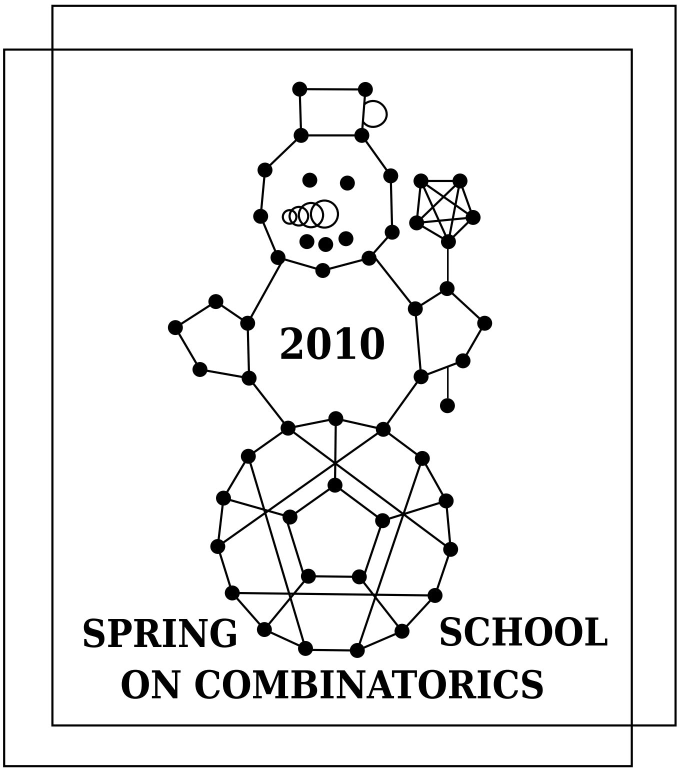

SPRING SCHOOL
ON COMBINATORICS

Zuzka Safernová, Jan Volec (eds.)

Preface

Spring school on Combinatorics has been a traditional meeting organized for faculty and students participating in the Combinatorial Seminar at Charles University for nearly 30 years. It is internationally known and regularly visited by students, postdocs and teachers from our cooperating institutions in the DIMATIA network. As it has been the case for several years, this Spring School is generously supported by the Institute of Theoretical Computer Science (ITI) of Charles University and the Department of Applied Mathematics (KAM) of Charles University.

The Spring Schools are entirely organized and arranged by our undegraduate students. The lecture subjects are selected by supervisors from the Department of Applied Mathematics (KAM) and Institute for Theoretical Computer Science (ITI) of Charles University as well as from other participating institutions. In contrast, the lectures themselves are almost exclusively given by students, both undergraduate and graduate. This leads to a unique atmosphere of the meeting which helps the students in further studies and their scientific orientation.

This year the Spring School is organized in Borová Lada, a mountain village in Šumava hills with a great variety of possibilities for outdoor activities like snow-shoe hiking or cross-country skiing.

We thank Mrs. Eva Jirkov (Pension Kavalier) who provided us rich and welcoming athmosphere during this and many past editions of Spring School.

Dan Král', Jan Kratochvíl, Jaroslav Nešetřil

Index of Talk Titles

Index of Speakers

Geometry

Miroslav Klimoš
xklimos@fi.muni.cz

Presented paper by Igor Pak

Hilbert's third problem

(*Lectures on Discrete and Polyhedral Geometry, chapter 15*)

Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second? That's the question we talk about. Hilbert's problems were posed in 1900. This is the easiest of them, it was resolved the same year.

First of all, let's focus on so-called scissor congruence in two dimensions: Is it possible to cut the polygon and rearrange the pieces into the square? Both the problem and the positive solution formulated in the following theorem are very old (beginning of 19th century).

Theorem 1. (Bolyai, Gerwien) *Two convex polygons in the plane are scissor congruent if and only if they have equal area.*

Then the book moves to higher dimensions and talks about the Dehn's theorem, which resolves the original Hilbert's problem in negative. To prove it, unfortunate polytopes and Bricard condition are used. Stronger results, such as the Dehn's invariant, can be found in the following chapters of the book.

Theorem 2. (Dehn) *A cube and a regular tetrahedron of the same volume are not scissor congruent.*

Definition 3. Let $E = e_1, ..., e_n$ be the set of edges of a convex polytope $P \subset \mathbb{R}^3$. Denote by α_i the dihedral angle at edge e_i. We say that P is *fortunate* if π can be written as a rational combination of α_i:

$$c_1\alpha_1 + ... + c_n\alpha_n = \pi, \quad \text{where } 0 < c_i \in \mathbb{Q}$$

Otherwise, P is *unfortunate*.

Lemma 4. (Bricard condition) *An unfortunate polytope in* $\mathbb{R}^3$ *is not scissor congruent to a cube.*

Zuzka Safernová
`zuzka@saferna.net`

Presented paper by Igor Pak

Polytope algebra

(`Lectures on Discrete and Polyhedral Geometry, chapter 16`)

Introduction

This is the second talk on *scissor congruence*. Here we present an algebraic approach, so we show how to add, subtract, multiply and divide the polytopes.

More formally

Theorem 1. (Sydler 1) *A regular tetrahedron is not scissor congruent to a disjoint union of* $k > 2$ *regular tetrahedra, not necessarily of the same sizes.*

Theorem 2. (Sydler 2) *There is a continuous family* $\{P_t, t \in [0,1]\}$ *of polytopes of equal volume which are not pairwise scissor congruent:* $P_s \nsim P_t$ *for all* $s \neq t$.

Definition 3. We say that a polytope $P \subset \mathbb{R}^3$ is *rectifiable* if it is scissor congruent to a cube of equal volume. Denote by $P \oplus Q$ the disjoint union of $P, Q \subset \mathbb{R}^3$. Polytopes cP are said to be *similar* to P, for all $c > 0$. We say that a polytope $P \subset \mathbb{R}^3$ is *self-similar* if it is scissor congruent to a disjoint union of two or more polytopes similar to P.

The main result in this talk is the following theorem:

Theorem 4. (Sydler's criteria) *A polytope* $P \subset \mathbb{R}^3$ *is rectifiable if and only if* P *is self-similar. Alternatively,* P *is rectifiable if and only if* $P \sim cP \oplus R$, *for some* $c < 1$ *and a rectifiable polytope* R.

The first Sydler's theorem follows immediately from here.

Now we present two other useful properties of scissor congruence. They can be viewed as saying that rectifiability of polytopes is invariant under "subtraction" and "division".

Theorem 5. (Complementarity lemma) *Suppose polytopes* $A, B, C, D \subset \mathbb{R}^3$ *satisfy:* $A \oplus B \sim C \oplus D$ *and* $B \sim D$*. Then* $A \sim C$*.*

Theorem 6. (Tilling lemma) *Let* $P \subset \mathbb{R}^3$ *be a polytope such that* $P_1 \oplus \ldots \oplus P_m$ *is rectifiable, where every* P_i *is either congruent to* P *or a mirror image of* P*,* $1 \leq i \leq m$*. Then* P *is also rectifiable.*

The following technical lemma is important for proofs of previous theorems.

Lemma 7. *Let* $P \subset \mathbb{R}^3$ *be a polytope and let* $\alpha_1, \ldots, \alpha_k > 0$ *be fixed nonnegative real numbers, such that* $\alpha_1 + \ldots + \alpha_k = 1$*. Then there exists a cube* R *such that*

$$P \sim \alpha_1 P \oplus \ldots \oplus \alpha_k P \oplus R.$$

Standard, Hill and regular tetrahedra of equal volume are not scissor congruent. We can check this directly without computing dihedral angles.

Definition 8. For every $\lambda \in \left[0, \frac{1}{2}\right]$ define a λ-*truncated cube* $Q(\lambda) \subset \mathbb{R}^3$ by the following inequalities:

$$|x|, |y|, |z| \leq 1, \quad |x| + |y| + |z| \leq 3 - \lambda.$$

For example, $Q(0)$ is the cube, $Q\left(\frac{1}{2}\right)$ is the *cubeoctahedron*, and $Q(\frac{1}{2\sqrt{2}})$ is the (usual) *truncated cube.*

The following result immediately implies the second Sydler's theorem.

Lemma 9. *We have* $Q(\lambda) \nsim cQ(\mu)$ *for all* $0 \leq \lambda < \mu \leq \frac{1}{2}$ *and* $c > 0$*.*

If we have enough time, we will say few words about Π-congruence. Two polytopes P, Q are Π-*congruent*, write $P \asymp Q$, if each polytope is a disjoint union of similar polytopes: $P = \cup_{i=1}^m P_i, Q = \cup_{i=1}^m Q_i$, where $P_i \simeq cQ_i$, for some $c > 0$. All scissor congruent polytopes are obviously also Π-congruent, but the inverse is not true even if the polytopes have the same volume. Zylev's theorem states that every two polytopes $P, Q \subset \mathbb{R}^3$ are Π-congruent.

Michal Vaner

`vorner@ucw.cz`

Presented paper by Igor Pak

Dissections and valuations

(`Lectures on Discrete and Polyhedral Geometry, chapter 17`)

Introduction

First we restrict cutting operations we allow to cutting a single simplex of a polytope in a special way or gluing it back together (inverse). We prove that any dissection can be obtained from any other dissection by finite series of these restricted operations.

Then we define symmetric valuations, which is class of functions generalizing volume of polytope. We prove that these functions are invariant when we cut the polytope into pieces by our restricted operations, rearrange them using rigid motions (translation and rotation) and glue them back together. Therefore, if two polytopes are scissor congruent (eg. we can cut one into finitely many pieces, rearrange the pieces and glue them together to get the other one), they must give the same result for all valuations.

Formally

Definition 1. Let D_1, D_2 be two dissections of the same polytope to simplices. We say they are connected by an elementary move if they are the same except that one simplex of D_1 is divided into two in D_2.

Definition 2. Two dissections of the same polytope are elementary move equivalent if one can be obtained from the other by series of elementary moves.

Theorem 3. *Every two dissections of the same polytope are elementary move equivalent.*

We first prove it for the plane, by proving that each dissection is elementary move equivalent to a starred triangulation (triangulation with edges ending only in vertices and all the edges sharing a common vertex) and proving that all starred triangulations are elementary move equivalent.

Then we show how to extend the theorem to G by splitting the polytope to cones and proving that it holds because the cones have polygonal bases. Similar trick can be used to extend it to higher dimensions.

Definition 4. Let φ be a function from all simplices of given dimension to non-negative real numbers (we could use any abelian group, actually). We call it symmetric if it is invariant under rigid motions. It is valuation, if it is invariant under elementary moves.

For example, volume of the simplex is a symmetric valuation.

It is easy to observe that each symmetric valuation can be uniquely extended to all convex polytopes.

Corollary 5. *Let P, Q be two polytopes. If there is φ symmetric valuation such that $\varphi(P) \neq \varphi(Q)$, then P and Q are not scissor congruent.*

There exists a set of such symmetric valuations called Dehn invariants in space. They are of the form:

$$\varphi(P) = \sum_{e \in P} |e| \, f\left(\gamma_e\right)$$

where γ_e is dihedral angle at edge e and f is additive function (such that $f(a+b) = f(a) + f(b)$).

Theorem 6. *Two polytopes P, Q in space are scissor congruent if and only if $vol(P) = vol(Q)$ and for every φ Dehn invariant $\varphi(P) = \varphi(Q)$.*

We present this theorem only without proof, since it is too technical.

Matroid structure

Tomáš Gavenčiak
gavento@gmail.com

Introduction to Matroid Theory

Definition 1. A *matroid* $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$ of a finite set E and a family $\mathcal{I}$ of subsets of E with following properties:

(I1) $\emptyset \in \mathcal{I}$ (*nonempty*).

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$ (*closed on subsets*).

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

A set E is called the *ground set* of $\mathcal{M}$, the members of $\mathcal{I}$ *independent sets*. Any subset of E not included in $\mathcal{I}$ is called *dependent*.

A *base* is an inclusion-wise maximal independent set of a matroid.

A *circuit* is an inclusion-wise minimal dependent set of a matroid.

Lemma 2. *If B_1 and B_2 are two bases of matroid $\mathcal{M}$, then $|B_1| = |B_2|$.*

Lemma 3. *Let $\mathcal{M}$ be a matroid on E and let $\mathcal{B}$ be the family of its bases. The family $\mathcal{B}$ has the following properties:*

(B1) *$\mathcal{B} \neq \emptyset$ (nonempty).*

(B2) *if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, then there is an element $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$.*

Theorem 4. *Let E be a set and $\mathcal{B}$ be a family of subsets of E satisfying properties (B1) and (B2). Let $\mathcal{I}$ be the family of subsets of E that are contained in some member of $\mathcal{B}$. Then, the pair $(E, \mathcal{I})$ is a matroid and $\mathcal{B}$ are its bases.*

Definition 5. The *rank function* of a matroid $\mathcal{M}$ is a function $r_{\mathcal{M}}$ from 2^E to non-negative integers where $r_{\mathcal{M}}(X)$ is defined to be the size of a largest independent subset of X.

Lemma 6. *Let $\mathcal{M}$ be a matroid on a set E. The rank function r of $\mathcal{M}$ has the following properties:*

(R1) $0 \leq r(X) \leq |X|$ *for every* $X \subseteq E$

(R2) $r(X) \leq r(Y)$ *for every* $X \subseteq Y \subseteq E$

(R3) $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ *for every* $X, Y \subseteq E$.

Theorem 7. *Let E be a finite set and r a function mapping 2^E to non-negative integers that has properties (R1), (R2), (R3). Let $\mathcal{I}$ be a family of all subsets X of E with $r(X) = |X|$. Then, the pair $(E, \mathcal{I})$ is a matroid and r is its rank function.*

Definition 8. *Uniform matroid* is $\mathcal{U}_{k,l} = (E, \mathcal{I})$, where E is an l-element set and $\mathcal{I}$ the family of all subsets X of E with $|X| \leq k$.

Representability and graphic matroids

Definition 9. Let E be a multiset of vectors of a vector space V. Consider a matroid with the ground set E where a subset of E is independent if it is linearly independent in V. Then, the conditions **(I1)**, **(I2)** and **(I3)** are satisfied. Such matroid is called a *vector matroid*.

Definition 10. Let $\mathbb{F}$ be a field and let A be an $m \times n$ matrix over $\mathbb{F}$. If the vector matroid given by the columns of A is isomorphic to a matroid $\mathcal{M}$, then A is a *representation* of $\mathcal{M}$ over $\mathbb{F}$.

The matroids that have a representation over some field are called *representable matroids*.

Definition 11. Let $G = (V, E)$ be a graph (with loops and parallel edges allowed). The *graphic matroid*, $\mathcal{M}(G)$ of the graph G is the matroid on the ground set E with a subset $X \subseteq E$ independent if it is acyclic in G.

Lemma 12. *A set of edges $B \subseteq E$ is a base of graphic matroid $\mathcal{M}(G)$ if and only if edges in B form a spanning tree in G.*

Lemma 13. *A graphic matroid is representable over any field.*

Duality and minors

Let $\mathcal{B}$ be a family of bases of a matroid $\mathcal{M}$. Let $\mathcal{B}^*$ be the family of complements of members of $\mathcal{B}$, i.e., $X \in \mathcal{B}^*$ if and only if $E \setminus X \in \mathcal{B}$.

Definition 14. The family $\mathcal{B}^*$ is a family of bases of a matroid on E. This matroid is called the *dual* of $\mathcal{M}$ and is denoted by $\mathcal{M}^*$.

Theorem 15. *A graph G is planar if and only if $\mathcal{M}(G)^*$ is a graphic matroid.*

Theorem 16. *If G is a planar graph then $\mathcal{M}(G)^*$ is isomorphic to $\mathcal{M}(G^*)$.*

Definition 17. *Minors* of matroids are defined using two basic operations, a deletion and a contraction of elements and their sets.

- **Deletion:** the matroid $\mathcal{M} \setminus T$ obtained by *deleting* a set $T \subseteq E$ is the matroid with the ground set $E \setminus T$ whose independent sets are those subsets of $E \setminus T$ that are independent in $\mathcal{M}$.
- **Contraction:** the matroid $\mathcal{M}/T$ obtained by *contracting* of a set $T \subseteq E$ is defined through deleting in the dual matroid: the matroid $\mathcal{M}/T$ is equal to $(\mathcal{M}^* \setminus T)^*$.

Lemma 18. *Every minor of an $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable.*

Lemma 19. *Every minor of a graphic matroid is graphic.*

Branch decompositions

Definition 20. A *connectivity function* $\lambda_{\mathcal{M}}$ of a matroid $\mathcal{M}$ with groud set E is defined as:

$$\lambda_{\mathcal{M}}(X) = r_{\mathcal{M}}(X) + r_{\mathcal{M}}(E \setminus X) - r_{\mathcal{M}}(E) + 1.$$

Definition 21. A *branch-decomposition* of a matroid $\mathcal{M}$ with the ground set E is an unrooted cubic tree T with a bijection between the leaves of T and the elements of E.

An edge $e \in T$ splits T into two subtrees and the elements corresponding to the leaves of the two subtrees form a partition (E_1, E_2) of the ground set E.

- The *width of an edge* $e \in T$ is equal to $\lambda(E_1) = \lambda(E_2)$.

- The *width of the branch-decomposition* T is the *maximum* among widths of $e \in T$.
- The *branch-width of the matroid* $\mathcal{M}$ is the *minimum* width of a branch-decomposition of $\mathcal{M}$, and is denoted by $\mathrm{bw}(\mathcal{M})$.

Exercises

1. Prove Lemma 3 and Theorem 4.
2. Prove Lemma 6 and Theorem 7.
3. Express $r^*(X)$ (rank in dual) in terms of $r(X)$, $|X|$, $r(E)$, ... Hint: use $r(E) + r^*(E) = |E|$.
4. Show that uniform matroid $\mathcal{U}_{k,l}$ is a matroid for all $k, l \leq 0$. What is $\mathcal{U}^*_{k,l}$?
5. Prove that $\mathcal{M}(G)$ is a matroid.
6. Let $G = (V, E)$ be a graph. Let $\mathcal{I} = \{V' \subseteq V | V'$ induces an acyclic subgraph of $G\}$. Is $(E, \mathcal{I})$ a matroid?
7. Let $G = (A \cup B, E)$ be a bipartite graph and $\mathcal{I} = \{E' \subseteq E \,|\, E'$ is a partial matching$\}$. Is $(E, \mathcal{I})$ a matroid?
8. Let $G = (A \cup B, E)$ be a bipartite graph and $\mathcal{I} = \{A' \subseteq A \,|\,$ there is a partial matching in G using entire $A'\}$. Is $(E, \mathcal{I})$ a matroid?
9. Show that every graphic matroid is binary (i.e., is representable over $\mathbb{Z}_2$). Find a binary matroid that is not graphic.
10. Prove that $\mathcal{U}_{2,4}$ is not binary. Which $\mathcal{U}_{k,l}$ are binary?
11. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Let $\mathcal{I}^+ = \{I + e | I \in \mathcal{I}, e \in E\}$. Is $\mathcal{M}^+(E, \mathcal{I}^+)$ a matroid?
12. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Let $\mathcal{I}^- = \{I - e | I \in \mathcal{I}, I \neq \emptyset, e \in I\} \cup \{\emptyset\}$. Is $\mathcal{M}^-(E, \mathcal{I}^-)$ a matroid?
13. Find an optimal branch decomposition of $\mathcal{U}_{k,l}$.
14. What width can a branch decomposition of $\mathcal{M}(K_4)$ have?

References

Daniel Král', Ondřej Pangrác: *Introduction to Matroid Theory*, Matfyzpress, **2009**

James Oxley: *Matroid Theory*, Oxford University Press, **1992**

Jan Volec

`volec@kam.mff.cuni.cz`

Presented paper by J Geelen, B. Gerards, G. Whittle

Towards a matroid-minor structure theory

(`http://homepages.cwi.nl/~bgerards/personal/papers.html`)

This talk surveys recent work that is aimed at generalising the results and techniques of the Robertson and Seymour Graph Minors Project to matroids over finite fields.

Conjectures

A number of the most interesting and apparently difficult conjectures in matroid theory concern minor-closed classes. Perhaps the most famous such conjecture is Rota's Conjecture.

Definition 1. (Excluded minor) For a minor-closed class of matroids a minor-minimal matroid that does not belong to that class is its excluded minor (obstruction).

Conjecture 2. (Rota's Conjecture) *Let $\mathbb{F}$ be a finite field. There are, up to isomorphism, only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.*

In 1958 Lazarson showed that there are an inifinite number of excluded minors for representability over the reals and this true also for all other infinite fields. So, if true, Rota's conjecture is best possible. At the moment, Rota's conjecture is proved only for $GF(2), GF(3)$ and $GF(4)$.

Other conjectures are inspired by the work of Roberston and Seymour in their Graph Minors Project. A major outcome of their project is the proof of Wagner's Conjecture establishing that in any infinite sequence of graphs there is one that is isomorphic to a minor of another. In other words, graphs are well-quasi-ordered under the minor order. The following conjecture is a natural generalization to matroids. This conjecture was made by Robertson and Seymour, although apparently not in print.

Conjecture 3. (Well-Quasi-Ordering Conjecture) *Let $\mathbb{F}$ be a finite field. Then in any infinite set of $\mathbb{F}$-representable matroids there is one that is isomorphic to a minor of the other.*

As yet, the Well-Quasi-Ordering Conjecture has not been resolved for any finite field. For any infinite field it is possible to construct infinite antichains of matroids that are representable over that field. Note that Well-Quasi-Ordering Conjecture is equivalent to the following conjecture.

Conjecture 4. *Let $\mathbb{F}$ be a finite field. Any minor closed class of $\mathbb{F}$-representable matroids has a finite number of $\mathbb{F}$-representable excluded minors.*

The Graph Minors Project has also algoritmic consequences. In particular Robertson and Seymour proved that there is a polynomial time algorithm for recognising a given graph as a minor. As a consequence any minor-closed property of graphs can be recognised in polynomial time. Geleen, Gerards and Whittle conjecture that this result also extends to matroids representable over finite fields.

Conjecture 5. (Minor-Recognition Conjecture) *For any finite field $\mathbb{F}$ and any $\mathbb{F}$-representable matroid N, there is a polynomial time algorithm for testing whether an F-representable matroid contains an N-minor.*

Combined with the Well-Quasi-Ordering Conjecture, Minor-Recognition Conjecture implies that, for a given finite field $\mathbb{F}$, there is a polynomial time algorithm for testing any minor-closed property for $\mathbb{F}$-representable matroids.

Partial result – excluding a planar graph

In this section, for a finite field $\mathbb{F}$ and a planar graph H we provide a structural description of $\mathbb{F}$-representable matroids with no $M(H)$-minor. This description provides a partial results for conjectures mentioned above. This restricted solutions are the promising beginning.

In Graph Minors Project Robertson and Seymour proved the following theorem.

Theorem 6. (Grid Theorem) *For any planar graph H, there is an integer $k(H)$ such that if G is a graph of tree width at least $k(H)$, then G cotains an H-minor.*

Note that every planar graph is a minor of some grid, therefore it suffices to prove this theorem for the case that H is a grid.

While it is possible to extend tree width definition to matroids, the extenstion of branch width, initially also introduced for graphs, extends more naturally and is easier to work with. Moreover, branch width is equivalent to tree width in the following way.

Theorem 7. *A class of graphs (or matroids) has bounded tree width if and only if it has bounded branch width.*

Johnson, Robertson and Seymour conjectured that Grid Theorem extends to all finite fields and Geleen, Gerards and Whittle proved this conjecture.

Theorem 8. (Grid Theorem for matroids) *For any planar graph H and any prime power q, there is an integer $\omega(H, q)$ such that if M is a $GF(q)$-representable matroid of branch width at least $\omega(H, q)$, then M cotains an H-minor.*

An easy corollary of this theorem is that any minor-closed class of $GF(q)$-representable matroids that does not contain all planar graphs has bounded branch width. Geleen, Gerards and Whittle also proved the following.

Theorem 9. *Let $\mathbb{F}$ be a finite field and k be an integer. Then each infinite set of $\mathbb{F}$-representable matroids with branch width at most k has two members such that one is isomorphic to a minor of the other.*

If we put the last two results together, we obtain partial result towards the Well-Quasi-Ordering Conjecture.

Corollary 10. *Let $\mathbb{F}$ be a finite field and $\mathcal{C}$ be a minor-closed class of F-representable matroids that does not contain the cycle matroids of all planar graphs. Then $\mathcal{C}$ is Well-Quasi-Ordered under the minor order.*

In combination with results of Hliněný, we also obtain partial progress towards the Minor-Recognition Conjecture.

Corollary 11. *For any finite field $\mathbb{F}$ and any planar graph H, there is a polynomial time algorithm for testing whether or not an $\mathbb{F}$-representable matroid contains an $M(H)$-minor.*

Another result of Geleen, Gerards and Whittle is

Theorem 12. *For any finite field $\mathbb{F}$ and any integer k, there is a finite number of excluded minors for $\mathbb{F}$-representable matroids with branch width at most k.*

Corollary 13. *For any finite field $\mathbb{F}$ and any planar graph H, there is a finite number of excluded minors for $\mathbb{F}$-representable matroids that do not have $M(H)$-minor.*

Martin Böhm

`bohm@atrey.karlin.mff.cuni.cz`

Presented paper by Geleen, Gerards, Robertson, Whittle

Obstructions to branch-decomposition of matroids

(`http://homepages.cwi.nl/~bgerards/personal/papers.html`)

Introduction

We aim to find a structure that guarantees large branch-width in a matroid, ideally one similar to the idea of grids for graphs. We will define **nets** and prove that large branch width in a matroid implies a net of large size as a minor of such a matroid.

We are inspired by the following theorem:

Theorem 1. (Robertson, Seymour) *For any positive integer n there exists an integer k such that if G is a graph with branch-width at least k, then G contains a minor isomorphic to the n by n grid.*

Chapter One, Definitions

Definition 2. A **series class** C of a matroid M is a maximal subset of $E(M)$ such that every two distinct members of C are in series, that means, they form a cocircuit.

Definition 3. (Minimum separation) For disjoint subsets A, B of $E(M)$ we let

$$\kappa_M(A,B) = \min_{A \subseteq X \subseteq E(M)-B} \lambda_M(X).$$

Definition 4. Let X be a subset of $E(M)$. We call X an $[k,n]$**-connected set** if for each partition (X_1, X_2) of M with $|X_1|, |X_2| \geq n$ we have $\kappa_M(X_1, X_2) \geq k$.

Definition 5. A (δ, γ)**-net** of a matroid M is a pair $(N, \mathcal{P})$ where N is a minor of M, $\mathcal{P}$ is a collection of series classes of N, $|\mathcal{P}| \geq \delta$ and $\kappa_M(P, Q) \geq \gamma$ for all distinct pairs of sets $P, Q \in \mathcal{P}$.

Definition 6. Let λ be a connectivity function on $E = E(M)$. A **tangle** of λ of an order k is a collection $\mathcal{T}$ of subsets of E such that:

- $\forall B \in \mathcal{T} : \lambda(B) < k$.
- For each separation (A, B) of order less than k, $\mathcal{T}$ contains A or B.
- If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq E$.
- $\forall e \in E : E - \{e\} \notin \mathcal{T}$.

We also need three theorems on matroids:

Theorem 7. (Tutte's Linking Theorem) *If S and T are disjoint sets of elements in a matroid M, then there exists a minor N of M such that $E(N) = S \cup T$ and $\lambda_N(S) = \kappa_M(S, T)$.*

Definition 8. For a positive integer q, $\mathcal{U}(q)$ is a clas of matroids with no $U_{2,q+2}$ minor and $\mathcal{U}^*(Q)$ is a class of matroids without an $U_{q,q+2}$ minor.

Theorem 9. (Proven in [5]) *For $q \geq 2$, if M is a simple rank-r matroid in $\mathcal{U}(q)$, then*

$$|E(M)| \leq \frac{q^r - 1}{q - 1}.$$

Theorem 10. (Proven in [6]) *If M is a matroid of b-w. at least $\binom{m+1}{2}$, then M contains a circuit of length at least m.*

Our two main goals will be:

Theorem 11. *Let M be a $GF(q)$-representable matroid. If M contains a (q^k, k)-net, then M has branch-width at least k.*

Theorem 12. *For all positive integers δ, γ and a finite field $\mathbb{F}$ there exists an integer k such that if M is an $\mathbb{F}$-representable matroid with b-w. at least k, then M or M^* contains a (δ, γ) net.*

Chapter Two, Observations

Lemma 13. *Let λ be a connectivity function and let $k \in \mathbb{Z}$. Now let $\mathcal{T}$ be a collection of subsets of E that satisfies:*

- $\forall B \in \mathcal{T} : \lambda(B) < k$.
- *For each separation (A, B) of order less than k, $\mathcal{T}$ contains A or B.*
- *If $A \subseteq B, B \in \mathcal{T}$ and $\lambda(A) < k$, then $A \in \mathcal{T}$.*
- *If (A, B, C) is a partition of E, then $\mathcal{T}$ cannot contain all three sets A, B, C.*
- $\forall e \in E : E - \{e\} \notin \mathcal{T}$.

Then $\mathcal{T}$ is a tangle.

Lemma 14. *Let λ be a connectivity function on E. Then the maximum order of a tangle of λ is equal to the branch-width of λ.*

Lemma 15. *For all positive integers k and $q \geq 2$, if $M \in \mathcal{U}(q)$ and M contains a (q^k, k)-net, then M has branch-width at least k.*

Lemma 16. *Let X be a subset of $E(M)$. If X is an $[k, n]$-connected set and $|X| \geq 3n$, then M has branch-width at least $k + 1$.*

Definition 17. Let $\mathcal{T}$ be a tangle of M of order k. For $X \subseteq E(M)$, if $X \subseteq T, T \in \mathcal{T}$, then we define $\Phi_{\mathcal{T}}(X) = \min(\lambda_M(A) - 1 : X \subseteq A \in \mathcal{T})$. Otherwise, we set $\Phi_{\mathcal{T}}(X) = k - 1$.

Lemma 18. *Let M be a matroid and let $\mathcal{T}$ be a tangle of M of order k. Then $\Phi_{\mathcal{T}}$ is the rank function of a matroid of rank $k - 1$.*

Lemma 19. *If M is a matroid with a branch-width at least $3k + 1$, then there exists a $[k, k]$-connected subset X of $E(M)$ with $|X| \geq 3k$.*

Remark 20. *The amount of work needed to verify that a set is $[k, n]$-connected grows exponentially with respect to both k and n.*

Chapter Three, The Technical Chapter

Definition 21. For positive integers δ, γ we define a (δ, γ)**-frame** in M to be a pair $(N, \mathcal{P})$ such that N is a minor of M, $\mathcal{P}$ is a set of series classes of N, $|\mathcal{P}| \geq \delta$ and $\forall P \in \mathcal{P} : |P| \geq \gamma$.

Lemma 22. *There exists an integer-valued function* $f_1(\delta, \gamma, q, k)$*, such that for every integer* $\delta, \gamma, q \geq 2$*, and* $k \geq 1$*, if* M *is a matroid in* $\mathcal{U}^*(q)$ *with b-w.* $\geq 3(k+\delta)+1$*, then either* M *contains a* (δ, γ)*-frame or there exists* $Y \subseteq E(M)$ *such that* $M|Y$ *has b-w.* $\geq k$ *and* $|Y| \leq f_1(\delta, \gamma, q, k)$*.*

Lemma 23. *There exists an integer-valued function* $f_2(\gamma, q, t)$ *such that for any integer* $\delta, \gamma, q \geq 2$ *and* $t \geq 1$*, if* M *is a matroid in* $\mathcal{U}^*(q)$ *that does not contain a* (δ, γ)*-frame and* $A \subseteq E(M)$*, with* $\lambda_M(A) \leq t$*, then there exists* $X \subseteq E(M) - A$ *such that* $\lambda_{M/X}(A) \leq \delta$ *and* $|X| \leq f_2(\gamma, q, t)$*.*

Lemma 24. *There exists and integer-valued function* $f_3(\delta, \gamma, q, k_1, k_2, n)$ *such that for any integer* $\delta, \gamma, k_1, k_2, n, q \geq 2$*, if* M *is a matroid in* $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ *s.t.* M *has a b-w.* $\geq f_3(\dots)$ *and neither* M *nor* M^* *contains a* (δ, γ)*-frame, then there exists a restriction* N *of* M *and a partition* $(A_1, A_2, \dots A_n)$ *of* $E(N)$ *such that:*

- $N|A_1, N|A_2 \dots N|A_{n-1}$ *each have b-w. at least* k_1*.*
- $N|A_n$ *has b-w. at least* k_2*.*
- $\forall i \in \{1 \dots n-1\} : \lambda_N(A_1 \cup A_2 \cup \dots \cup A_i) \leq \delta$*.*

Lemma 25. (Main result of Chapter Three) *There exists an integer-valued function* $f_4(\delta, \gamma, k)$ *s.t. for all integers* $\delta, \gamma, k \geq 2$*, if* M *is a matroid in* $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ *with b-w.* $\geq f_4(\delta, \gamma, k)$*, then* M *or* M^* *contains a* (δ, γ)*-frame.*

Chapter Four, The Sweet Finish

Definition 26. Let f be an integer valued function defined on the set of the positive integers. A matroid M is called (m, f)-connected, if whenever (A, B) is a separation of order $l < m$, then either $|A| \leq f(l)$ or $|B| \leq f(l)$.

Lemma 27. (Proven in [3]) *Let* $g(l) = (6^{l-1} - 1)/5$ *for all positive integers* l*. If* M *is a minor-minimal matroid with b-w.* k*, then* M *is* $(k+1, g)$*-connected.*

After the publication of this paper, the following conjecture was proven in [11]:

Theorem 28. *For any positive integer* n *and prime power* q*, there exists an integer* k *such that if* M *is a* $GF(q)$*-representable matroid with branch-width at least* k*, then* M *contains a minor isomorphic to the cycle-matroid of the* n *by* n *grid.*

References

[1] R.E. Bixby, W.H. Cunningham, Matroid optimization and algorithms, in: R. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, Elsevier, Amsterdam, 1995.

[2] J.S. Dharmatilake, A min-max theorem using matroid separations, in: Matroid Theory, Seattle, WA, 1995, in: Con- temp. Math., vol. 197, Amer. Math. Soc., Providence, RI, 1996, pp. 333342.

[3] J.F. Geelen, A.M.H. Gerards, N. Robertson, G.P. Whittle, On the excluded-minors for the matroids of branch- width k, J. Combin. Theory Ser. B 88 (2003) 261265.

[4] T. Johnson, N. Robertson, P.D. Seymour, Connectivity in binary matroids, handwritten manuscript.

[5] J.P.S. Kung, Extremal matroid theory, in: N. Robertson, P.D. Seymour (Eds.), Graph Structure Theory, Amer. Math. Soc., Providence, RI, 1993, pp. 2162.

[6] B. Oporowski, Partitioning matroids with only small cocircuits, Combin. Probab. Comput. 11 (2002) 191197.

[7] J.G. Oxley, Matroid Theory, Oxford Univ. Press, New York, 1992.

[8] N. Robertson, P.D. Seymour, Graph minors V: excluding a planar graph, J. Combin. Theory Ser. B 41 (1986) 92 114.

[9] N. Robertson, P.D. Seymour, Graph minors X: Obstructions to tree-decomposition, J. Combin. Theory Ser. B 52 (1991) 153190.

[10] W.T. Tutte, Mengers theorem for matroids, J. Res. Nat. Bur. Standards, B. Math. Math. Phys. B 69 (1965) 4953.

[11] J. Geelen, B. Gerards, G. Whittle, Excluding a planar graph from GF(q)-representable matroids, J. Combin. Theory Ser. B 97 (2007) 971998.

Dusan Knop

`dusan.knop@gmail.com`

Presented paper by J Geelen, B. Gerards, G. Whittle

Tangles, tree-decompositions and grids in matroids

(`http://homepages.cwi.nl/~bgerards/personal/papers.html`)

Introduction

A tangle in matroid is an obstruction to small branch-width. I will show that there is a tree-decomposition of a matroid that "displays" all of its maximal tangles and when M is representable over a finite field, each tangle of sufficiently large order "dominates" a large grid minor.

More formally

Definition 1. Let $\lambda : 2^E \to \mathbb{Z}$ be an integer-valued, symetric and submodular function, then we say λ is connectivity function on E. A connectivity system is a pair $K = (E, \lambda)$ where λ is a connectivity function on E.

Definition 2. Let $K = (E, \lambda)$ be connectivity system. A partition (A, B) of E is called separation of order $\lambda(A)$.

Definition 3. Let K be a connectivity system. A tangle in K of order θ is a collection τ of subsets of $E(K)$ such that:
For each $B \in \tau$, $\lambda_K(B) < \theta$
For each separation (A, B) of order less than θ, τ contauins either A or B
If $A, B, C \in \tau$, then $A \cup B \cup C \neq E(K)$
For each $e \in E(K)$, $EK - \{e\} \notin \tau$

For matroid M and $X \subset E(M)$, we let $\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M) + 1$, where r_M is rank function of matroid M. We will use $K_M = (E(M), \lambda_M)$ as matroid connectivity system.

Lemma 4. *Let τ be tangle of order $\theta \geq 3$ in a matroid M. Then each subset of $E(M)$ with rank less than $\theta - 1$ is in τ.*

I will prove that next definition is correct.

Definition 5. Let τ be tangle of order θ in a connectivity system K and let $\acute{\theta} \leq \theta$. Now let $\acute{\tau}$ be the collection of all sets $A \in \tau$ with $\lambda_K(A) < \acute{\theta}$. We say that $\acute{\tau}$ is the truncation of τ to order $\acute{\theta}$.

Definition 6. Let $K = (E, \lambda)$ be a connectivity system and let $X \subset E$. Let $K \circ X = ((E - X) \cup \{e_X\}, \lambda')$ where for each $A \subset E - X$, $\lambda'(A) = \lambda(A)$ and $\lambda'(A \cup \{e_X\}) = \lambda(A \cup X)$.

Lemma 7. *If K is connectivity system and $X \subset E$*

Definition 8. A set X of elements in a connectivity system K is called titanic if each partition (A_1, A_2, A_3) of X satisfies $\lambda_K(A_i) \geq \lambda_K(X)$ for at leats one $i = 1, 2, 3$.

And now the partial converse of previous lemma will come:

Lemma 9. *Let K be a connectivity system, let $X \subset E(K)$ be titanic with $\lambda_K(X) < \theta$, and let τ' be a tangle of order θ in $K \circ X$. Now let τ be the collection of all $A \subset E(K)$ such that $\lambda_K(A) < \theta$ and either $A - X \in \tau'$ or $(A - X) \cup \{e_X\} \in \tau'$. Then τ is a tangle of order θ in K.*

Now we can move towards minors and I can present next definition.

Definition 10. Let N be a minor of M and let τ_N be a tangle in N of order θ. Now let τ_M be the collection of all sets $A \subset E(M)$ where $\lambda_M(A) < \theta$ and $A \cap E(N) \in \tau_N$. We say that τ_M is the tangle in M induced by τ_N.

Definition 11. Let f: $\mathbb{Z}_+ \to \mathbb{Z}_+$ be a function and $m \in \mathbb{Z}_+$ A matroid is called (m, f)-connected if whenever (A, B) is a separation of order $l < m$ we have either $|A| \leq f(l)$ or $|B| \leq f(l)$.

In the following we let $g(n) = (6^{n-1} - 1)/5$.

Theorem 12. *Let τ be a tangle of order θ in a matroid M. Then there exists a (θ, g)-connected minor N of M and a tangle τ' of order θ in N such that τ is the tangle in M induced by τ'.*

In the proof of thist theorem I will use lemma(without proof):

Lemma 13. *Let f: $\mathbb{Z}_+ \to \mathbb{Z}_+$ be a nondecreasing function. If e is an element of an (m, f)-connected matroid M, then M e or M/e is $(m, 2f)$-connected.*

Let G_n denote n by n grid graph. The goal is then to prove existence of natural tangle of order n in $M(G_n)$. For $i \in \{1, \ldots, n\}$ let P_i denote the horizontal paths in grid and Q_i denote the vertical path. Now let τ_n denote collection of all subsets $A \subset E(G_n)$ such that $\lambda_{M(G_n)}(A) < n$ and A does not contain any $E(P_i)$ for $i \in \{1, \dots, n\}$. Then this lemmma holds:

Lemma 14. *For $n \geq 3$. τ_n is a tangle in $M(G_n)$ of order n.*

We will need Tutte's linking Theorem (an extension of Menger's Theorem) and a lemma:

Theorem 15. *If S and T are disjoint sets of elements of a matroid M, then there exists a minor N of M such that $E(N) = S \cup T$ and $\lambda_N(S) = K_M(S, T)$.*

Lemma 16. *If S and T are disjoint sets of elements of a matroid M. Then there exist set $S_1 \subset S$ and $T_1 \subset T$ such that $|S_1| + 1 = |T_1| + 1 = K_M(S_1, T_1) = K_M(S, T)$.*

Let $[n]$ denote the set $\{1, \dots, n\}$

Lemma 17. *Let $i \in [n]$ and for each $j \in [n] - \{i\}$, let e_j and f_j be disjoint edges of P_j. Now let $X = \{e_j : j \in [n] - \{i\}\}$ and let $Y = \{f_j : j \in [n] - \{i\}\}$. Then $K_{M(G_n)}(X, Y) = n$.*

Definition 18. We call a set $A \subset (G_n)$ small if $\lambda_{M(G_n)}(A) < n$ and A does not contain any of $E(P_1)$ or $E(Q_i)$ for all $i \in [n]$.

Lemma 19. *For $n \geq 3$, $E(G_n)$ cannot be partitioned into three small sets.*

I will need following result to show first big theorem of this article:

Theorem 20. *There exist an integer-valued function $f(k, g)$ such that for any positive integer k and primepower q, if M is a $GF(q)$-representable matroid with branch-width at least $f(k, q)$, then M contains a minor isomorphic to $M(G_k)$.*

Theorem 21. *For each finite field $\mathbb{F}$ and possitive integer k there exists an integer θ such that, ij M is an $\mathbb{F}$-representable matroid and τ is a tangle in M of order θ, then τ dominates a minor N that is isomorphic to the cycle matroid of a k by k grid.*

Definition 22. Let E be a set, a partition of E into two sets is called a separation of E. Two separations (A_1, A_2) and (B_1, B_2) of set E are said to cross if $A_i \cap B_j \neq \emptyset$ for each $i, j \in [2]$. A collection S of separations of E is laminar if no two separations in S cross.

Definition 23. Let K be a connectivity system, a set $X \subset EK$ is robust if for each proper partition (X_1, X_2) of X either $\lambda_K(X_1) > \lambda_K(X)$ or

$\lambda_K(X_2) > \lambda_K(X)$. A separation (X, Y) of K is robust if X and Y are both robust.

Lemma 24. *Let K be a connectivity system and let S be the set of all robust separations of K. Then S is laminar.*

Theorem 25. *Let K be a connectivity system and let $\tau_1, \ldots, \tau_n$ be tangles in K, none of which is a truncation of another. Then there exists a tree-decomposition (T, P) of $E(K)$ such that $V(T) = [n]$ and such that folloving hold:*

1. *For each $i \in V(T)$ and $e \in E(T)$ if T' is the component of $T - e$ containing i then $P[V(T')]$ is not in τ_i.*
2. *For each pair of distionct vertices i and j of T, there exists a minimum-order distinguishing separation for τ_i and τ_j that is displayed by T.*

Corollary 26. *An m-element connectivity system has at most $\frac{m-2}{2}$ maximal tangles.*

Standalone Talks

Jakub Bulín

jakub.bulin@gmail.com

Presented paper by Dániel Marx and Barry O'Sullivan and Igor Razgon

Treewidth reduction for constrained separation and bipartization problems

(http://arxiv.org/PS_cache/arxiv/pdf/0902/0902.3780v3.pdf)

A problem is *fixed-parameter tractable* (FPT) with respect to a parameter k, if it can be solved in time $f(k)\cdot poly(n)$ for some (computable) function $f(k)$ depending only on k. We will present the following method of "treewidth reduction": given a graph $\mathbb{G}$, two terminal vertices s and t, and a parameter k, we can compute in a FPT-time a graph $\mathbb{G}^*$ having its treewidth bounded by a function of k while (roughly speaking) preserving all the minimal s–t separators of size at most k.

This method can be used to establish fixed-parameter tractability for various separation and bipartization problems. We will apply it to prove FPT for

- MINIMUM STABLE $s - t$ CUT problem (Is there an independent set of size $\leq k$ whose removal separates s and t?), and for
- STABLE BIPARTIZATION problem (Is there an independent set of size $\leq k$ whose removal makes the graph bipartite?).

Definition 1. Let $\mathbb{G} = (G, E)$ be a graph, $\mathbb{T}$ a tree and $\mathcal{V} = (V_t)_{t\in T}$ a family of subsets of G. The pair $(\mathbb{T}, \mathcal{V})$ is called a *tree decomposition* of $\mathbb{G}$, if

(T1) $G = \bigcup_{t\in T} V_t$,

(T2) $(\forall e \in E)(\exists t \in T)\, e \subseteq V_t$,

(T2) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the path between t_1 and t_3.

The *width* of $(\mathbb{T}, \mathcal{V})$ is $max\{|V_t| : t \in T\} - 1$. The *treewidth* of $\mathbb{G}$, $tw(\mathbb{G})$, is the minimal width of a tree decomposition of $\mathbb{G}$.

Theorem 2.(The treewidth reduction theorem) *Let $\mathbb{G}$ be a graph, $S \subseteq G$ and $k \in \mathbb{N}$. Let C be the set of all vertices of $\mathbb{G}$ participating in a minimal $s - t$ cut of size at most k for some $s, t \in S$. Then there is an FPT algorithm, parameterized by k and $|S|$, that computes a graph $\mathbb{G}^*$ having the following properties:*

(1) $C \cup S \subseteq G^*$

(2) For every $s, t \in S$, a set $K \subseteq G^*$ with $|K| \leq k$ is a minimal s–t separator of $\mathbb{G}^*$, if and only if $K \subseteq C \cup S$ and K is a minimal s–t separator of $\mathbb{G}$.

(3) The treewidth of $\mathbb{G}^*$ is at most $h(k, |S|)$ for some function h.

(4) For any $K \subseteq C$, $\mathbb{G}^*[K]$ is isomorphic to $\mathbb{G}[K]$.

The previous theorem, in combination with the well-known Courcelle's theorem (see below), provides a powerful tool to prove FPT for constrained separation and (using a little trick) also bipartization problems.

Theorem 3.(Courcelle's theorem) *All graph properties definable in monadic second-order logic can be decided in (linear) FPT-time w.r.t. the treewidth of the input.*

Robert Himmelmann

`rjh@mytum.de`

Presented paper by Andrew D. King

Hitting all maximum cliques with a stable set using lopsided independent transversals

(`http://arxiv.org/pdf/0911.1741v2`)

For certain colouring problems having a stable set S meeting every maximum clique in a graph G is helpful. Rabern proofed that a sufficient condition for the existence of such an S is $\omega(G) \geq \frac{3}{4}(\Delta(G)+1)$. In my talk I am going to present the best possible result of this type. For that purpose the following lemmas will be useful.

Intermediate Results

Lemma 1. *Let $\mathcal{Q}$ be a collection of maximum cliques in a graph G. Then we have $|\bigcap \mathcal{Q}| \geq 2\omega(G) - |\bigcup \mathcal{Q}|$.*

For a graph G let $\mathcal{C}(G)$ be the graph with the set $\mathcal{Q}$ of maximum cliques of G as vertices and edges between two cliques if their intersection is not empty. I.e. $\mathcal{C}(G) = (\mathcal{Q}, \{C_1, C_2 \in \binom{\mathcal{Q}}{2} : C_1 \cap C_2 \neq \emptyset\})$

Lemma 2. *Let G be a graph satisfying $\omega(G) > \frac{2}{3}(\Delta(G)+1)$. Then for every connected component $\mathcal{C}$ of $\mathcal{C}(G)$ we have $|\bigcap V(\mathcal{C})| \geq 2\omega(G) - (\Delta(G)+1)$*

Lemma 3. *Let $k \in \mathbb{N}$ and G be a graph partitioned into cliques $V_1, ..., V_r$. If for any $v \in V_i$, v has at most $\min\{k, |V_i| - k\}$ neighbours in $G - V_i$, then G contains a stable set of size r.*

Theorem

With these Lemmas the following can be shown:

Theorem 4. *If a graph G satisfies $\omega(G) > \frac{2}{3}(\Delta(G)+1)$ then it contains a stable set S, such that $\omega(G-S) < \omega(G)$.*

To see that this bound cannot be improved, consider the graph G_k obtained from a C_5 by substituting every vertex with a clique of size k. Then we have $\omega(G_k) = 2k = \frac{2}{3}(\Delta(G_k)+1)$ but there is no stable set meeting every maximum clique.

Proof Outline

Lemmas 1 and 2 have already been proven elsewhere, see for example [2]. Lemma 3 is an easy consequence of the following lemma.

Lemma 5. *Let G be a graph partitioned into stable sets $V_1, ..., V_r$. Let x be a vertex in V_r. Assume that for every $J \subseteq [r-1]$ there are no sets $X, Y \subseteq \{x\} \cup \bigcup_{j \in J} V_j$ such that*

- Every vertex in $\{x\} \cup \bigcup_{j \in J} V_j$ has a neighbour in $X \cup Y$.
- *X and Y are stable.*
- $\forall i. |Y \cap V_i| \leq 1$

- *Every vertex in Y has exactly one neighbour in X.*
- $x \in X$

Then there is a stable set S such that $\forall i.|V_i \cap S| = 1$ (i.e. S is an independent system of representatives) with $x \in S$.

To proof this lemma we will assume that there is a counterexample G. It can then be shown that there is an increasing sequence $(Y_i)_i$ of sets $Y_i \subseteq V(G)$, such that for any i we have $\forall j.|V_j \cap Y_i| = 1$. Since G is finite this is a contradiction.

References

[1] A. King: Hitting all maximum cliques with a stable set using lopsided independent transversals, *Preprint*, http://arxiv.org/abs/0911.1741v2

[2] L. Rabern: On hitting all maximum cliques with an independent set, *Journal of Graph Theory* (2010), http://dx.doi.org/10.1002/jgt.20487

Pavel Klavík

`pavel@klavik.cz`

Presented paper by K. S. Booth and G. S. Lueker

Testing for the Consecutive Ones Property, Interval Graphs, and Graph Planarity Using PQ-Tree Algorithms

(*Journal of Computer and System Sciences 13, 335-379 (1976)*)

Introduction

Interval graphs are intersection graphs of intervals on the real line. Every interval is represented by a vertex and two vertices are adjacent if and only if the corresponding intervals intersect. Studying of interval graphs is motivated by many real world applications—in molecular biology, scheduling, etc.

A natural problem to ask is whether a given graph is an interval graph, and in the positive case to find an interval representation of this graph. In this paper, Booth and Lueker present a linear-time algorithm for interval graph recognition, using a new data structure called PQ-trees.

PQ-tree

A *PQ-tree* is a data structure used to represent a so-called permissible permutation of a set U. For given subsets $S_i \subseteq U$, a permissible permutation is a permutation that contains all the elements of sets S_i consecutively (in any order). For example, permutations bac and cab contain the set $\{a, b\}$ consecutively but permutations acb and bca do not.

A PQ-tree represents one permissible permutation. Leaves correspond one-to-one to elements of U. There are two types of inner nodes: P-nodes and Q-nodes. The order of children of inner nodes is fixed. The permutation represented by this tree is an ordering of its leaves from left to right, see Figure 1.

Since sets S_i admit different permutations, we can reorder the children of inner nodes to get other permissible permutations. We can reorder the children of P-nodes arbitrarily. On the other hand, Q-nodes are more restricted—we can only reverse the order of their children. Two PQ-trees are called equivalent if there is a sequence of reorderings that changes one to another. For example, the trees shown in Figure 1 are equivalent.

Theorem 1. *For given sets S_i, there exists a PQ-tree, such that all the trees equivalent to it represent exactly all the permissible permutations for sets S_i.*

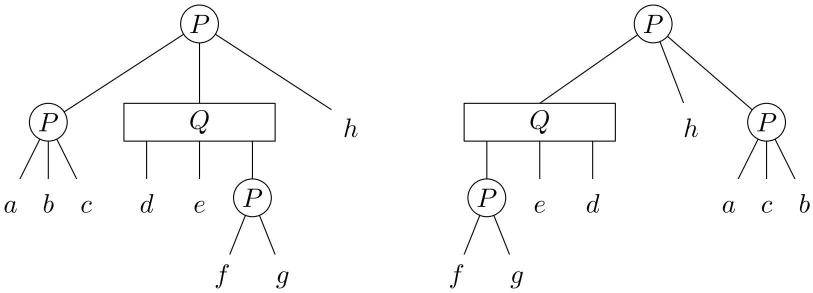

Figure 1: PQ-trees representing permutations $abcdefgh$ and $fgedhacb$.

Moreover, we can iteratively construct a PQ-tree in time linear to the sizes of U and S_i – in $\mathcal{O}(|U| + \sum |S_i|)$.

Recognizing interval graphs using PQ-trees

A 0-1 matrix has the *consecutive ones property* if there exists a permutation of its rows, such that all the ones in every column appear consecutively. This problem is easily solvable using PQ-trees.

Lemma 2. *For a given 0-1 $n \times m$ matrix with f ones, we can decide whether it has the consecutive ones property in time $\mathcal{O}(n + m + f)$.*

We use the following characterization of interval graphs. A graph is an interval graph if and only if it is chordal and there exists an ordering of its maximal cliques in a way that every vertex appears only in several consecutive cliques.

To find a permissible ordering of maximal cliques, we consider the incidence matrix of maximal cliques and vertices. The graph is an interval graph if and only if this matrix has the consecutive ones property. As shown in Figure 2, for a given ordering of maximal cliques, it is easy to construct an interval representation.

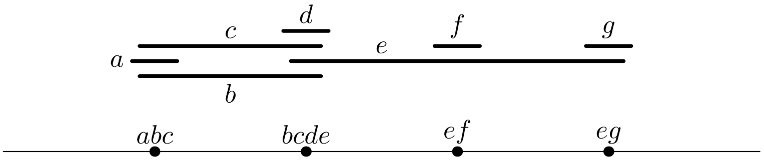

Figure 2: An interval representation for a given ordering of maximal cliques.

Marek Krčál and Martin Tancer

marek.krcal@gmail.com, tancer@kam.mff.cuni.cz

Combinatorial proof of colored Tverberg theorem

Introduction

This contribution is based on the following sources: *Optimal bounds for the colored Tverberg problem* by Blagojević, Matschke and Ziegler; *Chessboard complexes indomitable* by Vrećica and Živlajević; and a work in progress by Krčál, Matoušek, Tancer and Wagner.

During the talk we recall Tverberg theorem and we explain colored Tverberg problem. We emphasize the trick by Blagojević, Matschke and Ziegler which allowed to use topological methods for the colored problem and thus solve the problem for the case of prime numbers. Few months later, Vrećica and Živaljević simplified the topological proof. This simplification allows us to translate the proof into mostly combinatorial setting using just basic topological notions (which can be even avoided if necessary).

More formally

Theorem 1. (Tverberg's theorem) *Let d and r be given positive integers. For any set $A \subset \mathbb{R}^d$ of at least $(d+1)(r-1)+1$ points there exists r pairwise disjoint subsets $A_1, \ldots, A_r \subseteq A$ such that the intersection of their convex hulls is nonempty.*

Theorem 2. (Colored Tverberg theorem) *For any integers $r, d \geq 2$ there exists an integer t such that given any $t(d+1)$-point set $Y \subset \mathbb{R}^d$ partitioned into $d+1$ color classes $Y_1, \ldots, Y_{d+1}$ with t points each, there exist r pairwise disjoint sets $A_1, \ldots, A_r$ such that each A_i contains exactly one point of each Y_j, $j = 1, \ldots, d+1$ (that is, the A_i are rainbow), and $\cap_{i=1}^{r} conv(A_i) \neq \emptyset$.*

The task of colored Tverberg problem is to determine the smallest possible $t = t(d, r)$ for which the theorem above holds. Let us denote it by t_{col}. From general position $t_{col} \geq r$. Blagojević, Matschke and Ziegler have shown that $t_{col} = r$ if $r + 1$ is a prime number. In particular they proved the following more general result.

Theorem 3. *For an integer $d \geq 2$ and r a prime number given any $((r-1)(d+1)+1)$-point set $Y \subset \mathbb{R}^d$ partitioned into $d+2$ color classes $Y_1, \dots, Y_{d+1}, Y_{d+2}$ with $|Y_1|, \dots, |Y_{d+1}| = r-1$; $|Y_{d+2}| = 1$, there exist r pairwise disjoint sets $A_1, \dots, A_r$ such that each A_i contains at most one point of each Y_j, $j = 1, \dots, d+2$ (that is, the A_i are rainbow), and $\cap_{i=1}^{r} conv(A_i) \neq \emptyset$.*

Michaela Kukučková

kukuckaa@gmail.com

Presented paper by César Cérnandez-Célez, Gelasio Salazar, and Robin Thomas

Nested Cycles in Large Triangulations and Crossing-Critical Graphs

(`http://arxiv.org/abs/0911.4690`)

Introduction

The *crossing number* of a graph **G** ($cr(G)$) is the minimum, over all drawings γ of **G**, of the number of crossings in γ. We say, that **G** is *k-crossing critical*, if $cr(g) \geq k$, but $cr(G-e) < k$ for every edge e of **G**.

Many mathematicians have explored the *crossing critical* graphs and their properties. One of the questions that arose is, how many *k-crossing critical* graphs with given average degree do exist. The results of Richter, Thomassen, Pinontoan, Bokal and Hlineny are, that for every q rational from the interval 3, 6 there exists an integer k such that there is an infinite family of *k-crossing critical* graphs.

In my talk I will present the work of Célez, Salazar, and Thomas, who proved that:

Theorem 1. *For each fixed positive integer k, the collection of all k-crossing critical graphs with average degree at least six is finite.*

In fact, it turned out, that even stronger claim holds, and that is:

Theorem 2. *For all integers $k \geq 1$, $r \geq 0$ there is an integer n_0 such that if $\mathbf{G}$ is a k-crossing critical simple graph on n vertices with average degree at least $6 - \frac{r}{n}$, then $n < n_0$.*

The clue to the proof of these theorems is the discovery summed up in the following theorem:

Theorem 3. *For every integer k there exists an integer n such that every planar triangulation on at least n vertices has an s-nest of size k for some $s \in \{0, 1, 2\}$.*

Martin Kupec

`kupec@kam.mff.cuni.cz`

Presented paper by Gerth Stølting, Brodal and Rolf Fagerberg

Dynamic Representations of Sparse Graphs

(*WADS'99, LNCS 1663, pp. 342-351, 1999*)

This talk describes an algorithm for maintaining graphs with bounded arboricity in linear space under edge insertion, deletion and adjacency query. Representation of a graph takes $O(m+n)$ space. The data structure supports adjacency query in $O(c)$ worst-case time, edge insertion in $O(1)$ amortized time and edge deletion in $O(c + \log n)$ amortized time. Where n is the number of vertices in the graph and c the bound on the arboricity.

Definition 1. The arboricity c of the graph $G = (V, E)$ is defined by

$$c = max_J \frac{E(J)}{V(J) - 1}, J \subseteq V, |V(J)| \geq 2.$$

Operations considered on graph $G = (V, E)$ with arboricity c

- `Adjacent`(u, v), return true if and only if $(u, v) \in E$,
- `Insert`(u, v), $E := E \cup \{(u, v)\}$,
- `Delete`(u, v), $E := E \setminus \{(u, v)\}$,
- `Build`(V, E), $G := (V, E)$.

We define operations as follow:

```
Adjescent(u, v):
  return $v \in \text{adj}[u]$ or $u \in \text{adj}[v]$

Delete(u, v):
  $\text{adj}[u] := \text{adj}[u] \setminus \{v\}$
  $\text{adj}[v] := \text{adj}[v] \setminus \{u\}$

Build(V, E):
  for all $v \in V$ do
    $\text{adj}[v] = \emptyset$
  end for
  for all $(u, v) \in E$ do
    Insert(u, v)
  end for

Insert(u, v):
  $\text{adj}[u] := \text{adj}[u] \cup \{v\}$
  if $|\text{adj}[u]| = \Delta + 1$ then
    $S := \{u\}$
    while $S \neq \emptyset$ do
      $w :=$ Pop($S$)
      for all $x \in \text{adj}[w]$ do
        $\text{adj}[x] := \text{adj}[x] \cup \{w\}$
        if $|\text{adj}[x]| = \Delta + 1$ then
          Push($S, x$)
        end if
      end for
      $\text{adj}[w] := \emptyset$
    end while
  end if
```

Effectiveness and time bounds of presented algorithm follows non-constructive proof. The proof discusses existence of an algorithm based on edge reorientations and it's amortized time bounds.

We will use classical characterisation of arboricity bounded graphs by Nash-Williams:

Theorem 2 (Nash-Williams). *A graph $G = (V, E)$ has arboricity c if and only if c is the smallest number of sets $E_1, \ldots, E_c$, that E can be partitioned* *into, such that each subgraph* (V, E_i) *is a forest.*

Definition 3. Δ-orientation is an orientation of an undirected graph, such that each vertex has outdegree bounded by Δ.

Definition 4. An arboricity c preserving sequence is sequence of edge insertions and deletions on a graph G, initially of arboricity at most c, where the arboricity stays bounded by c during the entire sequence.

Using next lemmas we bound amount of extra work performed by edge reorientations.

Lemma 5. *Given an arboricity* c *preserving sequence* σ *of edge insertions and deletions on an initially empty graph, let* G_i *be the graph after the i'th operation, and let* k *be the number of edge insertions.*

If there exists a sequence $\bar{G}_0, \bar{G}_1, \ldots, \bar{G}_{|\sigma|}$ *of* δ*-orientations with at most* r *edge reorientations in total, then the algorithm performs at most*

$$(k + r)\frac{\Delta + 1}{\Delta + 1 - 2\delta}$$

edge reorientations in total on sequence σ*, provided* $\Delta \geq 2\delta$*.*

Lemma 6. *Let* $G = (V, E)$ *be a graph with arboricity at most* c*, let* $\bar{G} = (V, \bar{E})$ *an orientation of* G*, and let* $\delta > c$*. In* $\bar{G}$*, if* $u \in V$ *has outdegree at least* δ *then there exists a vertex* v *with outdegree less than* δ *and a directed path from* u *to* v *containing at most* $\lceil \log_{\delta/c} |V| \rceil$ *edges.*

Finally theorem bounding time consumption follows from previous lemmas.

Theorem 7. *In an arboricity* c *preserving sequence of operations starting with an empty graph, the algorithm for* $\Delta/2 \geq \delta > c$ *supports* `Insert`*(u, v) in amortized* $O(\frac{\Delta+1}{\Delta+1-2\delta})$ *time,* `Build`*(V, E) in amortized* $O(|V|+|E|\frac{\Delta+1}{\Delta+1-2\delta})$ *time,* `Delete`*(u, v) in amortized* $O(\Delta + \frac{\Delta+1}{\Delta+1-2\delta})$ *time, and* `Adjacent`*(u, v) in worst case* $O(\Delta)$ *time.*

Choosing $\Delta = 4c$ and $\delta = \frac{3}{2}c$ gives time bounds as stated at the beginning.

Robert Lukot'ka

`lukotka@dcs.fmph.uniba.sk`

Presented paper by Dan Hefetz

On two generalizations of the Alon-Tarsi polynomial method

(`http://arxiv.org/pdf/0911.2099`)

Introduction

Alon and Tarsi introduced an algebraic technique for proving the upper bounds of the choice number of graphs. For a graph G, the number obtained by this method was named the *Alon-Tarsi number* and denoted by $AT(G)$. Alon and Tarsi gave two interpretations of this parameter. The first interpretation states that $AT(G) \leq k$ if and only if there exist an orientation D, such that the out-degree of every vertex is less than k and the number of odd eulerian subdigraphs is not the same as the number of even eulerian subdigraphs of D. The second characterization applies only for line graphs of d-regular d-colorable line graphs and interprets $AT(G)$ in terms of signed colorings.

The first characterization is generalized by showing that there is an infinite family of weighting functions, each of which can be used to characterize $AT(G)$. The second characterization is generalized for all graphs (and hypergraphs). At last we use the second characterization to prove that $\chi(G) = ch(g) = AT(G)$ for certain families of graphs.

More formally

Let $G = (V, E)$ be a undirected multigraph with vertex set $\{1, \dots, n\}$. The graph polynomial is defined by

$$P_G(x_1, x_2, \dots, x_n) = \prod_{\substack{1 \leq i < j \leq n \\ (i,j) \in E}} (x_i - x_j).$$

This polynomial encodes the information about proper colorings of G. A graph G is k-colorable if and only if there exists an n-tuple $(a_1, \dots, a_n) \in$

$\{0,\dots,k-1\}$ such that $P_G(a_1,\dots,a_n)\neq 0$. Similarly G is k-choosable if and only if for any family of sets S_i of size at least k there exists an n-tuple $(a_1,\dots,a_n)\in S_1\times S_2\times\cdots\times S_n$ such that $P_G(a_1,\dots,a_n)\neq 0$.

The following theorem gives a sufficient condition for the existence of such an n-tuple

Theorem 1.*[(Combinatorial Nullstellensatz)] Let $\mathbb{F}$ be a field and let $p=p(x_1,\dots x_n)$ be a polynomial over $\mathbb{F}$. Suppose that deg$(p)=\sum_{i=1}^n t_i$ and the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in p is nonzero. Then if $S_1,S_2,\dots S_n$ are subsets of $\mathbb{F}$ with $|S_i|>t_i$, there exists an n-tuple $(a_1,\dots,a_n)\in S_1\times S_2\times\cdots\times S_n$ such that $p(a_1,\dots,a_n)\neq 0$.*

Let $f:V\to\mathbb{N}^+$. We say that G is f-*AT*, if a coefficient in P_G satisfies the above theorem with $t_i<f(i)$. We say that G is k-*AT* if G is f-*AT* for $f(i)\equiv k$.

The first characterization characterizes the f-*AT* graphs using eulerian subdigraphs. By the term eulerian, we mean that for every vertex we require the out-degree to be the same as the in-degree, we do not require connectivity.

Theorem 2. *Let $G=(V,E)$ be an undirected graph, $V=v_1,\dots,v_n$. Let D be an orientation such that the out-degree of v_i is d_i. The graph G is f-AT, $f(i)=d_i+1$, if and only if the number of odd eulerian subdigraphs is not the same as the number of even eulerian subdigraphs of D.*

We prove the following generalization.

Theorem 3. *With notation as in above theorem, the graph G is f-AT if and only if $\sum_{A\subseteq E}(-1)^{|A|}\prod_{i=1}^n\prod_{j=1}^{d_i}(d_A^+(v_i)-d_A^-(v_i)-u_j^i)\neq 0$, where the numbers u_j^i are arbitrary real numbers.*

Indeed if we choose $u_j^i=j$ we get the statement of the previous theorem. Other interpretation given by Alon and Tarsi is this:

Theorem 4. *Let H be a d-regular d-edge colorable graph and let $G=(\{v_1,\dots,v_n\},E)$ be its line graph. Let C_d be the set of proper colorings of G with colors $\{0,1,\dots d-1\}$. The sign of $c\in C_d$, denoted by sign(c), is defined to be 1 if $P(c(v_1),c(v_2),\dots,c(v_n))>0$ and -1 otherwise. Then $AT(G)\leq d$ if and only if $\sum_{c\in C_D}\neq 0$.*

We generalize it as follows.

Theorem 5. *Let $G=(V,E)$ be a graph, $V=\{1,2,\dots,n\}$ and $|E|=m$. Let $f:V\to\mathbb{N}^+$ be a function satisfying $\sum_{u\in V}f(u)=m+n$. Let C_d be* *the set of all proper colorings c of G with colors $\{0, 1, \ldots \max\{f(i) - 1 : 1 \leq i \leq n\}\}$. Then G is f-AT if and only if*

$$\sum_{c \in C_f} (-1)^{\sum_{i=1}^{n} c(i)} \prod_{i=1}^{n} \binom{f(i)-1}{c(i)} P_G(c(1), \ldots, c(n)) \neq 0.$$

We apply this theorem to uniquely colorable graphs.

Theorem 6. *Let G be a uniquely k-colorable graph with n vertices and $(k-1)n - \binom{k}{2}$ edges. Then $AT(G) = k$.*

Theorem 7. *Let G be a uniquely k-colorable graph with n vertices and m edges. Let the sets A_i be the color classes of the unique coloring. Let r_o (respectively r_e) be number of odd (even) sets A_i. Let p_o (respectively p_e) be number of pairs A_i, A_j, $1 \leq i < j leqn$, such that both A_i and A_j are odd (even) and $e(A_i, A_j)$ is odd. If $m \leq \max\{(n - r_o)(k-1) + \binom{r_o}{2} - p_e, (n - r_e)(k-1) + \binom{r_e}{2} - p_o\}$, then $AT(G) = k$.*

This thorem provides the following results

Corollary 8. *$AT(K_{2*n}) = n$, where K_{2*n} is complete n-partite graph with each part of size 2.*

Corollary 9. *$AT(C_n^p) = \chi(C_n^p) = p + 1$, where C_n^p is the p-th power of the n-cycle.*

Ján Mazák

`mazak@dcs.fmph.uniba.sk`

Presented paper by Jan van den Heuvel and Stéphan Thomassé

Cyclic Orderings and Cyclic Arboricity of Matroids

(`http://arxiv.org/pdf/0912.2929`)

We present a general result concerning cyclic orderings of the elements of a matroid. This result has a few interesting corollaries, namely we generalize a result of Nash-Williams on covering graphs by spanning trees and prove a conjecture of Gonçalves and a special case of conjecture of Kajitani et. al.

A matroid $(E, \mathcal{I})$ is formed by a finite *ground set* E together with a collection $\mathcal{I}$ of *independent sets*. If a set $S \subseteq E$ is independent, then also any of its subsets is independent. If A and B are two independent sets and A has more elements than B, then we can add an element of A to B obtaining again an independent set. A maximal independent set is called a *base*. Any two bases have the same cardinality. Any matroid has an associated *rank function*: the rank $r(F)$ of a set $F \subseteq E$ is the cardinality of a maximal independent set contained in F. The rank of a matroid is the cardinality of a base.

The *maximal density* of a matroid $\mathcal{M}$ is $\gamma(\mathcal{M}) = \max(|A|/r(A))$, the maximum taken over all non-empty subsets $A \subseteq E$. We call a matroid *uniformly dense* if $\gamma(\mathcal{M}) = |E|/r(E)$. A matroid $\mathcal{M}$ of rank r with ground set E is *cyclically orderable* if there exists a cyclic ordering of the elements of E such that any r consecutive elements form a base of $\mathcal{M}$.

Conjecture 1. *A matroid is cyclically orderable if and only if it is uniformly dense.*

We prove this conjecture for a special class of matroids.

Theorem 2. *Let $\mathcal{M}$ be a matroid such that $|E|$ and $r(E)$ are coprime. Then $\mathcal{M}$ is cyclically orderable if and only if it is uniformly dense.*

The *arboricity* $\Upsilon(\mathcal{M})$ of a matroid $\mathcal{M}$ is the minimum number of bases needed to cover all the elements of the matroid. The *fractional arboricity* $\Upsilon_f(\mathcal{M})$ of a matroid $\mathcal{M}$ is defined in a way similar to the fractional chromatic number of a graph: a non-negative weight is assigned to each base of $\mathcal{M}$ in such a way that the sum of the weights of the bases containing an element e of E is at least one; then $\Upsilon_f(\mathcal{M})$ is the minimal sum of the weights assigned to the bases under these conditions. A third kind of arboricity is circular arboricity resembling the circular chromatic number of a graph. Let S_d be a circle with circumference d. Given a matroid $\mathcal{M}$, we want to map the elements of E to S_d in such a way that for any unit-length arc (cyclic interval) of S_d, the elements mapped into this interval form an independent set. The *circular arboricity* $\Upsilon_c(\mathcal{M})$ is the infimum taken over all values of d for which such a mapping is possible.

Since every base can contain at most $r(A)$ elements of any subset A of the matroid, the arboricity of a matroid is always at least its maximal density. A theorem of Edmonds assures that $\Upsilon(\mathcal{M}) = \lceil \gamma(\mathcal{M}) \rceil$. An easy corollary of this theorem is that $\Upsilon_f(\mathcal{M}) = \gamma(\mathcal{M})$. Our second main result is captured in the following theorem.

Theorem 3. *For a matroid $\mathcal{M}$ we have $\Upsilon_c(\mathcal{M}) = \Upsilon_f(\mathcal{M})$.*

A *weighted matroid* is a matroid $\mathcal{M}$ together with a weight function ω from E into the set of non-negative rational numbers. The weight $\omega(A)$ of a set $A \subseteq E$ is the sum of the weights of its elements. Similarly to the unweighted case, the *maximal density* $\gamma(\mathcal{M}, \omega)$ is the maximum of $\omega(A)/r(A)$ taken over all non-empty subsets A of E. A weighted matroid is *uniformly dense* if E achieves the maximal density.

Let Φ be a mapping from E to S_d. The *cyclic interval* (arc of the circle S_d) associated with an element e of E is $J(e) = [\Phi(e),\Phi(e) + \omega(e))$. Thus, to every point x of S_d, we can associate the set $E_\Phi(x) \subseteq E$ such that $e \in E_\Phi(x)$ whenever $x \in J(e)$. The mapping Φ is a *fractional cyclic independence order* of $(\mathcal{M}, \omega)$ if $E_\Phi(x)$ is independent for every point x of S_d.

Both Theorem 2 and Theorem 3 can be derived from the following theorem.

Theorem 4. *Let $(\mathcal{M}, \omega)$ be a weighted matroid, and d a positive rational number. Then there exists a fractional cyclic independence order of $(\mathcal{M}, \omega)$ into S_d if and only if d is greater than or equal to the maximal density $\gamma(\mathcal{M}, \omega)$.*

For a graph G, let $V(G)$ be the set of its vertices, $E(G)$ the set of its edges and $c(G)$ the number of its components.

Corollary 5. *Let G be a connected graph and d a positive real number such that for every set of edges A that form a cut-set of G we have $|A| \geq d \cdot c(G - A)$. Then there exists a mapping Φ from the edge set $E(G)$ into the circle S_d such that for every unit-length arc of S_d, the edges of G mapped into this arc form a spanning connected subgraph of G.*

The last corollary generalizes a well-known result of Tutte and Nash-Williams.

Theorem 6.*[Tutte, Nash-Williams] A multigraph contains k edge-disjoint spanning trees if and only if for every partition P of its vertex set it has at least $k(|P| - 1)$ edges with ends lying in different partitions.*

Guillem Perarnau
`guillem.perarnau@ma4.upc.edu`

Presented paper by C. Lee, J. Lee, S. Oum [3]

Rank-width of Random Graphs

(`http://arxiv.org/pdf/1001.0461`)

Introduction

The *rank-width* of a graph G, $\mathbf{rw}(G)$, is a width parameter that was introduced by Oum and Seymour[5] in order to study the *clique-width*. Let $G = G(n,p)$ a random graph following the Erdős-Rényi model. The *tree-depth* of G is another parameter that has a crucial paper on the theory of bounded expansion classes and has been introduced under numerous names. The definition we will use is that given by Nesetril and Ossona de Mendez [4]. The asymptotic behaviour of both parameters will be presented in the case of dense and sparse random graphs. The first part is presented in [3] and the second part has been developed by the speaker and O. Serra.

Rank-width of random graphs

Let V_1 and V_2 two disjoint subsets of nodes of G. We define their cut-rank $\rho_G(V_1, V_2)$ as the rank of $(0,1)$-matrix that gives the adjacencies between both subsets.

A rank-decomposition of G consist in a pair (T, L) where T is a sub-cubic tree and L a surjective application between the vertices of G, and the leaves of T. For each $uv \in E(T)$ we get a partition of G by looking on $A_{uv} = L^{-1}(C_u)$ and $B_{uv} = L^{-1}(C_v)$, where C_u and C_v are the leaves of the disjoint components of $T\backslash\{uv\}$.

In this context we can define the rank-width as,

$$\mathbf{rw}(G) = \min_{(T,L)} \max_{uv \in E(T)} \rho_G(A_{uv}, B_{uv})$$

Theorem 1.*[Dense random graphs] Asymptotically almost surely* $G = G(n,p)$ *satisfies:*

1. *if* $p \in (0,1)$ *is fixed, then* $\lceil \frac{n}{3} \rceil - O(1) \leq \boldsymbol{rw}(G) \leq \lceil \frac{n}{3} \rceil$
2. *if* $\frac{1}{n} \ll \min(p, 1-p)$*, then* $\lceil \frac{n}{3} \rceil - o(n) \leq \boldsymbol{rw}(G) \leq \lceil \frac{n}{3} \rceil$

Theorem 2.*[Sparse random graphs] Suppose that* $p = c/n$ *with* $c > 0$*, then a.a.s.* $G = G(n,p)$ *satisfies:*

1. *if* $0 < c < 1$ *, then* $\boldsymbol{rw}(G)$ *is at most* 2
2. *if* $c = 1$ *, then* $\boldsymbol{rw}(G)$ *is at most* $O(n^{2/3})$
3. *if* $c > 1$*, then there exists* $r = r(c)$ *such that* $\boldsymbol{rw}(G)$ *is at least* rn

This last theorem improves the bounds given by Kloks[2] and Gao[1] for the threshold of linear behaviour of the *tree-width* of a random graph,

Corollary 3. *Let* $c > 1$ *and* $p = c/n$*. Then there exists* $t = t(c)$ *such that a.a.s.* $\boldsymbol{tw}(G(n,p)) \geq tn$

Tree-depth of sparse random graphs

Let T a rooted tree, we define its closure as the graph having the same vertex set, and edges between vertices if one is ancestor of the other. Then $\mathbf{td}(G)$ is the minimum height of such a T whose closure contains G as a subgraph.

Theorem 4.*[Sparse random graphs] Suppose that* $p = c/n$ *with* $c > 0$*, then a.a.s.* $G = G(n,p)$ *satisfies:*

1. *if* $0 < c < 1$ *, then* $\boldsymbol{td}(G) = \Theta(\log \log n)$
2. *if* $c = 1$ *, then* $\boldsymbol{td}(G)$ *is at most* $O(\log n)$
3. *if* $c > 1$*, then there exists* $s = s(c)$ *such that* $\boldsymbol{td}(G)$ *is at least* sn

Then taking into account easy inequalities between tree-like parameters

Corollary 5. *Let* $c = 1$ *and* $p = c/n$*. Then a.a.s.* $\boldsymbol{rw}(G(n,p)) = O(\log n)$*.* This result improves Theorem 2.2.

References

[1] Gao, Yong On the threshold of having a linear treewidth in random graphs, *Computing and combinatorics, Lecture Notes in Comput. Sci.*,Springer , Berlin 2006.

[2] Kloks, Ton Treewidth, *Lecture Notes in Computer Science, Vol.842*, Springer, Berlin 1994.

[3] Lee, C, Lee, J and Oum,S Rank-width of Random Graphs, *Preprint*, 2010.

[4] Nešetřil, Jaroslav and Ossona de Mendez, Patrice Tree-depth, subgraph coloring and homomorphism bounds, *European J. Combin., Vol.27*, 2006.

[5] Oum, Sang-il and Seymour, Paul Approximating clique-width and branch-width, *J. Combin. Theory Ser. B, Vol. 96*, 2006.

List of All Participants

Ondřej Bílka
Charles University, Prague
Czech Republic
`neleai@seznam.cz`

Martin Böhm
Charles University, Prague
Czech Republic
`bohm@atrey.karlin.mff.cuni.cz`

Jakub Bulín
Charles University, Prague
Czech Republic
`jakub.bulin@gmail.com`

Josef Cibulka
Charles University, Prague
Czech Republic
`cibulka@kam.mff.cuni.cz`

Jiří Fink
Charles University, Prague
Czech Republic
`fink@kam.mff.cuni.cz`

Tomáš Gavenčiak
Charles University, Prague
Czech Republic
`gavento@kam.mff.cuni.cz`

Robert Himmelmann
TU München
Germany
`rjh@mytum.de`

František Kardóš
Pavol Jozef Šafárnik University, Košice
Slovakia
`frantisek.kardos@upjs.sk`

Pavel Klavík
Charles University, Prague
Czech Republic
pavel@klavik.cz

Miroslav Klimoš
Masaryk University, Brno
Czech Republic
myreggg@gmail.com

Tereza Klimošová
Charles University, Prague
Czech Republic
matfyzacka@gmail.com

Dušan Knopp
Charles University, Prague
Czech Republic
dusan.knop@gmail.com

Dan Král'
Charles University, Prague
Czech Republic
kral@kam.mff.cuni.cz

Jan Kratochvíl
Charles University, Prague
Czech Republic
honza@kam.mff.cuni.cz

Marek Krčál
Charles University, Prague
Czech Republic
krcal@kam.mff.cuni.cz

Petr Kučera
Czech University of Agriculture, Prague
Czech Republic
kucera@pef.czu.cz

Michaela Kukučková
Charles University, Prague
Czech Republic
kukuckaa@gmail.com

Martin Kupec
Charles University, Prague
Czech Republic
`kupec@kam.mff.cuni.cz`

Robert Lukot'ka
Comenius University, Bratislava
Slovakia
`lukotka@dcs.fmph.uniba.sk`

Lukáš Mach
Charles University, Prague
Czech Republic
`lukas.mach@gmail.com`

Jiří Matoušek
Charles University, Prague
Czech Republic
`matousek@kam.mff.cuni.cz`

Ján Mazák
Comenius University, Bratislava
Slovakia
`mazak@dcs.fmph.uniba.sk`

Jaroslav Nešetřil
Charles University, Prague
Czech Republic
`nesetril@kam.mff.cuni.cz`

Jitka Novotná
Charles University, Prague
Czech Republic
`copcas@gmail.com`

Pavel Paták
Charles University, Prague
Czech Republic
`ppatak@seznam.cz`

Guillem Perarnau
UPC, Barcelona
Spain
`guillem.perarnau@ma4.upc.edu`

Zuzka Safernová
Charles University, Prague
Czech Republic
`zuzka@saferna.net`

Martina Šimunková
Charles University, Prague
Czech Republic
`martina.simunkova@tul.cz`

Marek Sterzik
Charles University, Prague
Czech Republic
`marek@milimetr.org`

Rudolf Stolař
Charles University, Prague
Czech Republic
`ruda@kam.mff.cuni.cz`

Ondřej Suchý
Charles University, Prague
Czech Republic
`suchy@kam.mff.cuni.cz`

Martin Tancer
Charles University, Prague
Czech Republic
`tancer@kam.mff.cuni.cz`

Jakub Teska
University of West Bohemia, Pilsen
Czech Republic
`teska@kma.zcu.cz`

Tomáš Toufar
Charles University, Prague
Czech Republic
`tomas.toufar@gmail.com`

Tomáš Valla
Charles University, Prague
Czech Republic
`valla@kam.mff.cuni.cz`

Pavel Valtr
Charles University, Prague
Czech Republic
`valtr@kam.mff.cuni.cz`

Michal Vaner
Charles University, Prague
Czech Republic
`vorner@ucw.cz`

Jan Volec
Charles University, Prague
Czech Republic
`janvolec@jikos.cz`