such that $f(m) = n$ and for every $A \subseteq M$ it holds that $A \in S$ if and only if $f(A) \in S$ (Definition 8.1). A typical example of a transitive σ-XOR-system is the cut system of a lattice. Let $\mathrm{P}\left(\mathcal{G}_m\right)$ be the probability that there exist two ω-groundstates whose symmetric difference contains $m \in M$. It is not surprising that $\mathrm{P}\left(\mathcal{G}_m\right) = \mathrm{P}\left(\mathcal{G}_n\right)$ for every $m, n \in M$ in a transitive XOR-system and ω chosen from an independent and identical distribution (Lemma 8.2). It is more interesting that $\mathrm{P}\left(\mathcal{G}_m\right) = 0$ if and only if ω-groundstate is almost surely unique under the assumptions that ω chosen from an independent and identical distribution (Theorem 8.3). Furthermore, $\mathrm{P}\left(\mathcal{G}_m\right) < 1$ if the weight function is chosen from the independent and identical Gaussian distribution (Theorem 8.5).

We study the probabilities that a given state is a groundstate and also a symmetric difference of two groundstates. Both probabilities are zero in infinite σ-XOR-systems under the following assumptions. By Proposition 7.6, a state of S is not almost surely an ω-groundstate if the distribution of ω is unique, symmetric and independent. By Theorem 9.5, for every state of $A \in S$ the probability that there exist two ω-groundstates $B, C \in S$ such that $A = B \triangle C$ is zero, if (M, S) is an infinite transitive σ-XOR-system or a cut system of a graph with bounded degrees and the distribution of ω is the independent and identical Gaussian distribution.

Our study of XOR-systems convince us to believe in uniqueness of groundstate.

Conjecture 2.1. *Let (M, S) be a transitive σ-XOR-system and weight function ω be chosen from the independent and identical Gaussian distribution. Then, ω-groundstate is almost surely unique.*

Furthermore, we do not know any example of a σ-XOR-system having multiple
ω-groundstates with positive probability if the distribution of ω is independent, unique and symmetric.

3 Definition of Cut system

For simplicity, we translate some terms of statistical physics to terms of graph theory. Instead of a state σ we consider the set of vertices $T := \{v \in V \mid \sigma(v) = -1\}$ which gives us a natural one-to-one correspondence between states and subsets of vertices. A *T-cut* is the set of edges $\mathcal{C}_T$ of G which have exactly one end-vertex in $T \subseteq V$. Note that $uv \in \mathcal{C}_T$ if and only if $\sigma_u \sigma_v = -1$ where $uv \in E$. We consider a weight function on edges $\omega : E \to \mathbb{R}$ instead of coupling constants J.

Definition 3.1. Let $G = (V, E)$ be a graph on countable many vertices V such that every vertex has finite degree and $E \neq \emptyset$. The family of all cuts of G is denoted by $\mathcal{C}_G$. The pair $(E, \mathcal{C}_G)$ is called a *cut system*.

In this paper we use the following notation.

Definition 3.2. Let $B \subseteq A$ be two sets and $\omega : A \to \mathbb{R}$ be a function. By ω_B we denote the function obtained from ω by switching the sign on elements that belong to B, that is

$$\omega_B(x) = \begin{cases} -\omega(x) & \text{if } x \in B \\ \omega(x) & \text{otherwise} \end{cases}$$

for all $x \in A$. By $\omega(B)$ we denote the sum $\sum_{x \in B} \omega(x)$.

Note that $\omega_B(B) = -\omega(B)$. The sum $\omega(B)$ is well-defined if B is finite. We use the sum $\omega(B)$ for an infinite set B only for physical motivation of the Hamiltonian but we avoid it in mathematical proofs.

Let $A, B \subseteq E$ and $S, T \subseteq V$. Let ω^T denotes $\omega_{\mathcal{C}_T}$. Observe that $(\omega_A)_B = \omega_{A \triangle B}$ where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is called the *symmetric difference of* A *and* B. From $\mathcal{C}_S \triangle \mathcal{C}_T = \mathcal{C}_{S \triangle T}$ it follows that $(\omega^S)^T = \omega^{S \triangle T}$. This simplifies the notation of the Hamiltonian:

$$H_J(\sigma) = -\sum_{uv \in E} J_{uv}\sigma_u\sigma_v = -\sum_{e \in E} \omega^T(e) = -\omega^T(E).$$

We are interested in a state σ (or T in the new notation) with minimal Hamiltonian. If the graph is finite we can enumerate the Hamiltonian for every state and choose the minimal one. But for a graph on infinitely many vertices, the sum $\omega^T(E)$ is not well defined. Therefore, we restrict the condition of minimality of the Hamiltonian only for finite changes: we say that T is an ω-groundstate if $-\omega^{T \triangle T'}(E) \geq -\omega^T(E)$ for every finite set $T' \subseteq V$. For a finite graph this condition already says that there is no state $T \triangle T'$ of smaller Hamiltonian. Since $\omega^{T \triangle T'}(e) = \omega^T(e)$ for every $e \in E \setminus \mathcal{C}_{T'}$, we change the last inequality to $\omega^{T \triangle T'}(\mathcal{C}_{T'}) \leq \omega^T(\mathcal{C}_{T'})$ which is well-defined even for infinite graph. Observation that $\omega^{T \triangle T'}(e) = -\omega^T(e)$ for every $e \in \mathcal{C}_{T'}$ simplifies our condition: A state $T \subseteq V$ is an ω-groundstate if $\omega^T(\mathcal{C}_{T'}) \geq 0$ for every finite set $T' \subseteq V$. For further simplification we use a definition which is a little bit stronger on some graphs.

Definition 3.3. Let $(E, \mathcal{C}_G)$ be the cut system of a graph $G = (V, E)$ and $\omega : E \to \mathbb{R}$ be a weight function. A state $T \subseteq V$ is an ω-groundstate if $\omega^T(C) \geq 0$ for every finite cut $C \in \mathcal{C}_G$.

Observe that $\mathcal{C}_T$ is finite for every finite set $T \subseteq V$ since every vertex of G has finite degree. On the other hand, it does not generally hold that $T \subseteq V$ is finite if $\mathcal{C}_T$ is finite. For example, let us consider the omnidirectional infinite path P_∞. Every finite set of edges F of P_∞ forms a cut $\mathcal{C}_T$ but the set of vertices T has to be infinite if $|F|$ is odd.

Later, we show that there always exists an ω-groundstate in more general concept. We are interested whether ω-groundstate is unique. From the observation that $\omega(e) = \omega^V(e)$ for every $e \in E$, it follows that T is an ω-groundstate if and only if $V \setminus T$ is an ω-groundstate. Such two groundstates are called *groundstate pairs* and we do not consider them as different groundstates. Note that a state $T \subseteq V$ can be also represented by a cut $\mathcal{C}_T$ which is more convenient for us because it avoids the ambiguity of groundstate pairs.

An edge $e \in E$ is *frustrated* in a state $T \subseteq V$ if $\omega^T(e) < 0$. Another problem in Edwards-Anderson Ising model is determining how large the symmetric difference of the sets of frustrated edges of two groundstates can be. Let $\mathcal{F}(\omega, T) = \{ e \in E \mid \omega^T(e) < 0 \}$ be the set of frustrated edges. We show that symmetric difference of frustrated edges in two states forms a cut.

Lemma 3.4. *If* $T_1, T_2 \subseteq V$ *and* $\omega : E \to \mathbb{R} \backslash \{0\}$, *then* $\mathcal{F}(\omega, T_1) \triangle \mathcal{F}(\omega, T_2) = \mathcal{C}_{T_1 \triangle T_2}$.

Proof. An edge e belongs to $\mathcal{C}_{T_1 \triangle T_2}$ if and only if e belongs to exactly one cut of $\mathcal{C}_{T_1}$ and $\mathcal{C}_{T_2}$ which means that $\omega^{T_1}(e)$ and $\omega^{T_2}(e)$ have different signs. $\square$

If ω is chosen randomly from the Gaussian distribution, then an edge of weight 0 occurs with probability 0.

4 Definition of a σ-XOR-system

We generalize our definition of a groundstate of graphs into a special type of set systems. Let us note that every cut system is closed under symmetric difference; and this property is crucial for u.

Definition 4.1. Let M be a countable set and S be a family of subsets of M such that both M and S are nonempty and $\bigcup_{A \in S} A = M$. We say that (M, S) is a *XOR-system*, if $A \triangle B \in S$ for every $A, B \in S$. Let S_k be the family of finite sets of S.

The set systems (M, S), where M or S is the empty set, are not interesting for us. If some element $m \in M$ does not occur in any set of S, then we can remove m from M, so we require that $\bigcup_{A \in S} A = M$. Note that the cut system of every graph with at least one edge forms a XOR-system.

Let us observe that S_k is countable. Indeed, S_k is countable for the complete XOR-system $(\mathbb{N}, S)$ where $S = 2^{\mathbb{N}}$. That is because S_k is countable union of countable sets R_n where R_n the set all subsets of $\mathbb{N}$ of size n; and R_n is countable because there exists an injection from R_n to the Cartesian product $\mathbb{N}^n$ which is countable.

One may ask why we require only the symmetric difference of finitely many sets in the definition of a XOR-system. That is because it is not obvious what the symmetric difference of countably many subsets of M is. First, we need to define the limit of a sequence of sets.

Definition 4.2. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a set M. The sequence $(A_n)_{n\in\mathbb{N}}$ converges to $A \subseteq M$ if for every $m \in M$ the number of $n \in \mathbb{N}$ satisfying $m \in A \mathbin{\triangle} A_n$ is finite. This is denoted by $\lim_{n\to\infty} A_n = A$ or $A_n \xrightarrow{n} A$.

Note that a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of a set M converges to $A \subseteq M$ if and only if for every finite set $B \subseteq M$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ it holds that $(A_n \mathbin{\triangle} A) \cap B = \emptyset$. We say that $x \in M$ is an *alternating item* for the sequence $(A_n)_{n\in\mathbb{N}}$ if both sets $\{n \mid x \in A_n\}$ and $\{n \mid x \notin A_n\}$ are infinite. Clearly, the sequence $(A_n)_{n\in\mathbb{N}}$ converges if and only if it has no alternating item.

Now, we define the symmetric difference of an infinite sequence $(A_n)_{n\in\mathbb{N}}$ as limit of partial symmetric differences $\triangle_{i=1}^{n} A_i$.

Definition 4.3. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a set M and $B_n = \triangle_{i=1}^{n} A_i$ for $n \in \mathbb{N}$. The symmetric difference of the sequence $(A_n)_{n\in\mathbb{N}}$ converges to A if $B_n \xrightarrow{n} A$. This is denoted by $\triangle_{n\in\mathbb{N}} A_n = A$.

As usual, every sequence has at most one limit and at most one symmetric difference. Our main tool is compactness on a XOR-system (M, S), which states that every sequence $(A_n)_{n\in\mathbb{N}}$ of S has a subsequence $(A_{k_n})_{n\in\mathbb{N}}$ converging to $A \subseteq M$. We need to know whether the limit of a sequence of S remains in S which is provided by the following definition.

Definition 4.4. We say that a XOR-system (M, S) is closed under limits if the family S contains the limit of every converging sequence of S. We say that a XOR-system (M, S) is closed under symmetric differences if for every sequence $(A_n)_{n\in\mathbb{N}}$ of S such that $\triangle_{n\in\mathbb{N}} A_n = A$, the family S contains A.

Now, we show that it suffices to consider only the closure under limits.

Proposition 4.5. *A XOR-system (M, S) is closed under limits if and only if it is closed under symmetric differences.*

Proof. If the XOR-system (M, S) is closed under limits, then it contains limits of convergent partial symmetric differences of sequences. On the other hand, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of S converging to A. Let $B_1 = A_1$ and $B_n = A_n \mathbin{\triangle} A_{n-1}$ for $n \geq 2$. From $\triangle_{i=1}^{n} B_i = A_n$ it follows that $\triangle_{n\in\mathbb{N}} B_n = A \in S$. $\square$

We use limits of sequences instead of their symmetric differences because it is more convenient for us.

Definition 4.6. If a XOR-system (M, S) is closed under limits, then we define S_σ be the set of all limits of converging sequences of S_k. A XOR-system (M, S) is called σ-XOR-system if it is closed under limits and $S = S_\sigma$.

In another words, a XOR-system (M, S) is a σ-XOR-system if S contains a limit of every converging sequence of S and for every set $A \in S$ there exists a sequence of S_k converging to A.

For example, a complete system $(M, 2^M)$ is σ-XOR-system. On the other hand, $(\mathbb{N}, \binom{\mathbb{N}}{even})$ is not σ-XOR-system where $\binom{\mathbb{N}}{even}$ is the family of all subset of even size because the limit of the sequence $(\{1, n+1\})_{n\in\mathbb{N}}$ is $\{1\} \notin \binom{\mathbb{N}}{even}$. Later, we present examples of XOR-systems which are closed under limits but does not satisfy $S = S_\sigma$.

Let us show properties and relations between S_k and S_σ.

Lemma 4.7. *Let (M, S) be a XOR-system which is closed under limits. It holds that $S_k \subseteq S_\sigma$ and $S_k = (S_\sigma)_k$.*

Proof. If $A \in S_k$, then the constant sequence $(A)_{n\in\mathbb{N}}$ converges to A which implies that $A \in S_\sigma$ and the first part of the statement holds.

If $A \in S_k$, then $A \in S_\sigma$ which also implies that $A \in (S_\sigma)_k$. It remains to prove that $S_k \supseteq (S_\sigma)_k$.

Let $A \in (S_\sigma)_k$ which means that $A \in S_\sigma$ and A is finite. Since $S_\sigma \subseteq S$, we have $A \in S_k$. □

We say that a XOR-system (M, S) is finite if M is finite; otherwise, it is infinite. Now, we prove that every finite XOR-system is σ-XOR-system. In Section 5 we present more details about the boundary between finite and infinite XOR-systems.

Proposition 4.8. *Let (M, S) be a finite XOR-system. Then, (M, S) is a σ-XOR-system and $S = S_k = S_\sigma$ and they are finite.*

Proof. Since all sets of S are finite, we know that $S = S_k$. Moreover, S is finite because $|S| \le 2^{|M|}$.

Let $(A_n)_{n\in\mathbb{N}}$ be a converging sequence of S. If $(A_n)_{n\in\mathbb{N}}$ contains two different sets $A, B \in S$ infinitely many times, then every element of $A \triangle B$ is alternating, which contracts convergency of $(A_n)_{n\in\mathbb{N}}$. So, $(A_n)_{n\in\mathbb{N}}$ has only one set A infinitely many times which implies that $A_n \to A$. This proves that (M, S) is closed under limits.

Finally, $S_k \subseteq S_\sigma \subseteq S$ by definition and Lemma 4.7 which concludes the proof since $S = S_k$. □

Now, we prove that every converging sequence of S_σ has its limit in S_σ. It implies that a XOR-system (M, S) is σ-XOR-system if S contains a limit of every converging sequence of S_k and for every set $A \in S$ there exists a sequence of S_k converging to A.

Lemma 4.9. *Let (M, S) be a XOR-system which is closed under limits. For every sequence $(A_n)_{n\in\mathbb{N}}$ of S_σ converging to $A \in S$ there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of S_k converging to A.*

Proof. If M is finite, then $S = S_\sigma = S_k$ by Proposition 4.8, and the statement holds. So, we assume that M is infinite and $(d_n)_{n\in\mathbb{N}}$ is a sequence of all elements of M. Let M_n be $\{d_1, \dots, d_n\}$ for every $n \in \mathbb{N}$.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of S_σ such that $A_n \xrightarrow{n} A \in S$.

Since $A_n \in S_\sigma$, there exists a sequence $(A_n^k)_{k\in\mathbb{N}}$ of S_k such that $A_n^k \xrightarrow{k} A_n$.

Since $A_n \xrightarrow{n} A$, there exists i_m such that $(A_{i_m} \triangle A) \cap M_m = \emptyset$ for every $m \in \mathbb{N}$.

Since $A_{i_m}^k \xrightarrow{k} A_{i_m}$, there exists j_m such that $(A_{i_m}^{j_m} \triangle A_{i_m}) \cap M_m = \emptyset$ for every $m \in \mathbb{N}$.

We prove that $B_n = A_{i_n}^{j_n}$ is the requested sequence which satisfies $B_n \xrightarrow{n} A$, that is for every $m \in \mathbb{N}$ there exists $n' \in \mathbb{N}$ such that for every $n \geq n'$ it holds that $(B_n \triangle A) \cap M_m = \emptyset$. For given $m \in \mathbb{N}$ we choose $n' = m$. Let $n \geq m$. From $M_m \subseteq M_n$ it follows that

$$(B_n \triangle A) \cap M_m \subseteq (B_n \triangle A) \cap M_n = (A_{i_n}^{j_n} \triangle A) \cap M_n =$$

$$((A_{i_n}^{j_n} \triangle A_{i_n}) \triangle (A_{i_n} \triangle A)) \cap M_n \subseteq ((A_{i_n}^{j_n} \triangle A_{i_n}) \cap M_n) \cup ((A_{i_n} \triangle A) \cap M_n)$$

By definition of the sequence $(i_n)_{n\in\mathbb{N}}$ it holds that $(A_{i_n} \triangle A) \cap M_n = \emptyset$.

By definition of the sequence $(j_n)_{n\in\mathbb{N}}$ it holds that $(A_{i_n}^{j_n} \triangle A_{i_n}) \cap M_n = \emptyset$.

Therefore, $(B_n \triangle A) \cap M_m = \emptyset$. □

Now, we study the condition that σ-XOR-system has to be closed under limits. Later, we define a groundstate in a XOR-system and prove that (M, S) has always a groundstate and show that it suffices to study groundstates in S_σ.

If a XOR-system (M, S) is closed under limits and $S = S_\sigma$, then $(M, S) = (M, S_\sigma)$ is a σ-XOR-system. But we prove that (M, S_σ) is a σ-XOR-system even if $S_\sigma \neq S$.

Proposition 4.10. *If* (M, S) *is a XOR-system which is closed under limits and* $\bigcup_{A \in S_\sigma} = M$*, then* (M, S_σ) *is a* σ*-XOR-system.*

Proof. First, we prove that (M, S_σ) is a XOR-system. From $\emptyset \in S$ it follows that $\emptyset \in S_\sigma$. For $A, B \in S_\sigma$ there exist sequences A_n and B_n of S_k such that $A_n \xrightarrow{n} A$ and $B_n \xrightarrow{n} B$. It follows from $A_n \triangle B_n \xrightarrow{n} A \triangle B$ that $A \triangle B \in S_\sigma$, and therefore (M, S_σ) is a XOR-system.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of S_σ converging to A. By Lemma 4.9 there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of S_k converging to A. Since (M, S) is closed under limits, A belongs to S_σ. Therefore, (M, S_σ) is closed under limits too.

Clearly, $(S_\sigma)_\sigma \subseteq S_\sigma$ by definition. On the other hand, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of $(S_\sigma)_k$ converging to $A \in (S_\sigma)_\sigma$. By Lemma 4.7 we know that $(A_n)_{n\in\mathbb{N}}$ is a sequence of S_k and therefore, $A \in S_\sigma$. All together, $(S_\sigma)_\sigma = S_\sigma$. ☐

Now, we prove that the cut system $(E, \mathcal{C}_G)$ of a graph $G = (V, E)$ forms a σ-XOR-system. Recall that we consider only graphs on countably many vertices with finite degree of every vertex.

Theorem 4.11. *If* $G = (V, E)$ *is a connected graph with at least one edge, then* $(E, \mathcal{C}_G)$ *is a XOR-system which is close under limits. Moreover, if the maximum degree of* G *is finite, then* $(E, \mathcal{C}_G)$ *is a* σ*-XOR-system.*

Proof. From $\mathcal{C}_\emptyset = \emptyset$ and $\mathcal{C}_{T_1} \triangle \mathcal{C}_{T_2} = \mathcal{C}_{T_1 \triangle T_2}$, it follows that $(E, \mathcal{C}_G)$ is a XOR-system.

Now, we prove that $\mathcal{C}_G$ is closed under limits. Let $(C_n = \mathcal{C}_{T_n})_{n\in\mathbb{N}}$ be a sequence of cuts of G such that $C_n \xrightarrow{n} C \subseteq E$. We prove that C is a cut of G. Let x be a vertex of G. We suppose that $x \notin T_n$; otherwise, we may consider $V \setminus T_n$ instead of T_n because $\mathcal{C}_{T_n} = \mathcal{C}_{V \setminus T_n}$.

Does the sequence $(T_n)_{n\in\mathbb{N}}$ of a subsets of V converge? If it does not, then $(T_n)_{n\in\mathbb{N}}$ contains an alternating item $u \in V$. Note that $x \neq u$. Let us consider a path from x to u and let v be the first alternating vertex in $(T_n)_{n\in\mathbb{N}}$ on that path and w be the previous vertex which is not alternating. Hence, vw is an alternating edge on $(C_n)_{n\in\mathbb{N}}$ but $(C_n)_{n\in\mathbb{N}}$ is convergent which is a contradiction. Therefore, $T_n \xrightarrow{n} T \subseteq V \setminus \{x\}$.

If $\mathcal{C}_T$ and C are different, then cut sequences $(\mathcal{C}_{T_n})_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ are different for sufficiently large n which contradicts the assumption that $\mathcal{C}_{T_n} = C_n$ for all $n \in \mathbb{N}$. Hence, $C = \mathcal{C}_T \in \mathcal{C}_G$ and $\mathcal{C}_G$ is closed under limits.

Finally, we prove that for every $\mathcal{C}_T = C \in \mathcal{C}_G$ there exists a sequence $(C_n)_{n\in\mathbb{N}}$ of finite cuts of G converging to C. If T is finite, then $\mathcal{C}_T$ is finite and we consider the constant sequence $(\mathcal{C}_T)_{n\in\mathbb{N}}$. Otherwise, let $(v_n)_{n\in\mathbb{N}}$ be a sequence of all vertices of T. Clearly, the sequence $\left(\mathcal{C}_{\{v_1,\ldots,v_n\}}\right)_{n\in\mathbb{N}}$ converges to $\mathcal{C}_T$ and $\mathcal{C}_{\{v_1,\ldots,v_n\}}$ is finite for every $n \in \mathbb{N}$. ☐

It is obvious that common lattices satisfy all conditions of the last theorem. On the other hand, the request that the maximum degree of G is finite, is necessary. Let $K_{\mathbb{N}}$ be the complete graph with infinitely countably many vertices.

Proposition 4.12. *The only finite cut of* $K_{\mathbb{N}}$ *is the empty one. Moreover,* $(\mathcal{C}_{K_{\mathbb{N}}})_\sigma = \{\emptyset\}$.

Proof. For a contradiction, let us suppose that there exists a finite cut $\mathcal{C}_T$ containing edge uv. Assume that $u \in T$ and $v \notin T$. For every vertex $x \in V \setminus \{u, v\}$, either xu or xv belongs into $\mathcal{C}_T$ and we denote it by e_x. Clearly, edges $\{e_x \mid x \in V \setminus \{u, v\}\}$, are pairwise different. Hence, we have infinitely many edges in $\mathcal{C}_T$ which is a contradiction.

The only sequence of finite cuts of $K_{\mathbb{N}}$ is $(\emptyset)_{n\in\mathbb{N}}$ which implies that $(\mathcal{C}_{K_{\mathbb{N}}})_\sigma = \{\emptyset\}$. ☐

5 Finite XOR-systems

One of the main tools in this article is compactness which guarantees that every sequence has a convergent subsequence. Recall that a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of a set M converges to $A \subseteq M$ if for every $m \in M$ the number of $n \in \mathbb{N}$ satisfying $m \in A \triangle A_n$ is finite. A sequence $(A_{k_n})_{n\in\mathbb{N}}$ is called a *subsequence* of a sequence $(A_n)_{n\in\mathbb{N}}$ if $(k_n)_{n\in\mathbb{N}}$ is an increasing sequence of $\mathbb{N}$.

Theorem 5.1 (Compactness)**.** *Every sequence of subsets of a set has a converging subsequence.*

The proof of this theorem follows from well know Tychonoff Theorem [16]. It holds for every set of arbitrary cardinality; nevertheless, we use it only for countable sets in this article.

A XOR-system (M, S) is finite if M is finite. Proposition 4.8 states that a finite XOR-system (M, S) satisfies $S = S_k = S_\sigma$. Such property is not expected in infinite XOR-system. But it does not hold generally that $S_k \neq S_\sigma$ if M is infinite; for example, Proposition 4.12 shows an example of XOR-system (M, S) where M is infinite but $S_\sigma = S_k = \{\emptyset\}$. We need to require that (M, S) is σ-XOR-system.

Proposition 5.2. *Let (M, S) be a σ-XOR-system. For every $m \in M$ there exists $A \in S_k$ such that $m \in A$.*

Proof. The definition of a XOR-system requires that $\bigcup_{A \in S} A = M$ which implies that there exists $A \in S$ such that $m \in A$. The definition of σ-XOR-system requires that $S = S_\sigma$ which implies that $A \in S_\sigma$. Hence, there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of S_k converging to A which implies that $m \in A_n$ for sufficiently large n. ☐

Theorem 5.3. *Let (M, S) be a σ-XOR-system. The following statements are equivalent.*

(a) M is infinite.

(b) S_k is infinite.

(c) S_σ is not countable.

(d) $S_k \neq S_\sigma$.

(e) There exists a sequence $(A_n)_{n\in\mathbb{N}}$ of S_k such that $\bigcup_{n\in\mathbb{N}} A_n$ is infinite.

(f) There exists a sequence $(B_n)_{n\in\mathbb{N}}$ of S_k of pair-wise disjoint nonempty sets.

Proof. First, we prove that the statements (a), (b) and (e) are equivalent. The statement (e) implies (b) because $\bigcup_{n\in\mathbb{N}} A_n$ is not infinite if S_k is finite. From Proposition 4.8 it follows that (b) implies (a). Let $(m_n)_{n\in\mathbb{N}}$ be a sequence of all elements of M. By Proposition 5.2 there exists $A_n \in S_k$ such that $m_n \in A_n$ for every $n \in \mathbb{N}$. Therefore, $M = \bigcup_{n\in\mathbb{N}} A_n$ is infinite and (a) imples (e).

Now, we prove that (c), (d), (e) and (f) are equivalent. Since S_k is always countable, (c) implies (d).

From (d) it follows that there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of S_k converging to $A \in S_\sigma \setminus S_k$. Therefore, A is infinite and $\bigcup_{n\in\mathbb{N}} A_n \supseteq A$ is also infinite which implies (e).

Now, we prove that (e) implies (f). Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of S_k such that $\bigcup_{n\in\mathbb{N}} A_n$ is infinite. Note that the sequence $(A_n)_{n\in\mathbb{N}}$ has infinitely many different sets. Suppose that sets $(A_n)_{n\in\mathbb{N}}$ are pair-wise different nonempty sets, since we may consider only the first occurrence of each set in the sequence. By compactness, the sequence $(A_n)_{n\in\mathbb{N}}$ has a subsequence $(A_{k_n})_{n\in\mathbb{N}}$ converging to a set $A \in S_\sigma$.

We define B_n by induction on n. Let $B_1 = A_{k_1}$.Assume that $B_1, \ldots, B_{n-1} \in S_k$ are pair-wise disjoint nonempty sets. Our aim is to find $B_n \in S_k$ such that $B_n \cap B = \emptyset$, where $B = \bigcup_{i=1}^{n-1} B_i$. Since $A_{k_n} \xrightarrow{n} A$, there exists m such that $(A_{k_{m'}} \triangle A) \cap B = \emptyset$ holds for all $m' \geq m$. Therefore,

$$(A_{k_m} \triangle A_{k_{m+1}}) \cap B = ((A_{k_m} \triangle A) \triangle (A_{k_{m+1}} \triangle A)) \cap B \subseteq \\ ((A_{k_m} \triangle A) \cap B) \cup ((A_{k_{m+1}} \triangle A) \cap B) = \emptyset.$$

So, we define $B_n = A_{k_{m+1}} \triangle A_{k_m}$, which is non-empty and disjoint with all sets $B_1, \ldots, B_{n-1}$. This proves (f).

Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of pair-wise disjoint nonempty sets by (f). For a sequence $(a_n)_{n\in\mathbb{N}}$ of $\{0, 1\}$ we define a set $Z(a_n)$ to be the symmetric difference of all B_n where $a_n = 1$. For different sequences $(a_n)_{n\in\mathbb{N}}$ we obtain different sets $Z(a_n) \in S_\sigma$. The system S_σ is uncountable because there are uncountably many sequences $(a_n)_{n\in\mathbb{N}}$ which implies (c). □

Proposition 5.2 and Theorem 5.3 give us basic characterization of finite and infinite σ-XOR-systems. They also say that S_σ is either finite or uncountable for every σ-XOR-system (M, S).

6 Existence of a groundstate

In this section, we define a groundstate for a XOR-system and we prove that a σ-XOR-system has a groundstate. One can check that the following definition only extends the definition of groundstate for cut systems into a general XOR-system.

Definition 6.1. Let (M, S) be a XOR-system and $\omega : M \to \mathbb{R}$ be a weight function. Then, $A \in S$ is an ω-groundstate if $\omega_A(B) \geq 0$ for every $B \in S_k$.

Before we study groundstates, we need two simple lemmas. The first one directly follows from the fact that $(\omega_A)_B = \omega_{A \triangle B}$.

Lemma 6.2. *Let (M, S) be a XOR-system, $A, B \in S$ and $\omega : M \to \mathbb{R}$. Then, B is an ω_A-groundstate if and only if $A \triangle B$ is an ω-groundstate.*

Lemma 6.3. *Let A and B be finite subsets of a set M and $\omega : M \to \mathbb{R}$ be a weight function. Then, $\omega_A(B \triangle A) = \omega(B) - \omega(A)$.*

Proof.

$$\begin{aligned}\omega_A(B \triangle A) &= \omega_A((B \setminus A) \cup (A \setminus B))\\ &= \omega_A(B \setminus A) + \omega_A(A \setminus B)\\ &= \omega(B \setminus A) - \omega(A \setminus B)\\ &= (\omega(B \setminus A) + \omega(B \cap A)) - (\omega(A \setminus B) + \omega(B \cap A))\\ &= \omega(B) - \omega(A).\end{aligned}$$

□

We prove that there always exists a groundstate in a σ-XOR-system. But we need to start with a groundstate in a finite XOR-system.

Lemma 6.4. *Every finite XOR-system (M, S) has an ω-groundstate for every $\omega : M \to \mathbb{R}$.*

Proof. Since S is finite by Proposition 4.8, we choose $A \in S$ such that $\omega(A)$ is minimal. Then, A is an ω-groundstate because $\omega_A(B) = \omega(A \triangle B) - \omega(A) \geq 0$ for every $B \in S_k$ by Lemma 6.3. □

A closure of a finite set $Z \subseteq S$ in a XOR-system (M, S) is the minimal subset of S containing Z that is closed under finite symmetric differences. Note that $(\bar{M}, \bar{Z})$ is a XOR-system where $\bar{M} = \bigcup_{A \in \bar{Z}} A$.

Theorem 6.5. *Let (M, S) be a XOR-system that is closed under limits. Then, (M, S) contains an ω-groundstate in S_σ for every $\omega : M \to \mathbb{R}$.*

Proof. Since S_k is a countable set, there exists a sequence $(X_n)_{n \in \mathbb{N}}$ of all sets of S_k. Let Z_n be the closure of $\{X_1, \dots, X_n\}$. Let set A_n be an ω-groundstate of the finite XOR-system $(\bigcup_{D \in Z_n} D, Z_n)$. By compactness (Theorem 5.1) there exists a subsequence $(A_{k_n})_{n \in \mathbb{N}}$ of $(A_n)_{n \in \mathbb{N}}$ converging to $A \in S_\sigma$. We show that A is an ω-groundstate of (M, S).

For a contradiction, let $X_l \in S_k$ such that $\omega_A(X_l) < 0$. Since $A_{k_n} \xrightarrow{n} A$, there exists $n \in \mathbb{N}$ such that $k_n \geq l$ and $(A_{k_n} \triangle A) \cap X_l = \emptyset$. Hence, $X_l \in Z_{k_n}$ and $\omega_{A_{k_n}}(X_l) = \omega_A(X_l) < 0$ but A_{k_n} is an ω-groundstate in $(\bigcup_{D \in Z_{k_n}} D, Z_{k_n})$ which is a contradiction. □

The condition that XOR-system (M, S) is closed under limits is necessary. For example, let S be the set of all finite subsets of an infinite set

M. Then, (M, S) is a XOR-system which is not closed under limits. If $\omega : M \to \{-1\}$, then there is no ω-groundstate because for every $A \in S$ we choose a finite and nonempty $B \subseteq M \setminus A$ to have $\omega_A(B) = \omega(B) = -|B| < 0$.

Let (M, S) be a XOR-system which is closed under limits. Let σ be the relation on S such that $A \sigma B$ if $A \bigtriangleup B \in S_\sigma$, where $A, B \in S$. The relation σ is an equivalence on S. Let us consider classes of equivalence S/σ. One of the classes of S/σ is S_σ.

One of the crucial properties of every XOR-system is called *invariance*, which says that $S = \{B \bigtriangleup A \mid B \in S\}$ for every $A \in S$. From the invariance it follows that it is really easy to find an ω-groundstate in every class S/σ.

Corollary 6.6. *A XOR-system (M, S), that is closed under limits, contains an ω-groundstate in every class of S/σ for every $\omega : M \to \mathbb{R}$.*

Proof. Let S' be a class of S/σ and $X \in S'$. By Theorem 6.5 there exists an ω_X-groundstate A in S_σ. By Lemma 6.2, $A \bigtriangleup X$ is an ω-groundstate. □

For example, let

$$\begin{aligned} S = &\{A \subset \mathbb{N} \mid |A \cap \{2n-1, 2n\}| \text{ is even } \forall n \in \mathbb{N}\} \\ &\cup \{A \subset \mathbb{N} \mid |A \cap \{2n-1, 2n\}| \text{ is odd } \forall n \in \mathbb{N}\}. \end{aligned}$$

Clearly, $(\mathbb{N}, S)$ is a XOR-system which is closed under limits and S/σ has two classes.

Let us consider the complete graph $K_\mathbb{N}$ on countably many vertices. By Proposition 4.12 we know that $S_\sigma = \{\emptyset\}$ which implies that every set of $\mathcal{C}_{K_\mathbb{N}}$ forms a class of $\mathcal{C}_{K_\mathbb{N}}/\sigma$. Since there are uncountably many cuts in $K_\mathbb{N}$, there are also uncountably many classes in $\mathcal{C}_{K_\mathbb{N}}/\sigma$.

Now, we know how groundstates behave between different classes in S/σ. By Corollary 6.6 we can consider only one class of S/σ; and by invariance groundstates have the same behaviour in every class S/σ. Moreover, by Theorem 4.11 common lattices satisfies $S = S_\sigma$ so we restrict our attention on σ-XOR-systems.

The following proposition proves that the limit of converging sequences of groundstates is also a groundstate.

Proposition 6.7. *Let (M, S) be a XOR-system and $\omega : M \to \mathbb{R}$ be a weight function. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of S such that A_n is an ω-groundstate for every $n \in \mathbb{N}$. If $A_n \xrightarrow{n} A$, then A is an ω-groundstate.*

Proof. Let $B \in S_k$. Since $A_n \to A$ there exists n such that $(A \bigtriangleup A_n) \cap B = \emptyset$. Hence $\omega_A(B) = \omega_{A_n}(B) \geq 0$ which proves that A is an ω-groundstate. □

7 Distributions of weight functions ω

As we mention in the beginning, the most common distributions for coupling constants J (or, weight function ω) are the Gaussian and $\pm J$ distributions. It is natural to consider that distribution of weight function ω is symmetric around zero.

The Hadamard product of vectors $u, v \in \mathbb{R}^M$ is the vector $u * v$ of $\mathbb{R}^M$ where $(u * v)_i := u_i v_i$ for all $i \in M$. Moreover, $u * B := \{u * x \mid x \in B\}$ where $B \subseteq \mathbb{R}^M$.

Definition 7.1. Let M be a countable set. The probability space $(\mathbb{R}^M, \mathbb{B}^M, \mathrm{P})$ is symmetric if $\mathrm{P}(B) = \mathrm{P}(s * B)$ for every $B \in \mathbb{B}^M$ and $s \in \{\pm 1\}^M$.

In $\pm J$ distribution the coupling constants are independently and uniformly chosen between two numbers $+1$ and -1. Crucial but physically unnatural property of this distribution is that it too often happens that a finite change F of a state A does not change the weight of the state A, i.e. $\omega_A(F) = 0$.

For example, let S be a family of all subsets of $\mathbb{N}$ that contains $2n$ if and only if it contains $2n-1$ for every $n \in \mathbb{N}$. Note that $(\mathbb{N}, S)$ is a σ-XOR-system which is generated by $\{\{2n-1, 2n\} \mid n \in \mathbb{N}\}$. We consider a random weight function $\omega : \mathbb{N} \to \{\pm 1\}$ with independent and uniform distribution. Let

$$I_\omega := \{n \in \mathbb{N} \mid \omega(2n-1) = -\omega(2n)\}.$$

Since the probability that $\omega(2n-1)$ and $\omega(2n)$ have the opposite sign is $\frac{1}{2}$, the set I_ω is infinite for almost every ω. Let A be an ω-groundstate. Since $I_\omega = I_{\omega_C}$ for every $C \in S$ we know that $I_\omega = I_{\omega_A}$. Observe that $B := A \triangle \{2n-1, 2n \mid n \in Z\}$ is an ω-groundstate too for every $Z \subseteq I_\omega$. Hence, for almost every ω we have uncountably many ω-groundstates.

On the other hand, if the distribution of ω is the Gaussian, then the probability that $\omega(2n-1) = \omega(2n)$ is zero. Therefore, ω-groundstate is almost sure unique in this XOR-system. For this reason we require that for almost every ω the probability that $\omega(F) = 0$ is zero for every finite $F \subseteq M$. Therefore, we consider the following property of the distribution of ω which is stronger than the condition $\mathrm{P}(\omega(A) \neq 0 \text{ for all } A \in S_k) = 0$, but it is more handy and Gaussian distribution satisfies it.

Definition 7.2. Let M be a countable set. Let $\mathcal{Z}$ be the set of all $x \in \mathbb{R}^M$ for which there exists $k \in \mathbb{Z}^M \setminus \{(0,0,\dots)\}$ such that $k_m \neq 0$ for finitely many $m \in M$ and

$$\sum_{m \in M, k_m \neq 0} k_m x_m = 0.$$

The probability space $(\mathbb{R}^M, \mathbb{B}^M, \mathrm{P})$ is unique if $\mathrm{P}\left(\mathcal{Z}\right) = 0$.

Note that $\omega \in \mathcal{Z}$ if $A \subseteq M$ finite and $\omega(A) = 0$. Moreover, for every $A \subseteq M$ it holds that $\omega \in \mathcal{Z}$ if and only if $\omega_A \in \mathcal{Z}$.

Note that if distribution of ω is unique and symmetric, then $\mathrm{P}\left(\omega(A) \geq 0\right) = \frac{1}{2}$ for every nonempty and finite $A \subseteq M$, since $\mathrm{P}\left(\omega(A) \geq 0\right) = \mathrm{P}\left(\omega(A) \leq 0\right)$ by symmetry and $\mathrm{P}\left(\omega(A) = 0\right) = 0$ by uniqueness.

Let us prove that the uniqueness is well defined.

Proposition 7.3. *The set* $\mathcal{Z}$ *is measurable and* $\mathrm{P}\left(\mathcal{Z}\right) = 0$ *for the Gaussian distribution.*

Proof. First, the set $\mathcal{Z}$ is measurable because it is a countable union of linear spaces
$\left\{x \in \mathbb{R}^M \mid kx = 0\right\}$ where $k \in Z^M \setminus \{(0,0,\dots)\}$ such that $k_m \neq 0$ for finitely many $m \in M$. Furthermore, every linear space $\left\{x \in \mathbb{R}^M \mid kx = 0\right\}$ has measure zero which implies that $\mathrm{P}\left(\mathcal{Z}\right) = 0$ for the Gaussian distribution. □

Now, we prove that every finite XOR-system has a unique groundstate almost surely. This can be proven directly but first we prove one stronger property of a XOR-system. It says that the symmetric difference of two groundstates does not contain any set of S_k as a subset almost surely.

Proposition 7.4. *Let* (M, S) *be a XOR-system and a weight function* ω *be chosen from a unique distribution. There is almost surely no* ω*-groundstates* $A, B \in S$ *such that* $A \triangle B$ *contains a non-empty set of* S_k *as a subset.*

Proof. Let $A, B \in S$ be an ω-groundstates and $K \in S_k$ such that $K \subseteq A \triangle B$. Since $\omega_B(K) = (\omega_A)_{A \triangle B}(K) = -\omega_A(K)$, we have $\omega_A(K) = 0$. Hence, $\omega \in \mathcal{Z}$ and

$$\mathrm{P}\left(\exists \omega\text{-groundstates } A, B \in S, \exists C \in S_k : C \subseteq A \triangle B\right) \leq \mathrm{P}\left(\mathcal{Z}\right) = 0.$$

□

Corollary 7.5. *Let (M, S) be a finite XOR-system and distribution of a weight function ω be unique. Then, ω-groundstate is almost surely unique.*

Proof. The symmetric difference of two groundstates belongs into S_k and last proposition implies the statement. $\square$

The proof of Proposition 7.4 is very simple but it has many other consequences. For example, the complete infinite σ-XOR-system $(\mathbb{N}, 2^{\mathbb{N}})$ has unique groundstate almost surely, because the symmetric difference of every two difference states of $2^{\mathbb{N}}$ contains a finite subset of $\mathbb{N}$. Note that the complete σ-XOR-system has a physical interpretation: It is the cut system of one-dimensional lattice.

Newman and Stein [13] proved that multiple domain walls between groundstates in 2D lattice does not exist. Their Lemma 1 states that a domain wall in 2D lattice is infinite and contains no loops or dangling ends. This lemma also follows from Proposition 7.4.

Now, we prove that every state in an infinite σ-XOR-system is a groundstate with zero probability.

Proposition 7.6. *Let (M, S) be a σ-XOR-system and distribution of ω be unique, symmetric and random variables $(\omega(m))_{m\in M}$ be mutually independent. Then, M is infinite if and only if* $\mathrm{P}\left(A \textit{ is } \omega\textit{-groundstate}\right) = 0$ *for every* $A \in S$.

Proof. Let M be infinite. Then by Theorem 5.3, there exists a sequence $(M_n)_{n\in\mathbb{N}}$ of S_k, of pairwise different nonempty sets. Since random variables $(\omega(m))_{m\in M}$ are mutually independent, random variables $(\omega(M_n))_{n\in\mathbb{N}}$ are also mutually independent. Hence,

$$\begin{aligned}\mathrm{P}\left(A \text{ is } \omega\text{-groundstate}\right) &\le \mathrm{P}\left(\omega_A(M_n) \ge 0 \ \forall n \in \mathbb{N}\right)\\ &= \prod_{n\in\mathbb{N}} \mathrm{P}\left(\omega_A(M_n) \ge 0\right) = \prod_{n\in\mathbb{N}} \frac{1}{2} = 0.\end{aligned}$$

Let M be finite. Then by Proposition 4.8, it holds that $S = S_k$ and S is finite. If $\mathrm{P}\left(A \text{ is } \omega\text{-groundstate}\right) = 0$ for all $A \in S$, then

$$1 = \mathrm{P}\left(\exists A \in S_\sigma : A \text{ is } \omega\text{-groundstate}\right) \le \sum_{A\in S_k} \mathrm{P}\left(A \text{ is } \omega\text{-groundstate}\right) = 0,$$

which is a contradiction. $\square$

Let us present an example which shows that we cannot avoid the independence in the last statement. Let (M, S) be an infinite σ-XOR-system. Assume that $M = \mathbb{N}$ to simplify the notation. Let $\omega(1)$ be a random variable chosen from the Gaussian distribution and let $\omega(n) = e^{n-1}\omega(1)$ for all $n \in \mathbb{N}$ and for almost every $\omega \in \mathbb{R}^{\mathbb{N}}$. This distribution is symmetric; and moreover, it is unique because e is Euler constant which is transcendental. Therefore, weights of all elements of M have the same sign almost surely which implies that P ($\emptyset$ is an ω-groundstate) $= \frac{1}{2}$.

Now, we present other unnatural properties of $\pm J$ distribution. In $\pm J$ distribution the coupling constants are independently and uniformly chosen between two numbers $+1$ and -1. Crucial but physically unnatural property of this distribution is that it too often happen that a finite change of states (or, set of S_k) has weight zero. Hence, one can consider a sharp inequality in the definition of groundstate.

Definition 7.7. Let (M, S) be a XOR-system and let $\omega : M \to \mathbb{R}$ be a weight function. A state $A \in S$ is a sharp ω-groundstate if $\omega_A(B) > 0$ for every non-empty state $B \in S_k$.

In a unique distribution we do not need to distinguish between groundstates and sharp groundstates because every groundstate is almost surely sharp. But the situation is more complicated in $\pm J$ distribution.

Proposition 7.8. *Let graph $G = (V, E)$ be a cycle and let $(E, \mathcal{C}_G)$ be its cut system. If weight function $\omega : E \to \{\pm 1\}$ is chosen from $\pm J$ distribution, then there is no sharp ω-groundstate in $(E, \mathcal{C}_G)$ with probability* $1/2$ *and there are multiple ω-groundstates with probability* $1/2$.

Proof. Let $neg(\omega)$ be the set of edges $e \in E$ with $\omega(e) = -1$. Observe that cuts in the cycle are exactly sets of edges of even size. Note that $|neg(\omega_A)|$ has the same parity for every cut A. The probability that $|neg(\omega)|$ is odd is 1/2.

Therefore, if $|neg(\omega)|$ is odd, then we find multiple ω-groundstates but no sharp one. Indeed, a cut A is an ω-groundstate if and only if A is even and $|A \triangle neg(\omega)| = 1$. On the other hand, if A would be a sharp ω-groundstate, then there exists an edge $e \in neg(\omega_A)$ and let f be any other edge but $\omega_A(\{e, f\}) = 0$.

If $|neg(\omega)|$ is even, then $A = neg(\omega)$ is the only ω-groundstate which is sharp because ω_A has no edge of negative weight. $\square$

In the last proposition we can notice that there is no sharp ω-groundstate if and only if there are multiple ω-groundstates for every $\omega : M \to \mathbb{R}$. This statement holds generally.

Proposition 7.9. *Let (M, S) be a XOR-system and $\omega : M \to \mathbb{R}$. There exists an ω-groundstate which is not sharp if and only if there exist ω-groundstates $A, B \in S$ such that $A \triangle B$ is finite and non-empty.*

Proof. Since A is not a sharp ω-groundstate, there exists $B \in S_k$ such that $\omega_A(B) = 0$. We prove that $A \triangle B$ is also an ω-groundstate. We use Lemma 6.3 to obtain

$$\omega_{A \triangle B}(C) = \omega_A(B \triangle C) - \omega_A(B) = \omega_A(B \triangle C) \geq 0,$$

since A is an ω-groundstate, where $C \in S_k$. Therefore, $\omega_{A \triangle B}(C) \geq 0$ for every $C \in S_k$ which proves that $A \triangle B$ is an ω-groundstate.

On the other hand, let $A, B \in S$ be ω-groundstates such that $A \triangle B$ is finite and non-empty. From definition it follows that $\omega(X) = -\omega_X(X)$ for every finite set $X \subseteq M$ which implies that

$$0 \leq \omega_A(A \triangle B) = -\omega_B(A \triangle B) \leq 0.$$

Hence, both A and B are ω-groundstates that are not sharp. □

Proposition 7.10. *Let (M, S) be a finite XOR-system and $\omega : M \to \mathbb{R}$. There is no sharp ω-groundstate if and only if there exist at least two ω-groundstates.*

Proof. If there is no sharp ω-groundstate, then there are at least two ω-groundstates by Theorem 6.5 and Proposition 7.9.

For a contradiction, let us assume that there exist ω-groundstates $A, B \in S$ such that A is sharp. So,

$$0 \leq \omega_A(A \triangle B) = -\omega_B(A \triangle B) \leq 0.$$

Hence, A is not a sharp ω-groundstate. □

Proposition 7.11. *Let graph $G = (V, E)$ be the 2-dimensional square lattice and let $(E, \mathcal{C}_G)$ be its cut system. If weight function $\omega : E \to \{\pm 1\}$ is chosen from $\pm J$ distribution, then there are at least two ω-groundstates in $(E, \mathcal{C}_G)$ almost surely but there is no sharp ω-groundstate almost surely.*

Proof. First, we prove that there is no ω-sharp groundstate almost surely. Then, the existence of at least two ω-groundstates follows from Proposition 7.9.

We split whole infinite lattice into infinitely many pair-wise disjoint finite sublattices. We present a particular configuration of weights ω on those finite sublattices which prevents the existence of a sharp ω-groundstate. Since we consider $\pm J$ distribution, such configuration occurs on each sublattice with positive probability. Hence, the probability that the configuration does not occur in any sublattice is zero.

It remains to present the configuration which prevents existence of a sharp ω-groundstate. Let $neg(\omega, E')$ be the set of edges $e \in E'$ with $\omega(e) = -1$ where $E' \subseteq E$. Recall that $|neg(\omega_A, C)|$ and $|neg(\omega, C)|$ have the same parity for every cut A where C is a finite cycle in G. In the configuration of weights ω we prescribe parity of $|neg(\omega, C)|$ on some squares C. The configuration of parities in squares is described in Figure 1. Those prescribed parities remains in a sharp ω-groundstates.

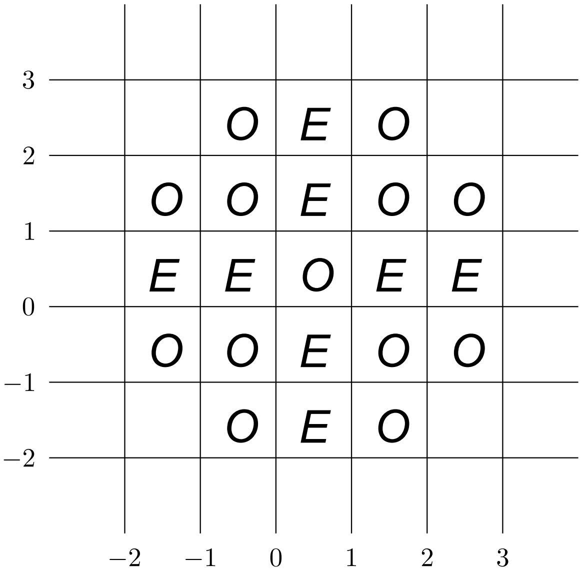

Figure 1: A configuration in a sublattice for Proposition 7.11. Letters "O" and "E" mean that the square has odd and even number of edges e with $\omega(e) = -1$, respectively.

In the rest of the proof, we use the following notation. Let $[a, b]$ be the vertex on coordinates a and b. Let $[a : a+1, b]$ be the edge between vertices $[a, b]$ and $[a + 1, b]$, similarly $[a, b : b + 1]$. Let $[a : a + 1, b : b + 1]$ be the square on edges $[a : a+1, b]$, $[a+1, b : b+1]$, $[a : a+1, b+1]$ and $[a, b : b+1]$.

Let A be a sharp ω-groundstate. We use two observations. First, every vertex is incident with at most one edge e with $\omega_A(e) = -1$ because every vertex has degree 4. Second, if $|neg(\omega_A, C)|$ is odd for a square C, then $|neg(\omega_A, C)| = 1$, because the only other possibility is $|neg(\omega_A, C)| = 3$ which volatiles the first observation for some vertex on the square C.

By the second observation we know that $|neg(\omega_A, [0 : 1, 0 : 1])| = 1$. Since the configuration is symmetric, we assume without lost of generality that $\omega_A([1, 0 : 1]) = -1$ and other edges on the square $[0 : 1, 0 : 1]$ have positive weight. From the first observation it follows that all edges incident with vertices $[1, 0]$ and $[1, 1]$ except the edge $[1, 0 : 1]$ have positive weight because $\omega_A([1, 0 : 1]) = -1$. Next, $\omega_A([2, 0 : 1]) = -1$ because $|neg(\omega_A, [1 : 2, 0 : 1])|$ is even and all other edges have known weight. From the first observation it follows that all edges incident with vertices $[2, 0]$ and $[2, 1]$ except the edge $[2, 0 : 1]$ have positive weight because $\omega_A([2, 0 : 1]) = -1$. Again, $\omega_A([3, 0 : 1]) = -1$ because $|neg(\omega_A, [2 : 3, 0 : 1])|$ is even and all other edges have known weight. For the last time, from the first observation it follows that all edges incident with vertices $[3, 0]$ and $[3, 1]$ except the edge $[3, 0 : 1]$ have positive weight because $\omega_A([3, 0 : 1]) = -1$. Finally, all edges on squares $[1 : 2, 1 : 2]$ and $[2 : 3, 1 : 2]$ except $[1 : 2, 2]$ and $[2 : 3, 2]$ have positive weight, which implies that $\omega_A([1 : 2, 2]) = \omega_A([2 : 3, 2]) = -1$ which contradicts the first observation on vertex $[2, 2]$. This concludes the proof. □

8 Transitive XOR-systems

A graph $G = (V, E)$ is vertex transitive if for every $u, v \in V$ there exists a graph automorphism $f : V \to V$ on G such that $f(u) = v$. Similarly, G is edge transitive if for every $d, e \in E$ there exists a graph automorphism $f : V \to V$ on G such that $f(d) = e$. A lattice is a vertex and edge transitive graph. In this section we describe this property on XOR-systems.

Definition 8.1. We say that XOR-systems (M_1, S_1) and (M_2, S_2) are isomorphic if there exists a bijection $f : M_1 \to M_2$ such that $A \in S_1$ if and only if $f(A) \in S_2$ for every $A \subseteq M_1$. Such function f is called an isomorphism between (M_1, S_1) and (M_2, S_2). An isomorphism between (M_1, S_1) and

(M_1, S_1) is called an automorphism on (M_1, S_1). A XOR-system (M_1, S_1) is transitive if for every $m, n \in M_1$ there exists an automorphism $f_{m,n}$ on (M_1, S_1) such that $f_{m,n}(m) = n$.

Clearly, cut systems of common lattices (i.e. n-dimensional square lattice, hexagonal lattice) form transitive σ-XOR-systems. Let

$$\mathcal{G} := \{\omega : M \to \mathbb{R} \mid \exists A, B \ \omega\text{-groundstates: } A \neq B\}$$

and

$$\mathcal{G}_m := \{\omega : M \to \mathbb{R} \mid \exists A, B \ \omega\text{-groundstates: } m \in A \triangle B\}$$

where $m \in M$. So, $\mathrm{P}(\mathcal{G})$ is the probability that there exist two different groundstates and $\mathrm{P}(\mathcal{G}_m)$ is the probability that there exist two groundstates whose symmetric difference contains given element. Note that $\mathrm{P}(\mathcal{G}) = \mathrm{P}(\cup_{m \in M} \mathcal{G}_m)$.

Lemma 8.2. *Let (M, S) be a transitive XOR-system and the distribution of weight function ω be independently and identically distributed. Then, $\mathrm{P}(\mathcal{G}_n) = \mathrm{P}(\mathcal{G}_m)$ for every $m, n \in M$.*

Proof. Let $f_{m,n}$ be an automorphism on (M, S) such that $f_{m,n}(m) = n$. Observe, that $\mathcal{G}_n$ is the set of all weight functions $\omega' : M \to \mathbb{R}$ such that $\omega'(x) = \omega(f_{m,n}(x))$ for all $x \in M$ where $\omega \in \mathcal{G}_m$. Therefore, $\mathrm{P}(\mathcal{G}_n) = \mathrm{P}(\mathcal{G}_m)$. □

Let (M, S) be a transitive σ-XOR-system. Let the weight function $\omega : M \to \mathbb{R}$ be chosen from an independent and identical distribution. Let $\alpha = \mathrm{P}(\mathcal{G}_n)$ for any $n \in M$. We prove that if $\alpha = 0$ then the probability that there exist two different groundstates is zero. It implies that there are no incongruent groundstates if $\alpha = 0$.

Theorem 8.3. *Let (M, S) be a transitive XOR-system and let the distribution of weight function ω be independent and identical. Then, $\alpha = 0$ if and only if $\mathrm{P}(\mathcal{G}) = 0$.*

Proof. If $\mathrm{P}(\mathcal{G}) = 0$, then $\alpha = 0$ because $\mathcal{G}_m \subseteq \mathcal{G}$ for every $m \in M$.

Let us assume that $\alpha = 0$. Since M is countable we can use the subadditivity to obtain

$$\begin{aligned}
&\mathrm{P}\left(\exists A, B\ \omega\text{-grounstates: } A \neq B\right)\\
&= \mathrm{P}\left(\exists A, B\ \omega\text{-grounstates } \exists m \in M : m \in A \triangle B\right)\\
&= \mathrm{P}\left(\bigcup_{m\in M} \mathcal{G}_m\right) \leq \sum_{m\in M} \mathrm{P}\left(\mathcal{G}_m\right) = \sum_{m\in M} \alpha = 0.
\end{aligned}$$

□

Note that there exist two different groundstates with probability at least α. Now, we prove that $\alpha < 1$.

Lemma 8.4. *Let (M, S) be a XOR-system, $K \in S_k$, $m \in K$ and $\omega : M \to \mathbb{R}$. If*

$$|\omega(m)| > \sum_{x\in K\setminus\{m\}} |\omega(x)|, \tag{2}$$

then there do not exist ω-groundstates $A, B \in S$ such that $m \in A \triangle B$.

Proof. First, observe that (2) is equivalent to (3).

$$\forall C \subseteq M :\ |\omega(m)| > \omega_C(K \setminus \{m\}) \tag{3}$$

For a contradiction, let us assume that there exist ω-groundstates $A, B \in S$ such that $m \in A \triangle B$. Without lost of generality, assume that $m \in A \setminus B$.

If $\omega(m) > 0$, then $0 \leq \omega_A(K) = \omega_A(m) + \omega_A(K \setminus \{m\}) = -\omega(m) + \omega_A(K \setminus \{m\})$ since A is an ω-groundstate which contradicts (3).

If $\omega(m) < 0$, then $0 \leq \omega_B(K) = \omega_B(m)+\omega_B(K\setminus\{m\}) = \omega(m)+\omega_B(K\setminus\{m\})$ since B is an ω-groundstate which also contradicts (3). □

Theorem 8.5. *Let (M, S) be a transitive σ-XOR-system and let distribution of $\omega : M \to \mathbb{R}$ be the independent and identical Gaussian distribution. Then, $\alpha < 1$.*

Proof. Let $m \in M$. By Proposition 5.2, there exists $K \in S_k$ containing m. Since we consider the Gaussian distribution, we know that $\mathrm{P}\left((2)\text{ holds}\right) > 0$. Therefore,

$$\alpha = \mathrm{P}\left(\mathcal{G}_m\right) \leq 1 - \mathrm{P}\left((2)\text{ holds}\right) < 1.$$

□

9 Domain walls

Proposition 7.6 states that every state is a groundstate with probability zero in infinite σ-XOR-systems. In this section we study the probability that a state is a domain wall, that is, a state is a symmetric difference of two groundstates. By Corollary 7.5, every finite XOR-system has a unique groundstate almost surely, so we restrict our attention in infinite σ-XOR-systems.

Let (M, S) be a XOR-system. We say that a sequence $(A_n)_{n\in\mathbb{N}}$ of S_k has bounded size if $\max_{n\in\mathbb{N}} |A_n|$ is finite.

We prove that a state is not a symmetric difference of two groundstates almost surely only for transitive σ-XOR-systems and cut systems (M, S) because our proof uses a sequence $(A_m)_{m\in M}$ of S_k with bounded size such that $m \in A_m$ for every $m \in M$. Such sequence does not exist in general σ-XOR-system, for example, the σ-XOR-system on $\mathbb{N}$ generated by sets

$$\{1\}, \{2, 3\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \ldots .$$

Lemma 9.1. *Let (M, S) be a transitive σ-XOR-system. There exists a sequence $(A_m)_{m\in M}$ of S_k with bounded size such that $m \in A_m$ for every $m \in M$.*

Proof. Let us choose $m' \in M$. By Proposition 5.2, there exists $A' \in S_k$ containing m'. Since (M, S) is transitive, there exists an automorphism f_m on (M, S) such that $f_m(m') = m$ for all $m \in M$. Let A_m be $f_m(A')$. Note that $m \in A_m$ for all $m \in M$ and all sets $(A_m)_{m\in M}$ have the same size. $\square$

Lemma 9.2. *Let $(E, \mathcal{C}_G)$ be a cut system of a graph $G = (V, E)$ with bounded degree. There exists a sequence $(A_m)_{m\in E}$ of $\mathcal{C}_G$ with bounded size such that $m \in A_m$ for every $m \in E$.*

Proof. For every edge $m = uv$ let A_m be the set of edges incident with the vertex u. Since the graph G has bounded degree, the sequence $(A_m)_{m\in E}$ has sets of bounded size. $\square$

Lemma 9.3. *Let (M, S) be an infinite σ-XOR-system which has a sequence $(A_m)_{m\in M}$ of S_k with bounded size such that $m \in A_m$ for all $m \in M$. Then, for every infinite set $C \subseteq M$ there exists a sequence $(C_n)_{n\in\mathbb{N}}$ of S_k of pairwise disjoint sets with bounded size such that $C_n \cap C \neq \emptyset$ for every $n \in \mathbb{N}$.*

Proof. First, we construct an infinite subsequence $(B_n)_{n\in\mathbb{N}}$ of the sequence $(A_m)_{m\in M}$ which moreover has pair-wise different sets. Note that the sequence $(A_m)_{m\in C}$ has infinitely many different sets because $C \subseteq \bigcup_{m\in C} A_m$ is infinite. We consider a subsequence $(B_n)_{n\in\mathbb{N}}$ of the sequence $(A_m)_{m\in C}$ which has exactly one occurrence of each set of $(A_m)_{m\in C}$. Since $m \in A_m$ for all $m \in M$, let $(b_n)_{n\in\mathbb{N}}$ be the sequence of corresponding elements of M which satisfies $b_n \in B_n$ for every $n \in \mathbb{N}$. Hence, the sequence $(B_n)_{n\in\mathbb{N}}$ has similar properties as $(A_n)_{n\in\mathbb{N}}$. It is a sequence of pair-wise different sets of S_k with bounded size and for all $n \in \mathbb{N}$ it holds that $b_n \in B_n \cap C$.

We construct the desired sequence $(C_n)_{n\in\mathbb{N}}$ by induction. Let C_1 be B_1. Let us assume that $C_1, \ldots, C_k \in S_k$ are pair-wise disjoint sets such that $C_n \cap C \neq \emptyset$ for every $n \in \{1, \ldots, k\}$. Let D_k be $\bigcup_{n=1}^{k} C_n$. We find $C_{k+1} \in S_k$ such that $C_{k+1} \cap D_k = \emptyset$ and $C_{k+1} \cap C \neq \emptyset$.

Let E_k be the set of indexes $n \in \mathbb{N}$ such that $b_n \notin D_k$. Let us consider the sequence $(B_n \cap D_k)_{n\in E_k}$. Since D_k is finite and E_k is infinite, $(B_n \cap D_k)_{n\in E_k}$ is an infinite sequence of finitely many sets. Therefore, there exists an infinite set $F_k \subseteq E_k$ such that $(B_n \cap D_k)_{n\in F_k}$ is a constant sequence. Hence, $(B_i \triangle B_j) \cap D_k = \emptyset$ for every $i, j \in F_k$.

It remains to find $i, j \in F_k$ such that $(B_i \triangle B_j) \cap C \neq \emptyset$. But it is easy to prove a stronger claim: For every $i \in F_k$ there exists $j \in F_k$ such that $b_j \notin B_i$. It holds because B_i is finite and F_k is infinite. Since $b_j \in B_j \cap C$ we know that $b_j \in (B_i \triangle B_j) \cap C$.

Note that $\max_{n\in\mathbb{N}} |C_n| \leq 2 \max_{m\in M} |A_m|$ because every set of $(C_n)_{n\in\mathbb{N}}$ is defined as a symmetric difference of (at most) two sets of $(A_m)_{m\in M}$. This finishes the proof. ☐

Proposition 9.4. *Let (M, S) be an infinite σ-XOR-system and let A be an infinite set of S. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of S_k of pair-wise disjoint sets with bounded size such that $A_n \cap A \neq \emptyset$ for every $n \in \mathbb{N}$. Let a weight function $\omega : M \to \mathbb{R}$ be chosen from the independent and identical Gaussian distribution. Then,*

$$\mathrm{P}\left(\exists B \in S : B, A \triangle B \text{ are } \omega\text{-groundstates}\right) = 0.$$

Proof. First, observe that

$$\begin{aligned} &\mathrm{P}\left(\exists B \in S : B, A \triangle B \text{ are } \omega\text{-groundstates}\right) \\ &\qquad \leq \mathrm{P}\left(\exists B \in S : \omega_B(A_n) \geq 0,\ \omega_{A \triangle B}(A_n) \geq 0\ \ \forall n \in \mathbb{N}\right) \\ &\qquad\qquad \leq \mathrm{P}\left(\exists B \subseteq M : \omega_B(A_n) \geq 0,\ \omega_{A \triangle B}(A_n) \geq 0\ \ \forall n \in \mathbb{N}\right). \end{aligned}$$

The formula "$\exists B \subseteq M$" says that we have to change signs of ω to satisfy conditions $\omega_B(A_n) \geq 0$ and $\omega_{A \bigtriangleup B}(A_n) \geq 0$. Since sets $(A_n)_{n\in\mathbb{N}}$ are pair-wise disjoint, we can choose signs of ω independently for every A_n. Hence,

$$\begin{aligned} &\mathrm{P}\left(\exists B \subseteq M : \omega_B(A_n) \geq 0,\ \omega_{A \bigtriangleup B}(A_n) \geq 0\ \ \forall n \in \mathbb{N}\right) \\ &\qquad\qquad = \prod_{n\in\mathbb{N}} \mathrm{P}\left(\exists B \subseteq M : \omega_B(A_n) \geq 0,\ \omega_{A \bigtriangleup B}(A_n) \geq 0\right). \end{aligned}$$

Let

$$P_n = \mathrm{P}\left(\exists B \subseteq M : \omega_B(A_n) \geq 0,\ \omega_{A \bigtriangleup B}(A_n) \geq 0\right).$$

If we prove that $P_n \leq c$ for some constant $c < 1$ which does not depend on n, then $\mathrm{P}\left(\exists B \in S : B, A \bigtriangleup B \text{ are } \omega\text{-groundstates}\right) = 0$.

In the probability P_n we have two conditions

$$\omega_B(A_n) = \omega_B(A_n \setminus A) + \omega_B(A_n \cap A) \geq 0$$

and

$$\omega_{B \bigtriangleup A}(A_n) = \omega_B(A_n \setminus A) - \omega_B(A_n \cap A) \geq 0$$

which we simplify into one condition

$$\omega_B(A_n \setminus A) \geq |\omega_B(A_n \cap A)|.$$

Let $t_n = |A_n \cap A|$ and $r_n = |A_n \setminus A|$. Let $X_1, \ldots, X_{t_n}$ and $Y_1, \ldots, Y_{r_n}$ be random variables $(\omega(m))_{m\in A_n\cap A}$ and $(\omega(m))_{m\in A_n\setminus A}$, respectively. Since $A_n \cap A$ is non-empty, $t_n \geq 1$. If $r_n = 0$, then $P_n = 0$ by Proposition 7.4. We assume that $r_n \geq 1$.

In order to prove that the probability that $|\omega_B(A_n \cap A)| \geq 1$ for all $B \subseteq M$ is positive, we consider the event that $|X_i|$ is close to 2^i for every $i \in \{1, \ldots, t_n\}$. Observe that if $|X_i|$ and 2^i differ less than $\frac{1}{t_n}$ for all $i \in \{1, \ldots, t_n\}$, then $|\omega_B(A_n \cap A)| > 1$. Because the distribution of ω is Gaussian, we know that

$$\mathrm{P}\left(\forall B \subseteq M : |\omega_B(A_n \cap A)| > 1\right) \geq \mathrm{P}\left(\left|X_i - 2^i\right| < \frac{1}{t_n}\ \ \forall i = 1, \ldots, t_n\right) > 0.$$

On the other hand, if $|Y_i| \leq \frac{1}{r}$ for every $i \in \{1, \ldots, r_n\}$, then $|\omega_B(A_n \setminus A)| < 1$ for every $B \subseteq M$. Hence,

$$\mathrm{P}\left(\forall B \subseteq M : |\omega_B(A_n \setminus A)| < 1\right) \geq \mathrm{P}\left(|Y_i| < \frac{1}{r_n}\ \ \forall i = 1, \ldots, r_n\right) > 0.$$

All together, we have

$$\begin{aligned}
P_n &= \mathrm{P}\left(\exists B \subseteq M : \omega_B(A_n \setminus A) \geq |\omega_B(A_n \cap A)|\right) \\
&= 1 - \mathrm{P}\left(\forall B \subseteq M : \omega_B(A_n \setminus A) < |\omega_B(A_n \cap A)|\right) \\
&\leq 1 - \mathrm{P}\left(\forall B \subseteq M : \omega_B(A_n \setminus A) < 1 < |\omega_B(A_n \cap A)|\right) \\
&= 1 - \mathrm{P}\left(\forall B \subseteq M : |\omega_B(A_n \setminus A)| < 1)\right) \cdot \mathrm{P}\left(\forall B \subseteq M : |\omega_B(A_n \cap A)| > 1\right) \\
&\leq 1 - \mathrm{P}\left(|Y_i| < \frac{1}{r_n} \;\; \forall i = 1, \ldots, r_n\right) \cdot \mathrm{P}\left(\left|X_i - 2^i\right| < \frac{1}{t_n} \;\; \forall i = 1, \ldots, t_n\right) \\
&< 1.
\end{aligned}$$

Hence, $P_n < 1$ and moreover, the probability P_n depends only on r_n and t_n. Since the sequence $(A_n)_{n \in \mathbb{N}}$ has sets of bounded size, there are finitely many different combination of values r_n and t_n, and we define c to be the maximal value of P_n. Hence, we have $c < 1$ such that $P_n \leq c$ for every $n \in \mathbb{N}$ which concludes the proof. □

Theorem 9.5. *Let (M, S) be an infinite transitive σ-XOR-system or a cut system of a graph with bounded degree. Let distribution of $\omega : M \to \mathbb{R}$ be the independent and identical Gaussian distribution. Let $A \in S$. Then,*

$$\mathrm{P}\left(\exists B \in S : B, A \triangle B \text{ are } \omega\text{-groundstates}\right) = \begin{cases} 0 & \text{if } A \text{ is non-empty,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. If A is the empty set, then the statement only says that there (almost surely) exists a groundstate which follows from Theorem 6.5. If A non-empty and finite, then the statement follows from Proposition 7.4. Let us assume that A is infinite.

By Lemmas 9.1 and 9.2 there exists a sequence $(A_m)_{m \in M}$ of S_k with bounded size such that $m \in A_m$ for every $m \in M$. By Lemma 9.3 there exists a sequence $(A_n)_{n \in \mathbb{N}}$ of S_k of pair-wise disjoint sets with bounded size such that $A_n \cap A \neq \emptyset$ for every $n \in \mathbb{N}$. Finally, from Proposition 9.4 we conclude that

$$\mathrm{P}\left(\exists B \in S : B, A \triangle B \text{ are } \omega\text{-groundstates}\right) = 0.$$

□

10 Conclusion

We study groundstates in a generalization of Edwards-Anderson Ising model. The original motivation of Ising model is ferromagnetism in critical structures which form lattices or grids but the concept of Hamiltonian

$$H(\sigma) = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j$$

and groundstates makes mathematical sense in an arbitrary graph. We develop abstract discreate mathematical theory of incongruency called XOR-system.

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