Midsummer Combinatorial Workshop 2009

Jiří Fink (ed.)

Midsummer Combinatorial Workshop 2009
Editor Jiří Fink
Published by KAM-DIMATIA series in March 2010.
Tato publikace vyšla v sérii KAM-DIMATIA v březnu 2010.
DIMATIA a Katedra aplikované matematiky
Malostranské náměstí 25, Praha

ISBN 978-80-254-6760-2

Preface

The 15. Prague Midsummer Combinatorial Workshop was held from July 27th to July 31st 2009 in our beautiful building Malostranské náměstí 25. This of course contributed to the comfort of the participants as all the activities (including the lunches) could be taken on the same site. Besides, as it was expressed by several participants, the renovated faculty building surely belongs to the most beautiful math and CS departments in the world! The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with the DIMATIA center. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Endre Szemerédi, Christian Krattenthaler and John Gimbel among the participants. The list of speakers is included in this booklet.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students. For example six undergraduate students from the USA and six undergraduate students from Charles University, together with their mentors Martin Tancer, Martin Mare from Prague side and Aaron D. Jaggard, David Duncan from US side took part in the workshop, within a joint DIMATIA-DIMACS program International REU (supported jointly by NSF and Czech Ministry of Education ME 09074).

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest some of the atmosphere at the workshop from these proceedings, and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

This volume was edited by Jiří Fink. Most of the contributions were supplied by the authors in an electronic form. In a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

The 15. Midsummer Combinatorial Workshops was supported by by our institute ITI (financed by the Ministry of Education of the Czech Republic as project 1M0545) while DIMATIA was the main organizer.

We hope to meet again in 2010 the same midsummer week!

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List of participants:

Marcel Abas	Peter Allen	Dominique Andres
Ondřej Bílka	Julia Boettcher	Josef Cibulka
Anna De Mier	David Duncan	Peter Erdős
Luis Esperet	Stefan Felsner	Jiří Fiala
Jiří Fink	Delia Garijo	John Gimbel
Andrew Goodall	Gena Hahn	Jan Hladký
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Pavel Rytíř	Zuzka Safernová	Marcus Schaefer
Jens M. Schmidt	Jiří Sgall	Mark Siggers
Jozef Skokan	Rudolf Stolař	Ondřej Suchý
Endre Szemerédi	Martin Tancer	Bhalchandra Thatte
Gabor Fejes Toth	Pavel Valtr	Jan van den Heuvel
Channing Verbeck	László Végh	Jan Volec
Doreen Yagnatinsky	Norma Yu	

Figure 1: The conference photo

Hamiltonian embeddings of K_n - Cayley maps

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Joint work with Naďa Krivoňáková [2].

Abstract

A Cayley map is an embedding of a Cayley graph such that the automorphism group of the map contains a subgroup acting regularly on the vertices of the map. A map - a cellular embedding of a connected graph - is said to be Hamiltonian if the length of every face of the embedding is equal to the number of vertices of the underlying graph and no vertex is repeated.

We show that for each n even there exists a Hamiltonian embedding of K_n such that the embedding is a Cayley map and that there is no Hamiltonian Cayley map of K_n if n is a prime. We also give a conjecture about the existence of Hamiltonian Cayley maps for odd numbers.

1 Introduction

The question for which pairs (n, r) there is an embedding of K_n with uniform face lenghts r is very interesting. From the fact that the complete graph K_n has $\frac{n(n-1)}{2}$ edges it follows that an embedding with face lengths r can only exists if r divides $n(n-1)$. For $r = 3$, the solution is a part of the solution of Heawood map coloring problem [5]. For $r = 4$ a solution is also known - in [3, 4] have been characterized all n with quadrilateral embeddings of K_n. In [2] is a characterization for $r = n$ - there all the embeddings are Hamiltonian.

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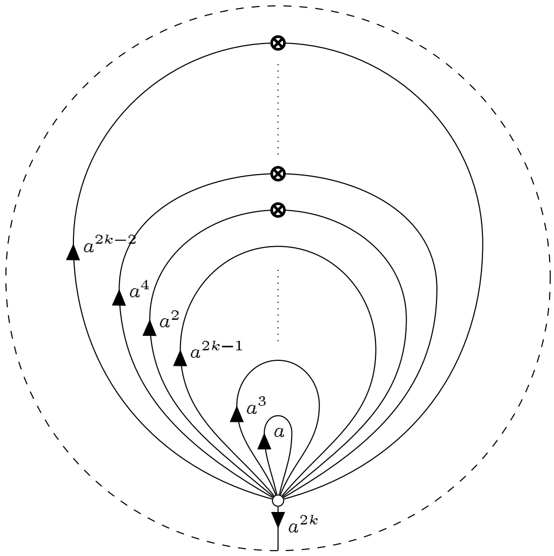

Figure 1: 1-vertex regular quotient on the nonorientable surface $\mathcal{N}_{k-1,1}$ with multiplicative cyclic voltage group Z_{4k}, which induces a hamiltonian Cayley map of K_{4k} in a nonorientable surface without boundary of non-orientable genus $h = \frac{(n-2)(n-3)}{2}$.

2 Results

We show that for each n even there is a Hamiltonian Cayley map of K_n and that there is no Hamiltonian Cayley map of K_n if n is a prime number. In figures 1 and 2 we can see 1-vertex regular quotients of Hamiltonian Cayley maps for even n. Therefore we obtain the following theorem:

Theorem 2.1. *[1] Let $n \geq 4$ be an even number. Then there exists a Hamiltonian Cayley map of K_n.*

On the other hand, the next theorem shows that there is no Hamiltonian Cayley map of K_p for p a prime.

Theorem 2.2. *[1] If p is a prime number greater than 3 then there is no Hamiltonian Cayley map of K_p.*

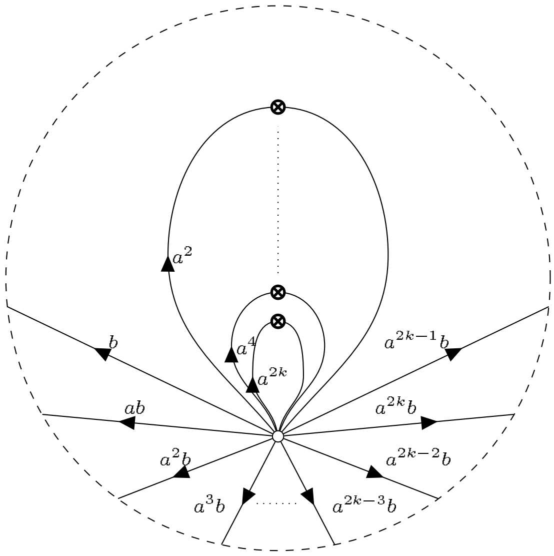

Figure 2: 1-vertex regular quotient on the nonorientable surface $\mathcal{N}_{k,1}$ with dihedral voltage group D_{2k+1}, which induces a hamiltonian Cayley map of K_{4k+2} in a nonorientable surface without boundary of non-orientable genus $h = \frac{(n-2)(n-3)}{2}$.

Proof. The only group of order p is the cyclic group $\mathcal{Z}_p$. If there exists a Hamiltonian Cayley map of K_p, every face of its 1-vertex quotient M_0 has to have either voltage 1 and length p or the order of its voltage is p and its length is 1. There are no elements of order two in $\mathcal{Z}_p$ - therefore the supporting surface of M_0 has no boundary and the sum of lengths of all faces of M_0 is $p-1$. So there are only faces of length 1 in M_0. But it is possible only if $p = 3$ - in this case the corresponding 1-vertex quotient map is a simple loop and it lifts to a 3-cycle - an embedding of K_3 in the 2-sphere. □

We conjecture the following:

Conjecture 2.3. *[1] If $n \geq 5$ is an odd number then there is no Hamiltonian Cayley map of K_n.*

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On Characterizing Game-Perfect Graphs by Forbidden Induced Subgraphs

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1 Introduction

Consider the following game, played on an (initially uncolored) graph $G = (V, E)$ with a color set C. The players, Alice and Bob, move alternately. A move consists in coloring a vertex $v \in V$ with a color $c \in C$ in such a way that adjacent vertices receive distinct colors. If this is not possible any more, the game ends. Alice wins if every vertex is colored in the end, otherwise Bob wins.

This type of game was introduced by Bodlaender [4]. He considers a variant, which we will call game g, in which Alice must move first and passing is not allowed. In order to obtain upper and lower bounds for a parameter associated with game g, two other variants are useful. In the game B Bob may move first. He may also miss one or several turns, but Alice must always move. In the other variant, game A, Alice may move first and miss one or several turns, but Bob must move. So in game B Bob has some advantages, whereas in game A Alice has some advantages with respect to Bodlaender's game (see [1]).

For any variant $\mathcal{G} \in \{B, g, A\}$, the smallest cardinality of a color set C, so that Alice has a winning strategy for the game $\mathcal{G}$ is called *$\mathcal{G}$-game chromatic number* $\chi_{\mathcal{G}}(G)$ of G.

During the last two decades, a lot of work has been done to determine upper bounds for the g-game chromatic number of classes of graphs, initiated by the papers of Bodlaender [4] and Faigle et al. [6]. A recent survey was given by Bartnicki et al. [3].

Let $\omega(G)$ be the clique number of a graph G. G is called *B-perfect* if, for any induced subgraph H of G, $\chi_B(H) = \omega(H)$. Analogously, we define *A-perfect* with respect to the game A and *g-perfect* with respect to Bodlaender's game. These concepts were introduced in [2] and are game-theoretic analoga of *perfect* graphs which are those graphs in which, for any induced subgraph H, the clique number equals the chromatic number $\chi(H)$.

Observation 1.1. *For any graph H,*

$$\omega(H) \leq \chi(H) \leq \chi_A(H) \leq \chi_g(H) \leq \chi_B(H).$$

In particular, B-perfect graphs are g-perfect, g-perfect graphs are A-perfect, and A-perfect graphs are perfect. We consider the problem of characterizing these classes of graphs. The (probably most difficult) case of perfect graphs has been solved by the Strong Perfect Graph Theorem [5]:

Theorem 1.2 (Chudnovsky, Robertson, Seymour, Thomas (2006))**.** *A graph is perfect if, and only if, it does neither contain an odd hole nor an odd antihole as induced subgraph.*

In Section 2 we will characterize B-perfect graphs. The characterization of A-perfect and g-perfect graphs is still open, some partial results are given in Section 3.

2 Characterizing B-perfect graphs

Theorem 2.1. *Let G be a (nonempty) graph. Then the following conditions are equivalent:*

(i) *G is B-perfect.*

(ii) *G does neither contain a C_4, nor a P_4, nor a split 3-star, nor a double fan as induced subgraph (see Fig. 1).*

(iii) *For every component H of G, there is $k \geq 0$, so that*

$$H = K_1 \vee (H_0 \cup H_1 \cup \ldots \cup H_k),$$

where the H_i are complete graphs for $i \geq 1$, and H_0 is either empty or there are $p, q, r \in \mathbb{N}$, so that $H_0 = K_p \vee K_r \vee K_q$ (see Fig. 2).

Proof. (i) $\Longrightarrow$ (ii): Winning strategies for Bob with ≤ 2 colors on C_4 resp. P_4 resp. with ≤ 3 colors on the split 3-star resp. the double fan are obvious.

(iii) $\Longrightarrow$ (i): We describe a winning strategy for Alice with $\omega(G)$ colors on a graph G as in (iii). This is sufficient since every induced subgraph of G is of the same type as described in (iii). For $H_0 = K_p \vee K_r \vee K_q$ let the K_p and the K_q be the *ears*. Alice always responds to Bob's moves in the same component H (if Bob passes, in an arbitrary component). As long as Bob does not play in an ear, Alice does not play in an ear; she first colors the universal vertex of H. If Bob plays in an ear K_p, Alice colors a vertex in the corresponding ear K_q with the same color (in case there is no uncolored vertex she uses the strategy described before). If Alice is forced to start coloring an ear, then all non-ear-vertices are colored, so a coloring of the ears is possible without creating danger for a non-ear-vertex.

(ii) $\Longrightarrow$ (iii): We examine the structure of a graph G without induced P_4, C_4, split 3-star, double fan. Let H be a component of G. We use the following lemma of Wolk [8].

Lemma 2.2 (Wolk (1965))**.** *A connected graph without induced C_4 and P_4 (a so-called* trivially perfect graph [7]*) has a universal vertex.*

So, H has a universal vertex v. Let $H_0, \ldots, H_n$ be the components of $H \setminus v$.

Claim 2.3. *At most one of the H_i is not complete.*

Proof. Assume H_1, H_2 are not complete. Then both contain a P_3. So H contains a double fan, which contradicts (ii). □

Let H_0 be the (only) component of $H \setminus v$ which is not complete. Let K be the largest clique of H_0. We are done if we show:

Claim 2.4. *$H_0 \setminus K$ induces a clique.*

Claim 2.5. *$H_0 \setminus K$ induces a module of H_0 (i.e. if $x \in K$, either x is adjacent to all $y \in H_0 \setminus K$ or to none.)*

Then $H_0 \setminus K$ corresponds to the K_p, its neighbors correspond to the K_r, and the rest of H_0 corresponds to the K_q. Thus G is as in (iii).

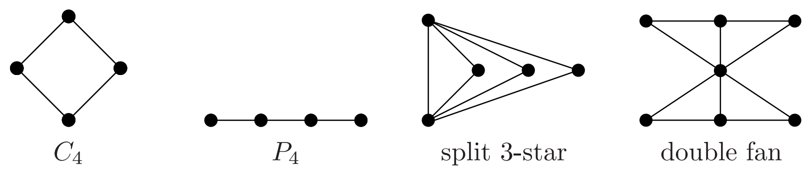

Figure 1: 4 forbidden induced subgraphs for B-perfect graphs

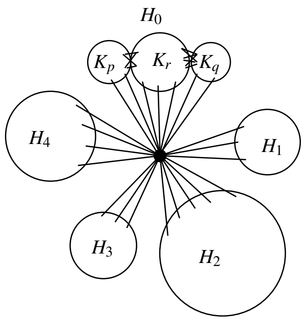

Figure 2: Structure of a component according to (iii)

Proof of Claim 2.4. Assume there are non-adjacent vertices $x, y \in H_0 \setminus K$. Since K is a maximal clique, there are $z, z' \in K$ such that neither x, z nor y, z' are adjacent. We note that, again by Lemma 2.2, H_0 has a universal vertex $w \in K$. If y, z are not adjacent, x, y, z, w, v induce a split 3-star, contradicting (ii). So we may assume that y, z are adjacent and, by symmetry, x, z' are adjacent. This implies that $z \neq z'$ and x, y, z, z' induce a P_4, contradicting (ii). □

Proof of Claim 2.5. Assume that there are $x \in K$, $s, t \in H_0 \setminus K$, so that s, x are not adjacent, but t, x are adjacent. By Claim 2.4, s, t are adjacent. Since K is a maximal clique, there is $y \in K$, so that t, y are not adjacent. This implies that s, t, x, y induces either a P_4 or a C_4, which contradicts (ii). □

This completes the proof of Theorem 2.1. □

3 On characterizing A-perfect graphs

There is not much known about A-perfect graphs. The graphs depicted in Fig. 3 are known to be minimal forbidden induced subgraphs, but there are probably many more. In a recent paper [2], the author proved the following two theorems.

Theorem 3.1. *A triangle-free graph G is A-perfect if, and only if, every component of G is either K_1 or $K_{m,n}$ or $K_{m,n} - e$, where e is an edge.*

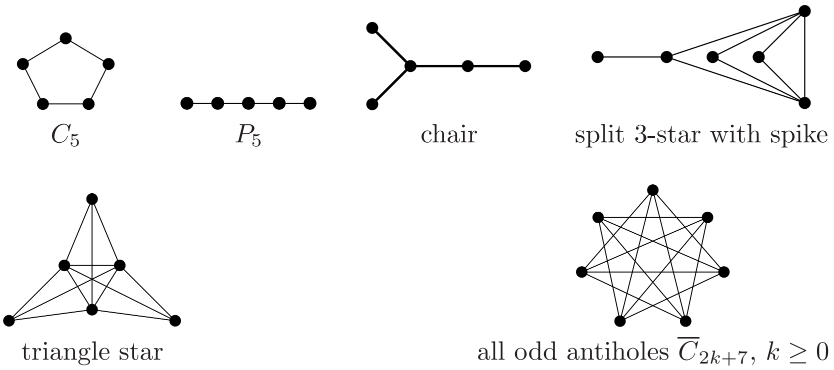

Figure 3: Some forbidden induced subgraphs for A-perfect graphs

Theorem 3.2. *Complements of bipartite graphs are A-perfect.*

In view of Observation 1.1 and the preceding results, the following open problem seems to be more difficult than Theorem 2.1.

Problem 3.3. *Characterize A-perfect graphs by forbidden induced subgraphs and/or explicitly.*

For other variants of the game, this question might be interesting, too.

Problem 3.4. *Characterize g-perfect graphs by forbidden induced subgraphs and/or explicitly.*

Even the following is an open question.

Problem 3.5. *Is the number of minimal forbidden induced subgraphs for A-perfectness that are different from odd antiholes finite?*

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Triangle Contact Representations

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Abstract

It is conjectured that every 4-connected plane triangulation has a triangle contact representation with homothetic triangles. We outline a roadmap for a proof of this conjecture and report on partial results and experimental evidence.

1 Introduction

Our interest in this paper are triangle contact representations of planar graphs with homothetic triangles, i.e, vertices are represented by a set of disjoint triangles that are identical up to scalings and translations, two triangles touch exactly if there is an edge between the corresponding vertices. See Figure 1. For brevity we will refer to such a representation as a *htc-representation*. Using an affine map a htc-representation can be transformed into a htc-representation with equilateral triangles. The big conjecture is:

Conjecture 1.1. *Every 4-connected planar triangulation has a triangle contact representation with homothetic triangles, i.e., a htc-representation.*

The conjecture came up during the Graph Drawing workshop in Bertinoro 2007. In [4] it was shown that max-tolerance graphs are the intersection graphs of homothetic triangles. Lehmann asked whether every planar graph is a max-tolerance graph. Kratochvíl asked for contact representations. A result of the workshop was that planar partial 3-trees (also known as subgraphs of stacked triangulations), and hence also series-parallel graphs, are contact graphs of homothetic triangles, see [1].

De Fraysseix et al. [2] have shown that relaxing the condition on the triangles from equilateral to isosceles allows a contact representation for every planar graph. See Figure 2. Actually, they show that such a representation is possible such that each contact is of the type corner vs. side, we call such a contact a *pure contact*. If we ask for a htc-representation of the octahedron graph, then we have to use triangles of equal size for the inner vertices

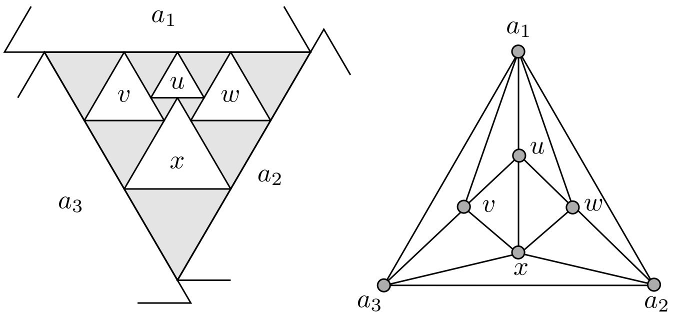


Figure 1: A homothetic triangle contact representation of a planar graph.

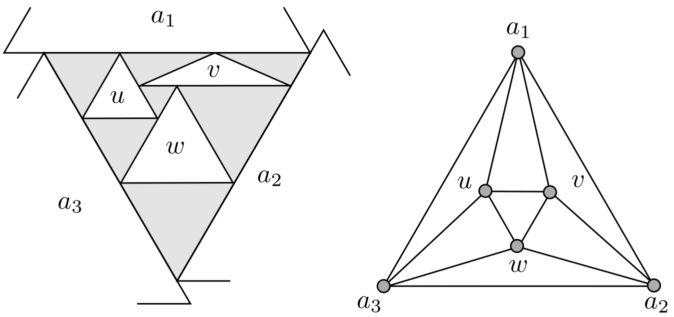


Figure 2: A isocseles triangle contact representation of the octahedron graph.

u, v and w. Consequently, there is a point where three corners meet and the 3-face formed by u, v and w is only represented by their mutual contact point, it is degenerated to size 0. This implies that graphs obtained from the octahedron by glueing a triangulation H into the face u, v, w can only have htc-representations where the triangles representing the inner vertices of H are of size 0. We shall not allow this. The kind of degeneracy described with this example of the octahedron graph depend on the existence of separating 3-cycles, i.e., they can only occur if the graph is not 4-connected. This is why we have the restriction in the conjecture.

An essential role in our investigations will be played by Schnyder woods:

Definition 1.2. An orientation and coloring of the inner edges of T with colors red, green and blue is a *Schnyder wood* if:

(1) All edges incident to a_1 are red, all edges incident to a_2 are green and all edges incident to a_3 are blue.

(2) Every inner vertex v has three outgoing edges colored red, green and blue in clockwise order. All the incoming edges in an interval between two outgoing edges are colored with the third color, see Figure 3 (left).

Figure 3: Left: Schnyder's edge coloring rule.
Right: Triangle contacts induce coloring and orientation of edges.

It was observed by de Fraysseix et al. [2] that a triangle contact representation of a triangulation where all contacts are pure induces a Schnyder wood. The construction is as indicated in Figure 3 (right): Color the corners of the triangles in the representation red, green, blue. Given an edge u, v, look at the contact of the corresponding triangles, if a corner of u's triangle is involved, then color the edge with the color of that corner and orient it from u to v.

The construction of a triangle contact representation of a planar graph, in [2], is as follows[1]: First augment the planar graph H to a triangulation G such that H is an induced subgraph of G. Compute a Schnyder wood of G and use this structure to build a pure triangle contact representation. The consequence is that every Schnyder wood of a triangulation G is induced by some triangle contact representation of G. This is not true for htc-representations.

The steps in our approach for htc-representations of triangulations are as follows:

[1]In [2] they speak about *canonical orderings* instead of Schnyder woods, but these are equivalent concepts.

- Compute a Schnyder wood S of the input graph G.
- Based on S build a system $\mathcal{A}_S$ of linear equations.
- Compute a solution x_S of $\mathcal{A}_S$.

If all entries of x_S are non-negative we are done; based on x_S we can build a htc-representation of G that induces S. If there are negative entries in x_S we use the sign information to transform S into another Schnyder wood S' and iterate. We conjecture that independent of the choice of S the sequence $S \to S' \to S'' \to$ has a finite length, i.e., there is a k such that the solution $x_{S^{(k)}}$ of the system corresponding to $S^{(k)}$ is non-negative.

There is strong experimental evidence that the conjecture is true. We have an implementation of the approach and computed thousands of htc-representations for planar graphs with up to 500 vertices. We have also restarted the computation for a fixed graph with alternate Schnyder woods and compared the result. This suggests that a 4-connected plane triangulation with a prescribed outer face has a *unique* htc-representation.

In the next section we give some details on the system $\mathcal{A}_S$ of linear equations and a sketch of the theoretical results we have so far.

2 Details for the Construction and Partial Results

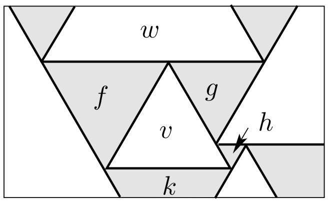
Let G be a plane triangulation with n vertices and a Schnyder wood S. The system $\mathcal{A}_S$ can be written as $A_S \cdot x = \mathbf{e_1}$ with a $(3n-8) \times (3n-8)$ matrix A_S and the first standard basis vector $\mathbf{e_1}$. The components of x are indexed by the $2n-5$ bounded faces and the $n-3$ inner vertices of G. The first equation is

$$\sum_{f \in \mathcal{F}(a_1)} x_f = 1,$$

where $\mathcal{F}(a_1)$ is the set of bounded faces incident to the special vertex a_1. Every inner vertex induces three equations, one for each color. For $c \in \{$red, green, blue$\}$ let $\mathcal{F}_c(v)$ be the set of bounded faces incident to v in the interval where edges of color c are incoming. The equation corresponding to (v, c) is

$$-x_v + \sum_{f \in \mathcal{F}_c(v)} x_f = 0.$$

From Figure 3 it is evident that the faces in $\mathcal{F}_c(v)$ are exactly the faces whose triangle has a side contained in the side of v's triangle opposite to the corner of color c. Therefore, the sum of sidelengths of triangles for faces in $\mathcal{F}_c(v)$ has to equal the sidelength of v's triangle. The scheme is illustrated in Figure 4.



$$x_w = x_f + x_g$$
$$x_v = x_f$$
$$x_v = x_g + x_h$$
$$x_v = x_k$$

Figure 4: A cutout of a htc-representation and some of the equations it implies. The equations from top to bottom are (w, red), (v, green), (v, blue) and (v, red).

The following result implies that the system $\mathcal{A}_S$ has a unique solution.

Fact 2.1. *The matrix A_S is non-degenerate, i.e.,* $\det(A_S) \neq 0$.

The idea for the proof is to show that $(-1)^{n-3}\det(A_S)$ is the number of perfect matchings of an auxiliary graph H_S. Multiplying the columns of A_S corresponding to vertices with -1 yields a 01-matrix $\hat{A}_S$. The graph H_S is the bipartite graph with adjacency matrix $\hat{A}_S$, i.e., it has $6n-16$ vertices, one for each equation of $\mathcal{A}_S$, one for each inner vertex of G and one for each bounded face of G. The non-vanishing summands $\prod_i \hat{a}_{i\ \sigma(i)}$ in the Leibniz-expansion of $\det(\hat{A}_S)$ are in bijection to the perfect matchings M_σ of H_S. The contribution of M_σ to $\det(\hat{A}_S)$ is $\text{sign}(\sigma)$. Define the sign of a matching M_σ as $\text{sign}(M_\sigma) = \text{sign}(\sigma)$. The crucial observations for the proof of Fact 2.1 are:

(1) If M and M' are perfect matchings of H_S, then $\text{sign}(M) = \text{sign}(M')$.

(2) H_S has a perfect matching.

The proof is based on properties of H_S: The graph H_S is planar and all its bounded faces are of length 6.

Fact 2.2. *If the unique solution x of $A_S \cdot x = \mathbf{e_1}$ is non-negative, then there is a htc-representation where the triangles of inner vertices and bounded faces have sidelengths as given by the vector x.*

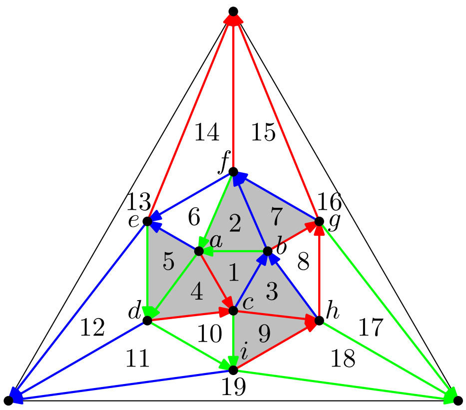


Figure 5: Schnyder wood of the icosahedron. The faces with negative values in the solution vector x are shaded. The boundary of the shaded area is a directed cycle.

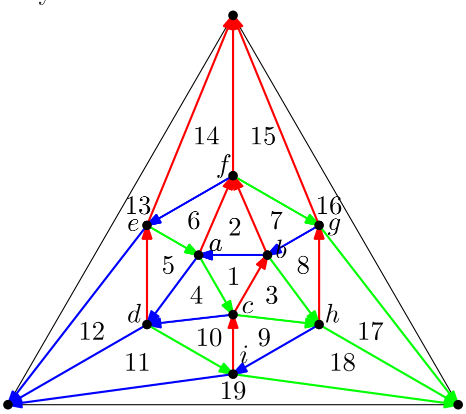


Figure 6: Schnyder wood of the icosahedron that results from reverting the cycle in Figure 2. The new solution vector is non-negative.

Fact 2.3. *If the unique solution x of $A_S \cdot x = \mathbf{e_1}$ has negative entries, then we can decompose the boundary between negative and non-negative faces into cycles that are directed in the Schnyder wood.*

From the theory of Schnyder woods it is know that the coloring of edges can be recovered if only the orientation of edges is given and indeed every 3-orientation, i.e., orientation such that every inner vertex has out-degree 3, corresponds to a Schnyder wood. This implies that a directed cycle of a Schnyder wood S can be reverted and appropriate recoloring yields another Schnyder wood S'.

Therefore, Fact 2.3 implies that whenever the solution x to the system $A_S \cdot x = \mathbf{e_1}$ has negative components, this solution can be used to move to another Schnyder wood S'. Figure 5 shows an example for Fact 2.3 and the transition $S \to S'$.

Let S and S' be Schnyder woods of a triangulation G. In [3] it is shown that S' can be reached from S via a series of triangle-flips, i.e., via a series of reversals of directed cycles of length three. Moreover if γ is a simple directed cycle in a Schnyder wood S, then $S' = \mathrm{flip}(\mathrm{S}, \gamma)$ can be obtained by flipping the triangles contained in γ.

Therefore it is particularly important to understand the effect of triangle-flips on the solution vectors.

Fact 2.4. *If Schnyder woods S and S' are related by a triangular-flip at a face f and x, x' are the solutions of the systems $\mathcal{A}_S$ and $\mathcal{A}_{S'}$, then*

$$\mathrm{sign}(x_f) \neq \mathrm{sign}(x'_f).$$

This suggests that starting with some Schnyder wood S and flipping negative faces may lead to Schnyder wood without negative faces, i.e., to a non-negative solution, hence, to a htc-representation. This is what happens in the experiments.

The proof of Fact 2.4 again uses the correspondence between determinants and matchings that was exploited for Fact 2.1. Indeed the solution x of $\hat{A}_S \cdot x = \mathbf{e_1}$ is explicitely given as the first column of the inverse of $\hat{A}_S$ wherefore the entry for a vertex or face z can be written in terms of the determinant of a cofactor: $\det(\hat{A}_S)\, x_z = \pm \det([\hat{A}_S]_{1,z})$.

Acknowledgments

First ideas and experiments concerning htc-representations where worked out at the Graph Drawing workshop in Bertinoro 2007. This was joint work with Jan Kratochvíl, Ileana Streinu and Alexander Wolff. Actually the basic idea of using Schnyder woods to generate a system of equations and flipping cycles to get rid of negative variables was born there. Figure 5 is taken from a memo written by Alexander Wolff.

A very good and useful implementation of the approach is due to my student Julia Rucker. I also thank Torsten Ueckerdt for discussions and his continuing interest in the topic.

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Covering line graphs with equivalence relations

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Joint work with Louis Esperet[2] and Andrew King[3].

Abstract

An equivalence graph is a disjoint union of cliques, and the equivalence number $eq(G)$ of a graph G is the minimum number of equivalence subgraphs needed to cover the edges of G. We consider the equivalence number of a line graph, giving improved upper and lower bounds: $\frac{1}{3}\log_2\log_2\chi(G) < eq(L(G)) \leq 2\log_2\log_2\chi(G)+2$. This disproves a recent conjecture that $eq(L(G))$ is at most three for triangle-free G; indeed it can be arbitrarily large.

To bound $eq(L(G))$ we bound the closely-related invariant $\sigma(G)$, which is the minimum number of orientations of G such that for any two edges e, f incident to some vertex v, both e and f are oriented out of v in some orientation. When G is triangle-free, $\sigma(G) = eq(L(G))$. We prove that even when G is triangle-free, it is NP-complete to decide whether or not $\sigma(G) \leq 3$.

Keywords: Equivalence covering, clique chromatic index, orientation covering, line graph, eyebrow number.

Given a binary relation $\sim$ over a set A, it is natural to consider expressing $\sim$ as a union of k transitive subrelations for the smallest possible value of k. If $\sim$ is reflexive and symmetric, each subrelation is an equivalence relation and we can restate the problem as a graph covering problem: We seek to cover the edges of a graph G with k *equivalence subgraphs*, i.e. subgraphs each of which is a disjoint union of cliques. This is an *equivalence covering* of G. The minimum k for which this is possible is the *equivalence number* of G, denoted $eq(G)$.

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The equivalence covering number was introduced by Duchet in 1979 [3]. Not surprisingly, it is NP-complete to compute, even for split graphs [2]. In [1], Alon proved upper and lower bounds for general graphs:

Theorem 1.1. *Let G be a graph on n vertices with minimum degree δ, and let $cc(G)$ be the minimum number of cliques needed to cover the edges of G. Then*

$$\log_2 n - \log_2(n-\delta-1) \leq eq(G) \leq cc(G) \leq 2e^2(n-\delta)^2 \log_e n.$$

Observe that if G is triangle-free, then every equivalence subgraph of G is a matching. It follows that in this case an equivalence covering of G is actually an edge coloring, and that $eq(G)$ is equal to the chromatic index $\chi'(G)$. Thus equivalence coverings can also be thought of as a generalization of edge colorings. In fact, McClain [7] formulated them seemingly independently of earlier work in precisely this context, calling $eq(G)$ the *clique chromatic index* of G.

In this paper we address the problem, first studied by McClain, of bounding the equivalence number of line graphs. For a graph G, the line graph $L(G)$ of G has a vertex corresponding to each edge of G, and two vertices of $L(G)$ are adjacent precisely if the two corresponding edges of G share an endpoint (i.e. are *incident*)[4]. McClain proved that for a graph G on n vertices, $eq(L(G)) \leq 4\left\lceil \frac{\log_e n}{\log_e 12} \right\rceil$, and asked if this bound could be improved [6]. We will prove that

$$\tfrac{1}{3}\left(\lceil \log_2 \log_2 \chi(G) \rceil + 1\right) \leq eq(L(G)) \leq 2\left(\lceil \log_2 \log_2 \chi(G) \rceil + 1\right),$$

where $\chi(G)$ is the chromatic number of G. We will actually prove a slightly better (but more unwieldy) lower bound. Since triangle-free graphs can have arbitrarily high chromatic number, our lower bound disproves a recent conjecture of McClain [7] stating that any triangle-free graph G has $eq(L(G)) \leq 3$.

Let $\overrightarrow{G_1}, \overrightarrow{G_2}, \ldots, \overrightarrow{G_k}$ be a set of orientations of G with the following property: For every vertex u of G with neighbors v and w, there is an i such that $\overrightarrow{uv}, \overrightarrow{uw} \in \overrightarrow{E}(\overrightarrow{G_i})$. In other words, for every $e, f \in E(G)$ sharing an endpoint v, some orientation $\overrightarrow{G_i}$ directs both e and f out of v. We call such

[4]We need only consider line graphs of simple graphs: If two vertices u and v of G have the same closed neighborhood, it is easy to see that $eq(G) = eq(G-v)$. Thus we can easily reduce the problem for line graphs of multigraphs.

a set of orientations an *orientation covering* of G, and accordingly define the *orientation covering number* of G, denoted $\sigma(G)$, as the size of a minimum orientation covering. Observe that $\sigma(G)$ and $eq(L(G))$ are the same if G is triangle-free. In the general case, we obtain the following inequalities:

Theorem 1.2. *For any graph G,*

$$eq(L(G)) \leq \sigma(G) \leq 3\, eq(L(G)).$$

Now, the fact that $\sigma(G) \leq 2\left(\lceil \log_2 \log_2 \chi(G) \rceil + 1\right)$ is just a simple consequence of a theorem of Kříž and Nešetřil [4], after a quick observation that if G has chromatic number k, then $\sigma(G) \leq \sigma(K_k)$. On the other hand, the lower bound on σ is a consequence of the following theorem:

Theorem 1.3. *For any graph G with $\chi(G) \geq 3$, $\chi(G) \leq 2^{2^{\sigma(G)-1}}$.*

Indeed, we prove the following stronger result:

Theorem 1.4. *Any graph G with $\sigma(G) = k \geq 3$ has $\chi(G) \leq k + 2^{2^{k-1}-k-1}$. Thus*

$$k \geq \log_2\left(\log_2(\chi(G) - k) + k + 1\right)$$

As a consequence of these inequalities, $\sigma(G)$ is precisely determined by $\chi(G)$ for $\chi(G) = 3, 4$. hence, $\sigma(G)$ is difficult to compute, as is $eq(L(G))$:

Theorem 1.5. *It is NP-complete to decide whether or not a triangle-free graph G has $\sigma(G) \leq 3$ (resp. $\sigma(G) \leq 4$). Equivalently, it is NP-complete to decide whether or not $eq(L(G)) \leq 3$ (resp. $eq(L(G)) \leq 4$).*

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Graph homomorphisms and contraction–deletion invariants

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Joint work with Delia Garijo[2] and Jaroslav Nešetřil [3].

This note is based on the extended abstract [3] and the paper [2]. Given a multigraph $G = (V, E)$, the multigraphs G/e and $G\backslash e$ are those obtained by respectively contracting and deleting an edge e. We are interested in graph parameters f that satisfy the following recurrence for any $e \in E$:

$$f(G) = \begin{cases} \alpha f(G/e) + \beta f(G\backslash e) & e \text{ not a bridge or loop,} \\ x f(G/e) & e \text{ a bridge,} \\ y f(G\backslash e) & e \text{ a loop,} \\ \gamma^{|V|} & E = \emptyset. \end{cases} \tag{1}$$

The *Tutte polynomial* of $G = (V, E)$ is defined by

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)},$$

where $r(A)$ is the rank of the spanning subgraph (V, A). The well-known "recipe theorem" says that if f satisfies the recurrence (1) for all multigraphs G then $f(G) = \gamma^{k(G)} \alpha^{r(G)} \beta^{n(G)} T(G; \frac{x}{\alpha}, \frac{y}{\beta})$. The *q-state Potts partition function*

$$P(G; q, y) = \sum_{\phi: V \to [q]} y^{|\{ij \in E : \phi(i) = \phi(j)\}|}$$

satisfies the recurrence $P(G) = (y-1)P(G/e) + P(G\backslash e)$ for any edge e, so that $P(G; q, y) = q^{k(G)} (y-1)^{r(G)} T(G; \frac{y-1+q}{y-1}, y)$. In particular, the case $y = 0$ is the *chromatic polynomial* $P(G; q) = q^{k(G)} (-1)^{r(G)} T(G; 1-q, 0)$.

[1] Partially supported by the hosting departments while visiting D. Garijo in December 2008 and J. Nešetřil in January 2009.

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The number of homomorphisms from a multigraph G to a multigraph H is denoted by $\hom(G, H)$. For an edge-weighted graph H with adjacency matrix $A(H) = (a_{u,v})$, we define

$$\hom(G, H) = \sum_{f:V(G)\to V(H)} \prod_{ij\in E(G)} a_{f(i),f(j)}.$$

Let $K_q^{a,b}$ denote the complete graph with edge weights b together with a loop of weight a at each vertex. Then $\hom(G, K_q^{y,1}) = P(G; q, y)$. For this reason we call $K_q^{a,b}$ a "Potts model graph".

The vector $(\hom(G, H) : H \in \mathcal{H})$ is called the *right $\mathcal{H}$-profile* of G, and $(\hom(G, H) : G \in \mathcal{G})$ the *left $\mathcal{G}$-profile* of H.

Theorem 1.1. *G is determined by its Tutte polynomial if and only if G is determined by its right $\{K_q^{y,1} : q, y \in \mathbb{N}\}$-profile.*

Definition 1.2. A multigraph is *q-state Potts unique* if it is determined by its right $\{K_q^{y,1} : y \in \mathbb{N}\}$-profile.

In [2] we show that cycles, theta graphs, squares of cycles, ladders and Möbius ladders are all q-state Potts unique. There are non-isomorphic graphs G and G' with $P(G; 2, y) = P(G'; 2, y)$ but $P(G; q, y) \neq P(G'; q, y)$ for $q \geq 3$, and also $P(G; q) \neq P(G'; q)$. On the other hand there are graphs G, G' with $P(G; 2, y) \neq P(G'; 2, y)$ while $P(G; q) = P(G'; q)$.

Problem 1.3. *Given $2 \leq q < q'$, are there non-isomorphic multigraphs G and G' such that $P(G; q, y) \neq P(G'; q, y)$ but $P(G; q', y) = P(G'; q', y)$?*

The dual of the cycle, C_k^*, consists of k parallel edges joining a pair of vertices. The dual of the star, $K_{1,k}^*$, consists of k loops on one vertex.

Theorem 1.4. *A Potts model graph $K_q^{a,b}$ is determined by its left $\{C_k, K_{1,k}\}$-profile and by its left $\{C_k^*, K_{1,k}^*\}$-profile.*

Theorem 1.5. *If $f(G) = \hom(G, H)$ satisfies the recurrence equations* (1) *for $G \in \{C_k, K_{1,k}, P_k, C_k^*, K_{1,k}^* : k \in \mathbb{N}\}$ then $x = \alpha + q\beta, y = \alpha + \beta, \gamma = q$ and H is isomorphic to the Potts model graph $K_q^{y,\beta}$.*

Corollary 1.6. *Suppose H is an edge-weighted graph such that* $\hom(G, H)$ $= h(G)T(G; x, y)$ *for $G \in \{C_k, K_{1,k}, P_k, C_k^*, K_{1,k}^* : k \in \mathbb{N}\}$, where $h(G)$ takes non-zero values such that the quotients $h(G)/h(G/e)$ and $h(G)/h(G\backslash e)$ only depend on whether e is a bridge, loop or neither. Then H is a Potts model graph.*

This extends [4, Theorem 2.7] and also a result given in [1, Example 3.3].

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Degrees of Perfection

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Abstract

The concept of a perfect graph has shown itself to be one of the fundamental concepts in graph theory. In this note we extend the notion of "perfectness" to a more general setting, involving set systems. Although perfect graphs are just one of the special cases in this setting, it appears that they are in fact central to most of the theory again.

Most of the results in this note are fairly straightforward observations. It is very likely that they have been made many times before. The author welcomes any pointers to earlier literature dealing with these topics.

1 Introduction

In this note a *set system* is a 2-tuple $(S, \mathcal{F})$ with S a non-empty finite set, and $\mathcal{F}$ a non-empty collection of subsets of S. We call a set system *good* if it satisfies the following two properties:

(i) The collection $\mathcal{F}$ is closed under taking subsets (if $F_1 \in \mathcal{F}$ and $F_2 \subseteq F_1$, then $F_2 \in \mathcal{F}$).

(ii) For every $s \in S$, there is an $F \in \mathcal{F}$ with $s \in F$.

Note that by (i), condition (ii) could also be formulated as: For every $s \in S$ we have $\{s\} \in \mathcal{F}$. We also have that always $\varnothing \in \mathcal{F}$.

Good set systems are known in the literature under several other names, often related to the context they have been studied (hypergraphs, simplicial geometry, etc.). Since we (initially) prefer to view them as just a 'collection of sets with little structure', we've chosen to use a neutral name.

The good set systems we will look at mostly are the *stable set systems* $(V_G, \mathcal{S}_G)$ of (finite, simple) graphs $G = (V_G, E_G)$[1]. A *stable set* of a graph G is a set of vertices $S \subseteq V_G$ so that there is no edge $uv \in E_G$ with $u, v \in S$. And then we use $\mathcal{S}_G$ to denote the collection of all stable sets of G.

[1] We refer to the new Bondy & Murty [1] for most definitions and notation regarding graphs.

It is obvious that a stable set system is good. But not all good set systems correspond to the stable set system of some graph, although we will later show that there is an easy characterisation of those good set systems that are stable set systems.

1.1 Covering and Packing

The *covering number* $\mathrm{Cov}(S,\mathcal{F})$ of a good set system is the minimum integer k so that there exist $F_1,\ldots,F_K \in \mathcal{F}$ with $S = \bigcup_{i=1}^{k} F_i$. And the *packing number* $\mathrm{Pack}(S,\mathcal{F})$ is the largest size of a subset $T \subseteq S$ so that for all $F \in \mathcal{F}$ we have $|T \cap F| \leq 1$ (in other words: the largest set $T \subseteq S$ so that no two elements from T appear in one set of $\mathcal{F}$). It is not hard to see that for the stable set system $(V_G, \mathcal{S}_G)$ of a graph G we have that $\mathrm{Cov}(V_G, \mathcal{S}_G)$ is the *chromatic number* $\chi(G)$ of the graph, and $\mathrm{Pack}(V_G, \mathcal{S}_G)$ is the *clique number* $\omega(G)$.

By their definitions, we immediately have for any good set system:

$$\mathrm{Pack}(S,\mathcal{F}) \;\leq\; \mathrm{Cov}(S,\mathcal{F}). \tag{1}$$

We can also define the covering number and the packing number as solutions to two integer programming problems. For a good set system, $\mathrm{Cov}(S,\mathcal{F})$ and $\mathrm{Pack}(S,\mathcal{F})$ are the solutions to, respectively:

$$\begin{aligned} \text{minimise} \quad & \sum_{F \in \mathcal{F}} x_F; \\ \text{subject to} \quad & \sum_{F \ni s} x_F \;\geq\; 1, \quad \forall s \in S; \\ & x_F \;\in\; \mathbb{Z},\; x_F \;\geq\; 0, \quad \forall F \in \mathcal{F}, \end{aligned}$$

and

$$\begin{aligned} \text{maximise} \quad & \sum_{s \in S} y_s; \\ \text{subject to} \quad & \sum_{s \in F} y_s \;\leq\; 1, \quad \forall F \in \mathcal{F}; \\ & y_s \;\in\; \mathbb{Z},\; y_s \;\geq\; 0, \quad \forall s \in S. \end{aligned}$$

By removing the integrality condition we get the fractional versions of these two parameters, obtained as the solutions of:

$$\begin{aligned} \text{minimise} \quad & \sum_{F \in \mathcal{F}} x_F; \\ \text{subject to} \quad & \sum_{F \ni s} x_F \ \geq\ 1, \quad \forall s \in S; \\ & x_F \ \in \mathbb{R},\ x_F \ \geq\ 0, \quad \forall F \in \mathcal{F}, \end{aligned}$$

and

$$\begin{aligned} \text{maximise} \quad & \sum_{s \in S} y_s; \\ \text{subject to} \quad & \sum_{s \in F} y_s \ \leq\ 1, \quad \forall F \in \mathcal{F}; \\ & y_s \ \in \mathbb{R},\ y_s \ \geq\ 0, \quad \forall s \in S. \end{aligned}$$

The solution of the LP problem at the top gives the *fractional covering number* $\mathrm{Cov}_f(S,\mathcal{F})$, while the solution of the LP problem at the bottom is the *fractional packing number* $\mathrm{Pack}_f(S,\mathcal{F})$.

Since the fractional versions are relaxations of the integral versions, we have $\mathrm{Cov}_f(S,\mathcal{F}) \leq \mathrm{Cov}(S,\mathcal{F})$ and $\mathrm{Pack}_f(S,\mathcal{F}) \geq \mathrm{Pack}(S,\mathcal{F})$. Moreover, the two fractional versions are actually dual LP problems, and hence have the same solution. So we have for any good set system

$$\mathrm{Pack}(S,\mathcal{F}) \ \leq\ \mathrm{Pack}_f(S,\mathcal{F}) \ =\ \mathrm{Cov}_f(S,\mathcal{F}) \ \leq\ \mathrm{Cov}(S,\mathcal{F}). \qquad (2)$$

Much more information about fractional combinatorics on set systems can be found in the book of Scheinerman & Ullman [5].

We introduce one more parameter, the *circular covering number* $\mathrm{Cov}_c(S,\mathcal{F})$. Let S_d be the topological cycle with circumference d. When speaking about a (left-closed, right-open) *cyclic interval* $[x,y)$ of S_d, we implicitly assume that we start at x and follow S_d in the clockwise order until reaching y. Given a good set system $(S,\mathcal{F})$, we want to map the elements of S to S_d such that for every cyclic unit interval $[x,x+1)$, the elements mapped into that cyclic interval form a set in $\mathcal{F}$. Then we define $\mathrm{Cov}_c(S,\mathcal{F})$ as the infimum over all values of d for which such a mapping is possible. Since S is finite and every singleton $\{s\}$ is an element of $\mathcal{F}$, it is easy to see that this infimum is actually attained and is in fact a rational number.[2]

[2] We would have the same definition if cyclic intervals were open on both sides or left-open, right-closed. With closed cyclic unit intervals we get the same value for the circular covering number, but it would be a real infimum in that instance.

For the stable set system $(V_G, \mathcal{S}_G)$ of a graph G, $\mathrm{Cov}_c(V_G, \mathcal{S}_G)$ is equal to the *circular chromatic number* $\chi_c(G)$ of the graph. See the survey paper of Zhu [8] for almost everything one could like to know about the circular chromatic number of graphs.

We next show that the circular covering number fits nicely between the fractional and integral covering number.

Proposition 1.1.
For all good set systems $(S, \mathcal{F})$ *we have*

$$\mathrm{Pack}(S,\mathcal{F}) \le \mathrm{Pack}_f(S,\mathcal{F}) = \mathrm{Cov}_f(S,\mathcal{F}) \le \mathrm{Cov}_c(S,\mathcal{F}) \le \mathrm{Cov}(S,\mathcal{F}). \quad (3)$$

Proof. Because of equation (2), we only need to show that $\mathrm{Cov}_c(S,\mathcal{F}) \le \mathrm{Cov}(S,\mathcal{F})$ and $\mathrm{Cov}_f(S,\mathcal{F}) \le \mathrm{Cov}_c(S,\mathcal{F})$. The proofs below are straightforward generalisations from the standard versions for the fractional, circular and integral chromatic number of graphs.

For the first inequality, let $k = \mathrm{Cov}(S,\mathcal{F})$ and let $F_1, \ldots, F_k \in \mathcal{F}$ so that $\cup_i F_i = S$. By removing elements that appear in more than one F_i from all but one of the sets those elements appear in, we find k disjoint sets $F_1', \ldots, F_k' \in \mathcal{F}$ so that$\cup_i F_i' = S$. Now for each $s \in S$, if $s \in F_i'$, then map s to the point i on the cycle S_k. Then every cyclic unit interval contains exactly one of the independent sets F_i', and hence is an allowed mapping. This proves $\mathrm{Cov}_c(S,\mathcal{F}) \le k = \mathrm{Cov}(S,\mathcal{F})$.

Next set $d = \mathrm{Cov}_c(S,\mathcal{F})$. Suppose $|S| = m$ and $\Phi : S \longrightarrow S_d$ is a mapping satisfying the requirements for the circular covering number. By going around the cycle, this mapping gives an ordering $(s_1, \ldots, s_m)$ of S (by taking $(s_1, \ldots, s_m)$ such that $0 \le \Phi(s_1) \le \Phi(s_2) \le \cdots \le \Phi(s_m) < d$). With each element s_i we associate a set F_i from $\mathcal{F}$ by taking the elements that get mapped into $[\,\Phi(s_i), \Phi(s_i)+1\,)$. For $i = 1, \ldots, m$, let x_{F_i} be equal to the difference between $\Phi(s_{i+1})$ and $\Phi(s_i)$ (measuring by going from $\Phi(s_i)$ to $\Phi(s_{i+1})$ in clockwise direction), where we take $s_{m+1} = s_1$. Note that x_{F_i} is zero if $\Phi(s_i) = \Phi(s_{i+1})$. For all other sets F in $\mathcal{F}$ set $x_F = 0$. It is easy to check that for all elements s in S we have $\sum\limits_{F \ni s} x_F \ge 1$, and that $\sum\limits_{F \in \mathcal{F}} x_F = d$, proving that $\mathrm{Cov}_f(S,\mathcal{F}) \le d = \mathrm{Cov}_c(S,\mathcal{F})$. □

1.2 Perfect Set Systems

Equation (3) in Proposition 1.1 actually gives six inequalities. For each of these we can ask for what good set systems we have equality. It is very unlikely that we will be able to give a good answer to those questions in general. One of the stumbling blocks for such an answer is that although the different concepts are global in definition, in many instances they will be determined by just a small part of the set system. So for many set systems we will have equality (say Pack = Cov), but that equality is determined by a small subset (for which the packing and covering number is large and equal), while the rest of the set system can have any structure (as long as it gives a small covering number).

In order to overcome this major obstacle, we may attempt to answer the question for what set systems we have equality "through and through". A natural definition of this idea of "through and through" is the following. If $(S, \mathcal{F})$ is a good set system and T is a non-empty subset of S, then the *induced set system* $(T, \mathcal{F}_T)$ is obtained by taking

$$\mathcal{F}_T \ := \ \{\, F \cap T \mid F \in \mathcal{F} \,\} \ = \ \{\, F \in \mathcal{F} \mid F \subseteq T \,\}.$$

Here the second equality follows as $\mathcal{F}$ is closed under taking subsets.

It again follows that if G is a graph with stable set system $(V_G, \mathcal{S}_G)$, and $U \subseteq V_G$, then the induced set system $(U, (\mathcal{S}_G)_U)$ is the stable set system of the subgraph of G induced on the vertex set U.

For any one of the inequalities $A \leq B$ in (3), we say that a set system $(S, \mathcal{F})$ is $(A{=}B)$-*perfect*, if $(S, \mathcal{F})$ and all its induced set systems give an equality. As an example, a set system $(S, \mathcal{F})$ is $(\mathrm{Pack} = \mathrm{Cov}_f)$-perfect if for all non-empty $T \subseteq S$ we have $\mathrm{Pack}(T, \mathcal{F}_T) = \mathrm{Cov}_f(T, \mathcal{F}_T)$.

Because Proposition 1.1 contains six inequalities, we in fact have six concepts of perfection. In the next section we will show that, perhaps surprisingly, we can completely describe the set systems that are perfect for five of those six types of perfection.

2 Degrees of Perfection

One of the important concepts in graph theory is that of a *perfect graph*: A graph G is *perfect* if $\omega(H) = \chi(H)$ for all induced subgraphs H of G. We have already seen that for a graph $G = (V_G, E_G)$ we have $\omega(G) = \mathrm{Pack}(V_G, \mathcal{S}_G)$ and $\chi(G) = \mathrm{Cov}(V_G, \mathcal{S}_G)$, and for an induced subgraph $H = (V_H, E_H)$ of G we have that the induced set system $(V_H, (\mathcal{S}_G)_{V_H}) = (V_H, \mathcal{S}_H)$. This means that perfect graphs are just those graphs G whose stable set system is $(\mathrm{Pack} = \mathrm{Cov})$-perfect (and hence also perfect *for every other inequality*).

In this section we will show that stable set systems from perfect graphs are the only set systems that are perfect for five of the six inequalities.

Theorem 2.1.
Let $(S, \mathcal{F})$ *be a good system. Then* $(S, \mathcal{F})$ *is* $(\text{Pack} = \text{Cov})$*-perfect,* $(\text{Pack} = \text{Cov}_c)$*-perfect,* $(\text{Pack} = \text{Cov}_f)$*-perfect,* $(\text{Cov}_f = \text{Cov})$*-perfect, or* $(\text{Cov}_c = \text{Cov})$*-perfect, if and only if there is a perfect graph* G *so that* $(S, \mathcal{F}) = (V_G, \mathcal{S}_G)$*.*

The proof of this theorem follows from the following three steps:

1. We first observe that if G is a perfect graph, then $(V_G, \mathcal{S}_G)$ is perfect for all the five types in the theorem.
2. Next we show that if a graph G is not perfect, then $(V_G, \mathcal{S}_G)$ is not perfect for any of the five types.
3. Finally we show that if a set system $(S, \mathcal{F})$ is not the stable set system of some graph, then $(S, \mathcal{F})$ is not perfect for any of the five types.

As mentioned already, step 1 follows directly from the definition of a perfect graph.

Crucial in completing step 2 is the celebrated Strong Perfect Graph Theorem, proved recently by Chudnovsky, Robertson, Seymour and Thomas [2].

Theorem 2.2 (Chudnovsky, *et al.* [2])**.**
A graph G *is not perfect if and only if* G *contains an odd cycle* C_{2k+1}*,* $k \geq 2$*, or the complement of an odd cycle* $\overline{C_{2k+1}}$*,* $k \geq 2$*, as an induced subgraph.*

It is fairly straightforward to check that for an odd cycle C_{2k+1} we have

$$\begin{aligned}
\text{Pack}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) &= \omega(C_{2k+1}) = 2, \\
\text{Cov}_f(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) &= \text{Cov}_c(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 2 + \frac{1}{k}, \\
\text{Cov}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) &= \chi(C_{2k+1}) = 3;
\end{aligned} \tag{4}$$

while the complement of an odd cycle $\overline{C_{2k+1}}$, $k \geq 2$, satisfies

$$\begin{aligned}
\text{Pack}(V_{\overline{C_{2k+1}}}, \mathcal{S}_{\overline{C_{2k+1}}}) &= \omega(\overline{C_{2k+1}}) = k, \\
\text{Cov}_f(V_{\overline{C_{2k+1}}}, \mathcal{S}_{\overline{C_{2k+1}}}) &= \text{Cov}_c(V_{\overline{C_{2k+1}}}, \mathcal{S}_{\overline{C_{2k+1}}}) = k + \frac{1}{2}, \\
\text{Cov}(V_{\overline{C_{2k+1}}}, \mathcal{S}_{\overline{C_{2k+1}}}) &= \chi(\overline{C_{2k+1}}) = k + 1.
\end{aligned} \tag{5}$$

Combining these formulas with Theorem 2.2 immediately gives the proof of step 2.

For step 3 we need to look into the structure of good set systems that do not appear as the stable set system of a graph.

Proposition 2.3.
Let $(S, \mathcal{F})$ be a good set system that can not be represented as the stable set system $(V_G, \mathcal{S}_G)$ for some graph G. Then there exists a subset $T \subseteq S$ with at least 3 elements so that $T \notin \mathcal{F}$, but every proper subset $T' \subsetneq T$ is in $\mathcal{F}$.

Proof. Suppose $(S, \mathcal{F})$ is a good set system that contains no subset $T \subseteq S$ as in the proposition. This means that every minimal (for subset inclusion) subset $T \subseteq S$ with $T \notin \mathcal{F}$ has exactly 2 elements. (Fewer than 2 is impossible, since $\{s\} \in \mathcal{F}$ for every $s \in S$.) Define the graph G with $V_G = S$ and

$$uv \in E_G \quad \Longleftrightarrow \quad u \neq v \text{ and } \{u, v\} \notin \mathcal{F}.$$

We will show that for this graph we have $(V_G, \mathcal{S}_G) = (S, \mathcal{F})$.

Firstly, if $\{u, v\} \subseteq F \in \mathcal{F}$ for some $u, v \in S$ and $F \in \mathcal{F}$, then also $\{u, v\} \in \mathcal{F}$ and hence $uv \notin E_G$. This shows that every set in $\mathcal{F}$ is a stable set in G.

Next, suppose there is some stable set $S \in \mathcal{S}_G$ with $S \notin \mathcal{F}$. Choose S minimal with that property. Since every subset of a stable set is also a stable set, and every minimal set in in $\mathcal{F}$ has cardinality 2, this means S must have 2 elements, say $S = \{u, v\}$. But as S is stable, we must have $uv \notin E_G$, and hence, by definition of G, $S = \{u, v\} \in \mathcal{F}$. This contradiction shows that every stable set in G is a set in $\mathcal{F}$, which completes the proof of the proposition. □

Now we can complete step 3. Let $(S, \mathcal{F})$ be a good set system that can not be represented as the stable set system $(V_G, \mathcal{S}_G)$ for some graph G, and take $T \subseteq S$ as in Proposition 2.3. Hence $|T| = k \geq 3$ and every proper subset $T' \subsetneq T$ is in $\mathcal{F}$. It is then straightforward to show that the induced set system $(T, \mathcal{F}_T)$ satisfies

$$\begin{aligned} \mathrm{Pack}(T, \mathcal{F}_T) &= 1, \\ \mathrm{Cov}_f(T, \mathcal{F}_T) &= \mathrm{Cov}_c(T, \mathcal{F}_T) = 1 + \frac{1}{k-1}, \\ \mathrm{Cov}(T, \mathcal{F}_T) &= 2. \end{aligned} \tag{6}$$

As $k \geq 3$, this immediately gives that $(S, \mathcal{F})$ is not perfect for any of the five types in Theorem 2.1, completing the final step and hence the proof of the theorem. □

2.1 The Sixth Degree of Perfection

Following Theorem 2.1, the only case left are good set systems that are $(\mathrm{Cov}_f = \mathrm{Cov}_c)$-perfect. But here we no longer have just stable set systems of perfect graphs. In equations (4) and (5) we have seen that if G is an odd cycle C_{2k+1} or the complement of an odd cycle $\overline{C_{2k+1}}$, then $\mathrm{Cov}_f(V_G, \mathcal{S}_G) = \mathrm{Cov}_c(V_G, \mathcal{S}_G)$. Moreover, as induced subgraphs of odd cycles or their complements are perfect graphs (odd cycles and their complements are the minimal imperfect graphs), it follows that we also have equality for all induced subgraphs. We must conclude that the stable set systems of odd cycles and their complements are $(\mathrm{Cov}_f = \mathrm{Cov}_c)$-perfect.

A second very large class of $(\mathrm{Cov}_f = \mathrm{Cov}_c)$-perfect set systems appear to be set systems that are loopless matroids. A set system $(S, \mathcal{F})$ is a *loopless matroid* if it satisfies:

(a) $(S, \mathcal{F})$ is good, and

(b) for every $F_1, F_2 \in \mathcal{F}$ with $|F_1| > |F_2|$ there is an $s \in F_1 \setminus F_2$ so that $F_2 \cup \{s\} \in \mathcal{F}$.

For a matroid, the sets in $\mathcal{F}$ are usually called *independent sets*. (The "loopless" follows from the fact that we require all singletons $\{s\}$ to be independent.) See Oxley's book [4] for background knowledge on matroids.

Theorem 2.4 (Van den Heuvel & Thomassé [3])**.**
Let $(S, \mathcal{F})$ be a loopless matroid. Then $\mathrm{Cov}_f(S, \mathcal{F}) = \mathrm{Cov}_c(S, \mathcal{F})$.

Since an induced set system of a loopless matroid is easily seen to be a loopless matroid again, it follows from Theorem 2.4 that in fact loopless matroids are $(\mathrm{Cov}_f = \mathrm{Cov}_c)$-perfect.

A further source of $(\mathrm{Cov}_f = \mathrm{Cov}_c)$-perfect set systems is by combining two of those. If $(S, \mathcal{F})$ and $(T, \mathcal{G})$, $S \cap T = \varnothing$, are good set systems that are both perfect for the same type, then their *union* $(S \cup T, \mathcal{F} \cup \mathcal{G})$ is also perfect of that type. It is not unlikely that other operations can be used to produce new perfect set systems from old ones.

We finally like to point out that loopless matroids and stable set systems of graphs have little in common.

Proposition 2.5.
A set systems $(S, \mathcal{F})$ is both a loopless matroid and a stable set system of a graph if and only if $(S, \mathcal{F}) = (V_G, \mathcal{S}_G)$ for a graph G which is the disjoint union of cliques.

Proof. It is easy to check that if G is the disjoint union of cliques, then $(V_G, \mathcal{S}_G)$ is a loopless matroid.

On the other hand, if G is not the disjoint union of cliques, then there must exist 3 vertices u, v, w with $uv, vw \in E_G$ and $uw \notin E_G$. This means that $\{u, w\}, \{v\} \in \mathcal{S}_G$. But these two stable sets clearly violate condition (b) in the definition of loopless matroid (with $F_1 = \{u, w\}$ and $F_2 = \{v\}$), hence for such a graph G, $(V_G, \mathcal{S}_G)$ is not a loopless matroid. $\square$

3 Discussion

An obvious first open problem that arises from the previous section is the following.

Problem 3.1.
Characterise the good set systems that are $(\mathrm{Cov}_f = \mathrm{Cov}_c)$*-perfect.*

This is probably a very hard problem. The following might be easier.

Problem 3.2.
Characterise the graphs G for which the stable set system $(V_G, \mathcal{S}_G)$ *is* $(\mathrm{Cov}_f = \mathrm{Cov}_c)$*-perfect.*

The two largest classes of set systems we know that are $(\mathrm{Cov}_f = \mathrm{Cov}_c)$-perfect are loopless matroids and stable sets of perfect graphs. Because of Proposition 2.5, these two classes overlap in a small subclass only. So in order to understand $(\mathrm{Cov}_f = \mathrm{Cov}_c)$-perfect better, the following might be an interesting first step in its own right.

Problem 3.3.
Find a "natural" class of good set systems that contains both (loopless) matroids and stable set systems of perfect graphs.

And then, as part 2: show that set systems in that "natural" class are $(\mathrm{Cov}_f = \mathrm{Cov}_c)$*-perfect.*

In [9], Zhu introduced the *circular clique number* $\omega_c(G)$ of a graph G. And the question what graphs are *circular perfect*, i.e., satisfy $\omega_c(H) = \chi_c(H)$ for each induced subgraph H, is raised and studied. Some further relations between perfect and circular perfect graphs can be found in [7].

It is not immediately obvious how to generalise the definition from [9] to give a good definition of the *circular packing number* of a set system. A possible definition is the following. For a good set system $(S, \mathcal{F})$, map the subsets of $\mathcal{F}$ to S_d (the topological cycle with circumference d) such that for every cyclic unit interval $[x, x+1)$, there exists an $s \in S$ so that all subsets of $\mathcal{F}$ containing s are mapped into that cyclic unit interval. And then $\mathrm{Pack}_c(S, \mathcal{F})$ is the supremum over all values of d for which such a mapping is possible.

We'll leave it to reader to check that this means that $\mathrm{Pack}(S, \mathcal{F}) \leq \mathrm{Pack}_c(S, \mathcal{F}) \leq \mathrm{Pack}_f(S, \mathcal{F})$. We now also can introduce four more degrees of perfection. And similar to Theorem 2.1, a set system $(S, \mathcal{F})$ is $(\mathrm{Pack} = \mathrm{Pack}_c)$-perfect or $(\mathrm{Pack}_c = \mathrm{Cov})$-perfect if and only if $(S, \mathcal{F}) = (V_G, \mathcal{S}_G)$ for some perfect graph G. That leaves the additional two problems which set systems are $(\mathrm{Pack}_c = \mathrm{Cov}_f)$-perfect or $(\mathrm{Pack}_c = \mathrm{Cov}_c)$-perfect.

Zhu in [10] gives a large number of observations and results regarding graphs that are circular perfect. Bases on these, it seems that it is very hard to characterise those graphs, which makes us doubt the feasibility to characterise set systems that are $(\mathrm{Pack}_c = \mathrm{Cov}_c)$-perfect.

A final problem we like to raise is the following. Our proof of Theorem 2.1 depends heavily on Theorem 2.2, the Strong Perfect Graph Theorem, to draw conclusions about the stable set systems of imperfect graphs. But in fact, the conclusions that do not involve the circular covering number can be proved without using the Strong Perfect Graph Theorem (see [6, Chapters 64−67] for details and references). It would be interesting to see if the following can also be done without using the Strong Perfect Graph Theorem.

Problem 3.4.
Show that if a graph G is imperfect, then its stable set system $(V_G, \mathcal{S}_G)$ *is not* $(\mathrm{Cov}_f = \mathrm{Cov})$*-perfect (without using the Strong Perfect Graph Theorem).*

In other words, prove, without using the Strong Perfect Graph Theorem, that if a graph is not perfect, then there is an induced subgraph H so that $\chi_c(H) \neq \chi(H)$.

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A Hadwiger Conjecture for Hyperplane Arrangements

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1 Introduction

Hadwiger [7] called a *Streckenkomplex* a $K(k)$ if it has a simplex of order k, but none of order $k+1$ as a minor and conjectured that a $K(k)$ is always k-colorable:

> Verschiedene Feststellungen stützen nämlich die Vermutung, dass die chromatische Zahl eines $K(k)$ nicht grösser als k ausfällt.

In today's notation this is known as the following parametrized conjecture:

Conjecture 1.1 (H(k)[7]). *If a graph is not k-colorable, then it must have a K_{k+1}-minor.*

While H(1) and H(2) are trivial, Hadwiger proved his conjecture for $k = 3$ and pointed out that Klaus Wagner proved that H(4) is equivalent to the Four Color Theorem [22, 1, 15]. Robertson, Seymour and Thomas [16] reduced H(5) to the Four Color Theorem. The conjecture remains open for $k \geq 6$.

The purpose of this note is to point out a generalization of this conjecture to (projective) hyperplane arrangements or more general oriented matroids.

The paper is organized as follows. In the next section we review the famous flow conjectures of Tutte, generalizations to regular matroids and discuss their relation to Hadwiger's conjecture. Then we will sketch a possible generalization to oriented matroids. In the last section we will give a detailed interpretation of $H(3)$ in affine hyperplane arrangements which we pose as an open problem. We assume familiarity with the basics of graph theory, matroid theory and oriented matroids. Standard references are [5, 14, 2].

2 Tutte's flow Conjectures

Let $A \in \{0, +1, -1\}^{r \times E}$ be a totally unimodular matrix representing a regular matroid M of rank r on a finite set E and G an Abelian group. A *G-NZ-flow* in M is a vector $f \in (G \setminus \{0_G\})^E$ such that $Af = 0$. If $G = \mathbb{Z}$ and $0 < |f(e)| < k$ we call f a NZ-k-flow.

First we consider the case that A is the incidence matrix of a directed graph $D = (V, E)$. Tutte [18] pointed out that the Four Color Theorem is equivalent to the statement that every planar graph admits an NZ-4-flow. Generalizing this to arbitrary graphs he conjectured that

Conjecture 2.1 (Tutte's Flow Conjecture [18]). *There is a finite number $k \in \mathbb{N}$ such that every graph admits a NZ-k-flow.*

And moreover that

Conjecture 2.2 (Tutte's Five Flow Conjecture [18]). *Every graph admits a NZ-*5*-flow.*

Note that the latter is best possible as the Petersen graph does not admit a NZ-4-flow. Conjecture 2.1 has been proven independently by Kilpatrick [12] and Jaeger [11] with $k = 8$ and improved to $k = 6$ by Seymour [17].

Conjecture 2.2 has a sibling which is a more direct generalization of the Four Color Theorem.

Conjecture 2.3 (Tutte's Four Flow Conjecture [20, 21]). *Every graph without a Petersen-minor admits a NZ-*4*-flow.*

In [20, 21] Tutte cited Hadwiger's conjecture as a motivating theme and pointed out that while

> "Hadwiger's conjecture asserts that the only irreducible chain-group which is graphic is the coboundary group of the complete 5-graph"

Conjecture 2.3 means that

> "the only irreducible chain-group which is cographic is the cycle group of the Petersen graph."

The first statement refers to the case where the rows of A consist of a basis of signed characteristic vectors of cycles of a digraph.

Combining these we derive the following formulation in terms of regular matroids which can be seen to be equivalent to Conjecture 2.3. First let us call any integer combination of the rows of A a *coflow*. Clearly, by duality resp. orthogonality, flows and coflows yield the same concept in regular matroids. Note that the existence of a NZ-k-coflow in a graph is equivalent to k-colorability [20].

Conjecture 2.4 (Tutte's Four Flow Conjecture, matroid version)**.** *A regular matroid that does not admit a NZ-*4*-flow has a the cographic matroid of the* K_5 *or the graphic matroid of the Petersen graph as a minor or, equivalently, a regular matroid that is not* 4*-colorable, i.e. that does not admit a NZ-*4*-coflow has a* K_5 *or a Petersen-dual as a minor.*

Using the Four Color Theorem Lai, Li and Poon have proven that

Theorem 2.5 ([13])**.** *A regular matroid that is not* 4*-colorable has a* K_5 *or a* K_5*-dual as a minor.*

Tutte's Five Flow Conjecture now suggests the following matroid version of Hadwiger's conjecture:

Conjecture 2.6. *If a regular matroid is not k-colorable for $k \geq 5$, then it must have a* K_{k+1}*-minor.*

It is well known that

Theorem 2.7 ([4, 19])**.** *a) The number of G-NZ-coflows in a regular matroid depends only on the order of the group, not on its structure.*

b) The existence of a G-NZ-coflow is equivalent to the existence of a NZ-$|G|$-coflow.

While in the case of the Four Flow Conjecture the simple structure of the additive group of $GF(4)$ allows to combine 4-flows or 4-coflows along 2-sums and 3-sums and prove that Conjectures 2.3 and 2.4 are equivalent the author has no general argument why the same should be possible for k-flows and $k \geq 5$. While Conjecture 2.6 implies Tutte's Five Flow Conjecture as well as H(k) for $k \geq 5$ it is not clear whether the converse holds.

Problem 2.8. *Is Conjecture 2.6 equivalent to $H(k)$ and Conjecture 2.2 for $k \geq 5$?*

We conclude this section considering the remaining cases of k. Again the cases $k = 1$ and $k = 2$ are trivial.

Theorem 2.9 (H(3) for regular matroids)**.** *If a regular matroid M is not 3-colorable, then it has a K_4-minor.*

Proof. If M has no K_4-minor it is the matroid of some series-parallel network (see [14] Corollary 11.2.15). Hence, the assertion follows from H(3) for graphs. □

3 Oriented Matroids

In the sixties of the last century the term orientable matroid was used for what is now known as a regular matroid. A matroid is *regular*, if there is an orientation of its circuits and cocircuits such that for all circuits C and all cocircuits D

$$|C^+ \cap D^+| + |C^- \cap D^-| = k \iff |C^+ \cap D^-| + |C^- \cap D^+| = k. \tag{1}$$

This has changed with the appearance of oriented matroids [6, 3]. We call a matroid *orientable* if there is an orientation of its circuits and cocircuits such that for all circuits C and all cocircuits D

$$|C^+ \cap D^+| + |C^- \cap D^-| > 0 \iff |C^+ \cap D^-| + |C^- \cap D^+| > 0. \tag{2}$$

In general, this definition will destroy orthogonality between directed circuits and cocircuits and the *chain group* ([19]) generated by the signed characteristic vectors of cocircuits will no longer coincide with the integer kernel of the matrix the rows of which are the signed characteristic vectors of circuits. Even worse, the latter is frequently trivial. Thus, to define flows or coflows we go back to Tutte's original definition of chain groups which we prefer to call *integer lattices*.

Definition 3.1. Let $\mathcal{D}$ denote the set of signed cocircuits of an oriented matroid $\mathcal{O}$ on a finite set E. For each $D \in \mathcal{D}$ we define its *signed characteristic vector* $\vec{\chi}_D \in \{\pm 1, 0\}^E$ as

$$\vec{\chi}_D(e) = \begin{cases} 1 & \text{if} \quad e \in D^+ \\ -1 & \text{if} \quad e \in D^- \\ 0 & \text{if} \quad e \notin D. \end{cases} \tag{3}$$

The *lattice of coflows* $\mathcal{F}^*(\mathcal{O})$ is defined as

$$\mathcal{F}^*(\mathcal{O}) := \left\{ \sum_{D \in \mathcal{D}} \lambda_D \vec{\chi}_D \mid \lambda_D \in \mathbb{Z} \right\}. \tag{4}$$

We say that an oriented matroid $\mathcal{O}$ is *k-colorable*, if there exists a coflow $f^* \in \mathcal{F}^*(\mathcal{O})$ such that

$$\forall e \in E : 0 < |f(e)| < k.$$

The *chromatic number* $\chi(\mathcal{O})$ of an oriented matroid is the smallest k such that $\mathcal{O}$ is k-colorable.

Note that this definition is compatible with the case of regular matroids and graphs. The flow lattice of an oriented matroid has been introduced in [8]. The following theorem and numerical results from [9] suggest that the graphic case should be the worst case for the chromatic number:

Theorem 3.2 ([10]). *If $\mathcal{O}$ is an oriented matroid of rank k then $\chi(\mathcal{O}) \leq k+1$. Furthermore, equality holds if, and only if $\mathcal{O}$ is the oriented matroid of an orientation of K_{k+1}.*

This tempts us to replace the term “regular” in Conjectures 2.4 and 2.6 by “orientable”. Explicitely:

Problem 3.3. H(4) for oriented matroids: *Does an oriented matroid that is not 4-colorable, necessarily have an orientation of K_5 or of the Petersen-dual as a minor?*

H(k) for oriented matroids and $k \geq 5$: *Does an oriented matroid that is not k-colorable for $k \geq 5$, necessarily have an orientation of K_{k+1} as a minor?*

The next section will be devoted to a generalization of $H(3)$.

While two orientations of a regular matroid differ only by reorientation, in general this is not true. E.g. there are orientations $\mathcal{O}_1, \mathcal{O}_2$ of U_3^6, the uniform matroid of rank 3 on 6 elements, such that $\dim \mathcal{F}(\mathcal{O}_1) = 6$ while $\dim \mathcal{F}(\mathcal{O}_2) = 5$. Nevertheless, we do not know a matroid where the chromatic numbers differ for different orientations.

Very little is known about algebraic coflows in oriented matroids. The example $\mathcal{O}_1$ just mentioned together with $\mathbb{Z}_4$ and the additive group of $GF(4)$ form a counterexample to a generalization of Theorem 2.7 a) , while we do not have an immediate counterexample to an oriented matroid version of Theorem 2.7 b) .

4 H(3) for hyperplane arrangements

By the topological representation theorem for oriented matroids ([6], see also [2] 5.2.1) every oriented matroid can be represented as an arrangement of pseudohyperspheres on the sphere. To simplify the discussion we will consider only the linear case, i.e. hyperplane arrangements, here.

Thus, let $\mathcal{H} = (H_e)_{e \in E}$ be an arrangement of affine hyperplanes in $\mathbb{R}^d$. A *plane* of $\mathcal{H}$ is any non empty intersection of elements of $\mathcal{H}$. Planes of dimension zero or one are called *vertices* resp. *lines*. In order to avoid working in projective spaces we call an arrangement *proper* if no two lines of $\mathcal{H}$ are parallel. If $S, T \subseteq E$ are two disjoint subsets of E such that $P = \bigcap_{e \in S} H_e$ is a plane which is contained in no hyperplane H_t with $t \in T$ then the hyperplane arrangement defined on P by $(P \cap H_t)_{t \in T}$ is called a *minor* of $\mathcal{H}$.

The arrangement of the K_4 is derived from the hyperplanes defined by $x_i = 0$ and $x_i - x_j = 0$ for $1 \leq i < j \leq 3$ and $x_i = 0$ in $\mathbb{R}^4$ by dehomogenization and is depicted in Figure 1.

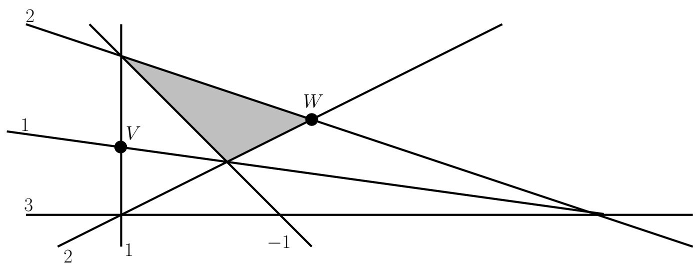

Figure 1: The arrangement of the K_4

Let $\mathcal{V}$ denote the set of vertices of a proper arrangement $\mathcal{H}$. A *coflow in* $\mathcal{H}$ consists of a maximal cell X and a map $z : \mathcal{V} \to \mathbb{Z}$. We say that a vertex $V \in \mathcal{V}$ is *positive* with respect to a hyperplane H_e, if $V \notin H_e$ and V lies on the same side of H_e as X, and *negative* if $V \notin H_e$ and V and X are on different sides of H_e. The *value* $f_z(e)$ *of a hyperplane* H_e *in a coflow* is defined as

$$f_z(e) = \sum_{V \text{ is positive wrt.} H_e} z(V) - \sum_{V \text{ is negative wrt.} H_e} z(V). \tag{5}$$

A NZ-k-coflow is a coflow such that $0 < |f_z(e)| < k$. The choice of X

corresponds to the choice of an acyclic orientation in a graph.

If we choose the shaded region in Figure 1 as X, set $z(V) = 2, z(W) = 1$ and $z(U) = 0$ for all other vertices U, we yield the indicated NZ-4-coflow. This is best possible, since $\chi(K_4) = 4$. Theorem 3.2 asserts that all (pseudo)-line arrangements, which are not projectively equivalent to Figure 1 admit a NZ-3-coflow.

Hadwiger's Theorem H(3) then becomes

Conjecture 4.1. *If an arrangement does not admit a NZ-3-coflow it must have a configuration projectively equivalent to Figure 1 as a minor.*

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Fractional total colourings of graphs of high girth

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Joint work with Tomáš Kaiser [1] [2] and Daniel Kráľ [3] [4].

Abstract

We prove that the fractional total chromatic number of any subcubic graph with girth at least 23000 equals 4. Furthermore, we prove that for any even positive integer Δ there exists a constant $g(\Delta)$ such that the fractional total chromatic number of any graph with maximum degree Δ and girth at least $g(\Delta)$ equals $\Delta + 1$. This yields a proof of a recent conjecture of Reed for $\Delta = 3$ and even $\Delta > 3$.

1 Introduction

Any graph G has a *line graph* $L(G)$ with vertex set $E(G)$; two vertices of $L(G)$ are adjacent precisely if their corresponding edges in G share an endpoint. Line graphs allow us to reformulate questions about the edges of G in terms of the vertices of $L(G)$, and they have proven to be fundamental to the study of both perfect graphs and claw-free graphs. The *chromatic index* $\chi'(G)$ and *fractional chromatic index* $\chi'_f(G)$ are the chromatic number and fractional chromatic number of $L(G)$, respectively.

Total graphs are a natural generalization of line graphs, taking into account both vertices and edges. For a graph G, the *total graph* $T(G)$ has vertex set $V(G) \cup E(G)$; two vertices of $T(G)$ are adjacent precisely if they correspond to two adjacent vertices, two edges sharing an endpoint, or an edge and one of its endpoints. The *total chromatic number* $\chi''(G)$

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[4]ITI is supported by the Ministry of Education of the Czech Republic as project 1M0545.

and *fractional total chromatic number* $\chi''_f(G)$ are the chromatic number and fractional total chromatic number of $T(G)$, respectively.

Thus the total graph combines the concepts of graphs and line graphs. We are interested in discovering situations in which fundamental properties of line graphs extend to total graphs. In particular, we investigate bounds on the fractional total chromatic number.

Trivially, $\Delta(G)+1 \leq \chi''(G) \leq 2\Delta(G)$. Behzad and Vizing independently conjectured that for any graph, $\chi''(G) \leq \Delta(G)+2$ [1, 6]. Kilakos and Reed proved that this holds for the fractional total chromatic number, i.e. that $\chi''_f(G) \leq \Delta(G) + 2$ [5]. This inequality is sharp unless G is K_{2n} or $K_{n,n}$ for some positive n [3]. Thus the Behzad-Vizing Conjecture is analogous to the Goldberg-Seymour Conjecture, which proposes that $\chi'(G) \leq \chi'_f(G)+1$. At the recent DIMACS Workshop on Graph Colouring and Structure, Reed conjectured that just as $\chi_f(G)$ approaches its trivial lower bound $\Delta(G)$ as the girth of G increases [4], so does $\chi''_f(G)$ approach $\Delta(G) + 1$:

Conjecture 1.1 (Reed)**.** *For any Δ and $\epsilon > 0$, there exists a constant $g(\Delta, \epsilon)$ such that a graph with maximum degree Δ and girth at least $g(\Delta, \epsilon)$ has fractional total chromatic number at most $\Delta + 1 + \epsilon$.*

This conjecture is easy to prove for $\Delta = 2$. We prove a stronger version when $\Delta = 3$ or Δ is even and greater than 3:

Theorem 1.2. *For any $\Delta = 3$ or even $\Delta > 3$, there is a constant $g(\Delta)$ such that a graph with maximum degree Δ and girth at least $g(\Delta)$ has fractional total chromatic number exactly $\Delta + 1$.*

We first prove this theorem for cubic bridgeless graphs, then extend it to subcubic graphs and finally graphs with even maximum degree greater than 3.

2 Cubic bridgeless graphs: An overview

We now briefly sketch our proof that for a cubic graph G of sufficiently high girth, $\chi''_f(G) = 4$. The first important observation to make is that the total neighbourhood (i.e. the neighbourhood in $T(G)$) of an edge of G consists of two vertices and four edges, which are covered by two cliques in $T(G)$. It follows that in a fractional total 4-colouring of G, every colour class T is a *full* total stable set, i.e. a stable set of $T(G)$ for which any vertex of G is either in T or an endpoint of an edge in T. Our approach is guided by the following observation:

Lemma 2.1. *Let G be a graph. The following are equivalent:*

1. $\chi_f(G) \leq k$,

2. *there exists an integer N and a multiset $\mathcal{W}$ of $k \cdot N$ independent sets in G, such that each vertex is contained in exactly N sets of $\mathcal{W}$,*

3. *there exists a probability distribution π on independent sets of G such that for each vertex v, the probability that v is contained in a random independent set (with respect to π) is at least $1/k$.*

It is well-known that a cubic bridgeless graph G has $\chi'_f(G) = 3$, i.e. a probability distribution on perfect matchings in G for which each edge of G is included with probability $\frac{1}{3}$. Consequently, we need only find a probability distribution on full total stable sets of G in which every edge is included with probability $a \geq \frac{1}{4}$ and every vertex is included with probability $(1-a)/3$.

Our approach is to give a randomized method for choosing a full total stable set that satisfies our requirements. To do this, we first decompose the graph randomly. The role of each vertex (resp. edge) depends on this decomposition, so we need to do this in such a way that each vertex (resp. edge) plays a given role with equal probability.

The first step of our decomposition is to choose a 2-factor F of G in which each cycle is oriented randomly. We do this by choosing the complement of a colour class in a fractional 3-edge colouring of G. The second step is to partition each cycle into induced paths of length between l and $9l$ for some sufficiently large constant value of l. We do this by partitioning the edges of F into sets of *boundary edges*, which we then remove. Our boundary edges must be sufficiently distant in $T(G)$ so we can avoid interference when resolving conflicts later in the process. To choose the partition, we appeal to Haxell's bound on the strong chromatic number of a graph [2]. The third and final decomposition step is to assign each resulting path to a *level* between 1 and k uniformly at random, for some sufficiently large constant k.

We assume that the girth of G is at least $g_3 = 19lk$. This allows us to ensure that each cycle of F can be decomposed into long *induced* paths, and that our choice of k levels (i) is large enough to guarantee that $a > 1/4$, and (ii) is small compared to the girth, guaranteeing independence between certain random events later in our method.

Given F and a boundary edge set B, we first randomly generate a set $T(F, B) \subset V(G) \cup E(G)$ which is a total stable set except near the points of B. For each level in turn, we consider the (directed) paths one at a time. Using carefully chosen "seed probabilities" that depend on the level of the path, we decide whether or not an imaginary edge and vertex at the beginning of the path are in $T(F, B)$. We then move along the path, placing each edge or vertex in the path whenever it does not create a conflict with a path of equal or lesser level. The next step is to resolve conflicts in $T(F, B)$, turning it into a full total stable set $\tilde{T}(F, B)$. The uniformly random choice of levels and the partition of edges in F into boundary sets allow us to prove that this method generates a full total stable set meeting our requirements, giving us the desired result for cubic bridgeless graphs.

3 Extending the result

Having proved the result for cubic bridgeless graphs, we extend it to all subcubic graphs. The only troublesome case is when G has all vertices of degree three except for one vertex of degree two. In this case we apply a modification of our previous method to auxiliary cubic graphs, then use an appropriate combined reweighting of the resulting total stable sets of G.

To extend this result to a graph G of even maximum degree $\Delta > 3$, we first embed G in a Δ-regular graph of large girth. We then proceed with a straightforward modification of our method for cubic bridgeless graphs, decomposing G into a set of 2-factors and proceeding from there. The analysis is more complicated, but the basic idea is the same.

The question of whether or not a Δ-regular graph of sufficiently high girth has $\chi''_f = \Delta + 1$ remains open for odd Δ greater than three.

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Lines in Graphs

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Joint work with Ondřej Bílka, Jozef Jirásek and Jan Volec.

Abstract

We consider lines in discrete metric spaces, particularly in spaces induced by graphs. A line is a subset of points satisfying the triangle equality. We describe some properties of these lines and also structural properties of the whole system of all the lines. We characterize graphs with exactly k different lines. We consider graphs with "all the lines different" (all pairs of points define pairwise different lines). These graphs are a parallel to points in general position and we present an open problem to find a better characterization for them. We also describe an open problem of reconstruction of a graph from its line system. We show a polynomial algorithm for reconstruction of trees, which implies that all trees are line reconstructible.

1 Introduction

We present properties of line systems in discrete metric spaces, particularly in spaces induced by graphs. Every connected graph $G = (V, E)$ induces a natural metric space (V, d) where $d(u, v)$ is the distance of vertices u and v (the number of edges on the shortest path between u and v). As in Chvátal [1], a line $\overleftrightarrow{ab}$ is a subset of V defined by two distinct vertices a and b:

$$\begin{aligned}\overleftrightarrow{ab} = \{x \mid d(a,x) + d(x,b) = d(a,b)\} \cup \\ \cup \{y \mid d(y,a) + d(a,b) = d(y,b)\} \cup \\ \cup \{z \mid d(z,b) + d(b,a) = d(z,a)\}\end{aligned}$$

We study properties of lines and their connections with the structure of the graph. For a given graph G, we denote the system of all its lines by $\mathcal{L}(G)$. Formally:

$$\mathcal{L}(G) = \{\overleftrightarrow{ab} | a, b \in V(G), a \neq b\}.$$

2 k-linear graphs

For a given graph G, how many different lines does it contain? Formally, what is the size of $\mathcal{L}(G)$? We call a graph G k-linear if $|\mathcal{L}(G)| = k$. For a given k, how large is the class of all k-linear graphs? For which values of k is this class infinite?

Observation 2.1 (Bridge pumping)**.** *Subdividing of a bridge does not change neither the structure nor the number of different lines of the graph.*

For a given k-linear graph with a bridge, we can easily construct an infinite class of all k-linear graphs by subdividing the bridge over and over. The following theorem says that this is the only way to get such an infinite class.

Theorem 2.2 (Infinite class and bridges)**.** *For any k, the class of all the k-linear graphs without a subdivided bridge is finite.*

Theorem 2.3. *There exists an infinite class of k-linear graphs if and only if $k \in \mathbb{N} \setminus \{2, 3, 5, 6\}$.*

3 Graphs with $\binom{|V|}{2}$ different lines

We present an open problem of characterizing the class of graphs, such that every pair of vertices defines a different line. Such graphs are a parallel to sets of points in general position. We present a list of some examples:

- Complete graphs K_n.
- Odd cycles C_{2n+1}.
- Complements of cycles C_n, for $n \notin \{3, 6\}$.
- Wheels W_n, for $n \neq 4$.
- The Petersen graph and other Moore graphs.

4 Line reconstruction and its complexity

For a given line system $\mathcal{L}(G)$, we would like to construct a graph G' with a line system $\mathcal{L}(G') \cong \mathcal{L}(G)$. Unfortunately, $\mathcal{L}(G)$ is not unique for all graphs, for example $\mathcal{L}(C_4) \cong \mathcal{L}(P_3)$. What is the complexity of such a reconstruction? Which graph classes are line reconstructible (all the graphs of such a class have pairwise non-isomorphic line systems)? We present one partial answer to both of these problems, regarding trees.

Theorem 4.1. *The reconstruction problem is polynomially solvable for trees.*

Corollary 4.2. *There are no two non-isomorphic trees with isomorphic line systems.*

Acknowledgments

We would like to thank Jan Kratochvíl and Pavel Valtr for introducing us to the idea of lines in metric spaces and for spending their time in discussions.

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Numbers of sum-closed permutations

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The number of mappings $f : [n] \to [n]$ (here $[n] = \{1, 2, \ldots, n\}$) is n^n. The requirement that f is injective makes f a permutation and brings the number down to $n!$—a decrease by the exponential factor $(1/e)^n$. Further condition that (besides injectivity) $f(1) < f(2) > f(3) < f(4) > f(5) < \ldots$ turns f into an alternating permutations and the number decreases further by the exponential factor $(2/\pi)^n$. Now if f is not only an alternating permutation of $[n]$ but even

$$f(2) = f(1) + f(3), f(4) = f(3) + f(5), f(6) = f(5) + f(7), \ldots$$

—we call such f sum-closed permutation—how many such f are there? Are there superexponentially many of them? Is there again an exponential decrease, that is, if s_n is the number of sum-closed permutations of $[n]$, is it true that $s_n < n^n(2/\pi e)^n c^n$ for some $c < 1$ (and $n > n_0$)? What is the exact asymptotics of the numbers s_n? How difficult is to compute them?

Complexity of covers by planar graphs

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We say that a graph G *covers* a graph H if there exists a vertex mapping $f : V(G) \longrightarrow V(H)$ (called a *covering projection*) such that for every $u \in V(G)$, f induces a bijection between $N_G(u)$ and $N_H(f(u))$. Bodlaeander [3] showed that deciding if G covers H is NP-complete if both G and H are part of the input. Abello et al. [1] asked for a characterization of the H-Cover problem for fixed target graphs H (i.e., only the source graph G is part of the input). Several partial results have been obtained by Kratochvíl et al. [5, 7] but the general characterization is still open. P/NP-co dichotomy is conjectured but also not proved.

The smallest graph for which H-Cover is NP-complete is the complete graph K_4 [4]. The question can be asked for multigraphs [6] and the smallest multigraph for which H-Cover is NP-complete is the weight graph $W(1, 1, 1)$ consisting of two vertices of degree 3 connected by a single edge and equipped with one loop each.

On several occasions I have asked what is the complexity of K_4-Cover and $W(1, 1, 1)$-Cover when restricted to planar inputs (among others during my talk at the "Nesetril 60th" conference in Prague in 2006). This summer both these problems have been showed NP-complete by our REU students Bílka, Jirásek, Klavík, Tancer, and Volec [2]. This immediately raises the following question:

Problem 1.1. Is H-Cover restricted to planar inputs NP-complete for every graph H such that i) H-Cover is NP-complete, and ii) H has a finite planar cover?

Note that a hardness proof would probably have some flavor of nonconstructiveness, since currently it is not known which graphs have finite planar covers (the famous conjecture of Negami [8] that these are precisely the projective planar graphs is still open).

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A factorization theorem for classical group characters, with applications to plane partitions and rhombus tilings

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Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ (i.e., a non-increasing sequence of non-negative integers) the *Schur function* $s_\lambda(x_1, x_2, \ldots, x_N)$ is defined by (see [2, p. 403, (A.4)], [3, Prop. 1.4.4], or [4, Ch. I, (3.1)])

$$s_\lambda(x_1, x_2, \ldots, x_N) = \frac{\det\limits_{1\le h,t\le N}\left(x_h^{\lambda_t+N-t}\right)}{\det\limits_{1\le h,t\le N}\left(x_h^{N-t}\right)}.$$

It is well-known (cf. [2, §24.2]) that $s_\lambda(x_1, x_2, \ldots, x_N)$ is an irreducible character of $SL_N(\mathbb{C})$ (respectively $GL_N(\mathbb{C})$). On the other hand, Schur functions are also known to carry rich combinatorial structure, which comes from the fact that they can alternatively be seen as certain generating functions for semistandard tableaux. We refer the reader to [3, 4, 5] for more information on symmetric functions and Schur functions in particular.

The purpose of this note is to present curious factorization properties for Schur functions of rectangular shape $(M^n) = (M, M, \ldots, M)$ (with n repetitions of M), which seem to have escaped the attention of previous authors. For a full account of this work, see [1].

Here are the corresponding results.

[1]Partially supported by the Austrian Science Foundation FWF, grants Z130-N13 and S9607-N13, the latter in the framework of the National Research Network "Analytic Combinatorics and Probabilistic Number Theory." This work was done during the authors' stay at the Erwin Schrödinger Institute for Physics and Mathematics, Vienna, during the programme "Combinatorics and Statistical Physics" in Spring 2008.

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[4]Partially supported by NSF grant DMS-0500616.

Theorem 1.1. *For any non-negative integers m and n, we have*

$$s_{((2m)^n)}(x_1,x_1^{-1},x_2,x_2^{-1},\dots,x_n,x_n^{-1}) \\ =(-1)^{mn}\,so_{(m^n)}(x_1,x_2,\dots,x_n)\,so_{(m^n)}(-x_1,-x_2,\dots,-x_n), \quad (1)$$

where, given a non-increasing sequence $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ of integers or half-integers[5]*, the* odd orthogonal character $so_\lambda(x_1,x_2,\dots,x_N)$ *is defined by (see [2, (24.28)])*

$$so_\lambda(x_1,x_2,\dots,x_N)=\frac{\det\limits_{1\le h,t\le N}\left(x_h^{\lambda_t+N-t+\frac12}-x_h^{-(\lambda_t+N-t+\frac12)}\right)}{\det\limits_{1\le h,t\le N}\left(x_h^{N-t+\frac12}-x_h^{-(N-t+\frac12)}\right)}.$$

Theorem 1.2. *For any non-negative integers m and n, we have*

$$s_{((2m+1)^n)}(x_1,x_1^{-1},x_2,x_2^{-1},\dots,x_n,x_n^{-1})= \\ sp_{(m^n)}(x_1,x_2,\dots,x_n)\,o^{even}_{((m+1)^n)}(x_1,x_2,\dots,x_n), \quad (2)$$

where, given a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$, the symplectic character $sp_\lambda(x_1,x_2,\dots,x_N)$ *is defined by (see [2, (24.18)])*

$$sp_\lambda(x_1,x_2,\dots,x_N)=\frac{\det\limits_{1\le h,t\le N}\left(x_h^{\lambda_t+N-t+1}-x_h^{-(\lambda_t+N-t+1)}\right)}{\det\limits_{1\le h,t\le N}\left(x_h^{N-t+1}-x_h^{-(N-t+1)}\right)},$$

and, given a non-increasing sequence $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_N)$ of positive integers or half-integers, the even orthogonal character $o^{even}_\lambda(x_1,x_2,\dots,x_N)$ is given by (see [2, (24.40) plus the remarks on top of page 411])

$$o^{even}_\lambda(x_1,x_2,\dots,x_N)=2\frac{\det\limits_{1\le h,t\le N}\left(x_h^{\lambda_t+N-t}+x_h^{-(\lambda_t+N-t)}\right)}{\det\limits_{1\le h,t\le N}\left(x_h^{N-t}+x_h^{-(N-t)}\right)}.$$

Odd and even orthogonal, and symplectic characters are as well irreducible characters for corresponding classical Lie groups. However, it seems difficult to find explanations of the identities in Theorems 1 and 2 within the framework of representation theory. (See [1] for more comments on this issue.)

[5]the latter being, by definition, positive odd integers divided by 2

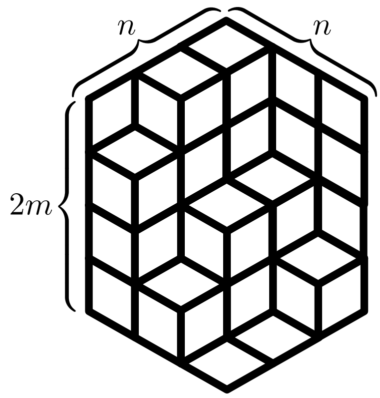

Figure 1: A rhombus tiling of a hexagon

The above identities are proved in [1] by reducing them to a certain polynomial identity, which, once figured out, can then be established by an inductive argument. Two more identities of a similar kind are also presented and proved in [1].

As we show in [1], there are attractive combinatorial interpretations of (1) and (2) in the special case $x_1 = x_2 = \cdots = x_n = 1$. For example, for this specialization, (1) becomes

$$PP(2m, n, n) = SPP(2m, n, n) \cdot TCPP(2m, n, n), \tag{3}$$

where $PP(2m, n, n)$ denotes the number of plane partitions contained in the $(2m)\times n\times n$ box (or, equivalently, the number of *rhombus tilings* of a hexagon with side lengths $2m, n, n, 2m, n, n$; see Figure 1 for an example in which $m = 2$ and $n = 3$), $SPP(2m, n, n)$ denotes the number of *symmetric plane partitions* contained in the $(2m) \times n \times n$ box (or, equivalently, the number of rhombus tilings of a hexagon with side lengths $2m, n, n, 2m, n, n$ which are symmetric with respect to the vertical symmetry axis of the hexagon), and $TCPP(2m, n, n)$ denotes the number of *transpose complementary plane partitions* contained in the $(2m) \times n \times n$ box (or, equivalently, the number of rhombus tilings of a hexagon with side lengths $2m, n, n, 2m, n, n$ which are symmetric with respect to the horizontal symmetry axis of the hexagon).

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Perfect matchings in cubic graphs

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Joint work with Louis Esperet[1] and František Kardoš[2].

By a classical theorem of Petersen [3], every cubic bridgeless graph has a perfect matching. In fact, every edge of a cubic bridgeless graph is contained in a perfect matching, and thus every n-vertex cubic bridgeless graph has at least three perfect matchings. Lovász and Plummer [2] conjectured in the mid 1980's that the number of perfect matchings of an n-vertex cubic bridgeless graph is exponentially in n. The conjecture has been verified for several special classes of graphs, most importantly bipartite [4] and planar [1] graphs. Though, until a year ago, the only known lower bound on the number of perfect matchings of a general cubic bridgeless graph was an estimate of $n/4+2$ given by the dimension of the perfect matching polytope. In this talk, we will present the first superlinear bound for the problem.

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A question about the q-chromatic function

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Let $G = (V, E)$ be a graph and n a positive integer. Let $V = \{1, \ldots, n\}$ and for $k \in \mathbf{P} = \{1, 2, \ldots\}$ let $V(G, k)$ denote the set of all vertex colourings $s : V \to \{0, \ldots, k-1\}$ such that $s(u) \neq s(v)$ whenever $uv \in E$. The q-chromatic function is defined by

$$M_q(G, k) = \sum_{s \in V(G,k)} q^{\sum_{v \in V} s(v)}.$$

Note that $M_q(G, k)|_{q=1}$ is the classical chromatic polynomial of G.

We recall some notation:

For $k \in \mathbf{P}$, let $(k)_q = q^{k-1} + \cdots + q + 1$ denote a *quantum integer*, with the convention that $(0)_q = 0$, and let $(k)!_q = \prod_{1 \leq n \leq k} (n)_q$, with the convention that $(0)!_q = 1$. For $0 \leq n \leq k$ the *quantum binomial coefficients* are defined by

$$\binom{k}{n}_q = \frac{(k)!_q}{(n)!_q (k-n)!_q}.$$

A simple quantum binomial formula

$$(a - z)(a - qz) \ldots (a - q^{k-1} z) = \sum_{i=0}^{k} (-1)^i \binom{k}{i}_q q^{i(i-1)/2} a^{k-i} z^i$$

leads to a well-known formula for the summation of the products of distinct powers. This gives the q-chromatic function for the complete graph:

For $k \in \mathbf{P}$, the q-chromatic function of the complete graph on $n \leq k$ vertices is given by

$$M_q(K_n, k) = n! \binom{k}{n}_q q^{n(n-1)/2}$$

and $M_q(K_n, k) = 0$ for $n > k$.

In my talk I ask to find a formula for the q-chromatic function of a complete bipartite graph.

The Pfaffian-Tree Theorem

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The classical Matrix-Tree Theorem allows one to list the spanning trees of a graph by monomials in the expansion of the determinant of a certain matrix. In 2001, A. Vaintrob and I discovered a similar result for 3-uniform hypergraphs, where the determinant is replaced by a Pfaffian, and the monomials come with signs.

Here is the result in its simplest form. Let $m \geq 1$ be an odd integer and introduce indeterminates y_{ijk} $(1 \leq i, j, k \leq m)$ which are totally antisymmetric in the indices i, j, k:

$$y_{ijk} = -y_{jik} = y_{jki} \quad \text{and} \quad y_{iij} = 0\ . \tag{1}$$

Consider the $m \times m$ matrix

$$\Lambda = (\lambda_{ij}),\ 1 \leq i, j \leq m, \quad \text{with} \quad \lambda_{ij} = \sum_k y_{ijk}. \tag{2}$$

Let $\Lambda^{(p)}$ denote the result of removing the pth row and column from Λ. It turns out that $(-1)^{p-1}\,\mathrm{Pf}(\Lambda^{(p)})$ does not depend on p, which allows us to define the polynomial

$$\mathcal{P}_m := (-1)^{p-1}\,\mathrm{Pf}(\Lambda^{(p)}) \tag{3}$$

in variables y_{ijk}.

Pfaffian-Tree Theorem. (Masbaum-Vaintrob [5]) *The polynomial $\mathcal{P}_m$ counts spanning hypertrees with signs: one has*

$$\mathcal{P}_m = \sum_T \ (\pm 1) \prod_{\{i,j,k\} \in E(T)} y_{ijk} \tag{4}$$

where the sum is over the spanning hypertrees of the complete 3-uniform hypergraph Γ_m on m vertices.

Here, Γ_m has vertex set $\{1, 2, \ldots, m\}$ and one hyperedge $\{i, j, k\}$ for every unordered triple of three distinct vertices i, j, k.

Pfaffian-Tree Theorem (Cont'd). *There is an explicit (and interesting) formula for the signs* ± 1 *appearing in (4). See [5] for details.*

For example, if $m = 3$, we have

$$\Lambda = \begin{pmatrix} 0 & y_{123} & y_{132} \\ y_{213} & 0 & y_{231} \\ y_{312} & y_{321} & 0 \end{pmatrix} = \begin{pmatrix} 0 & y_{123} & -y_{123} \\ -y_{123} & 0 & y_{123} \\ y_{123} & -y_{123} & 0 \end{pmatrix}$$

and

$$\mathcal{P}_3 = \operatorname{Pf} \Lambda^{(3)} = y_{123} \ .$$

This corresponds to the fact that the unique hyperedge $\{1, 2, 3\}$ of Γ_3 is a spanning hypertree.

If $m = 5$, we have

$$\mathcal{P}_5 = y_{123}\, y_{145} \ \pm \ \dots \ , \tag{5}$$

where the right-hand side is a sum of 15 terms corresponding to the 15 spanning hypertrees on Γ_5. Recall that hyperedges of a 3-uniform hypergraph have 3 vertices. We visualize them as Y-shaped objects

with the three vertices at their endpoints. For example, the spanning hypertree corresponding to the first term of (5) looks likes on Figure 1.

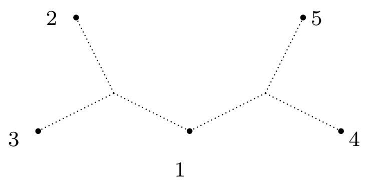

Figure 1: A spanning hypertree in the complete 3-uniform hypergraph Γ_5. It has two hyperedges, $\{1, 2, 3\}$ and $\{1, 4, 5\}$, and contributes the term $y_{123}\, y_{145}$ to $\mathcal{P}_5$.

Remarks. 1) The Pfaffian-Tree theorem can, of course, be stated not just for Γ_m but for arbitrary 3-uniform hypergraphs.

2) A. Vaintrob and I found the Pfaffian-Tree theorem by studying a problem in knot theory. The variables y_{ijk} associated with the hyperedges correspond to Milnor's triple linking numbers. The reason we draw them

as Y-shaped objects rather than as the more customary triangles is that we think of them as Feynman diagrams. A brief discussion of all this and of the knot-theoretical context that lead us to the Pfaffian-Tree formula can be found in Section 2 of [5] (see also [6]).

3) Our original proof of the Pfaffian-Tree theorem in [5] was by induction on the number of hyperedges, using a deletion-contraction formula. We also gave another proof, together with generalizations to Milnor numbers of higher order, in [6]. Abdesselam [1] found a proof using Grassmann variables; he also generalized the formula (in a different direction from ours). Finally, yet another proof of the Pfaffian-Tree theorem was given by Hirschman and Reiner [3], using a sign-reversing involution.

An application to computational complexity theory. As is well-known, the classical matrix-tree theorem (expressing the spanning tree generating function of a graph as a determinant) can be used to count the number of spanning trees on a graph in polynomial time. In contrast, because of the signs appearing in (4), we cannot use the Pfaffian-tree formula to count the number of spanning hypertrees on a 3-uniform hypergraph. Of course, this is not surprising, as counting spanning hypertrees on a 3-uniform hypergraph is known to be a ♯P-complete problem. On the other hand, we can use the Pfaffian-tree formula to give a randomized polynomial-time algorithm to decide the *existence* of a spanning hypertree on a 3-uniform hypergraph [2]. Note that, contrary to the case of ordinary graphs, where a spanning tree exists if and only if the graph is connected, deciding the existence of a spanning hypertree on a given 3-uniform hypergraph is a non-trivial problem. We have learned recently that there is also a polynomial time algorithm for this decision problem due to Lovász [4]. Details of our RP-algorithm and further references for the spanning hypertree problem are given in our joint preprint with S. Caracciolo, A. Sokal, and A. Sportiello [2].

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Vectors in a Box

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Joint work with Kevin Buchin [1] [2], Robin A. Moser [3] [4] and Dömötör Pálvölgyi [5].

Let $d \geq 1$ and let us consider the unit cube $[-1, 1]^d$; we will call it *the box* in this paper. We want to construct a large number t of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t$, each of them lying in the box, such that

(i) the sum $\mathbf{s} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_t$ also lies in the box, but

(ii) for every *proper* subset $S \subset [t]$ of indices[6] with $2 \leq |S| < t$, the sum $\sum_{i \in S} \mathbf{v}_i$ lies outside the box (we have to exclude $|S| = 1$, since every $\mathbf{v}_i$ itself does lie in the box).

So we are interested in long *minimal* sequences[7] with sum in the box. Let $\tau(d)$ denote the largest t such that a minimal sequence as above exists.

In order to illustrate this definition, let us check that $\tau(2) = 2$. We have $\tau(d) \geq 2$ for all d by definition. For proving $\tau(d) \leq 2$, we need to show that in every sequence $\mathbf{v}_1, \ldots, \mathbf{v}_t \in [-1, 1]^2$ with sum in the box there are two vectors with sum in the box.

If two of the vectors lie in opposite quadrants, then their sum is in the box and we are done. Otherwise, some two neighboring quadrants have

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[6]We use the notation $[t] = \{1, 2, \ldots, t\}$.

[7]Strictly speaking, the order of the vectors is irrelevant for the considered property, and so one should perhaps rather speak of sets or multisets of vectors. However, we find sequences easier to work with for notational reasons.

to be empty, which means that one of the two coordinates has the same sign for all the $\mathbf{v}_i$; w.l.o.g. we may assume that all the $\mathbf{v}_i$ have a positive y-coordinate. Then the y-coordinate can be ignored (since it lies in $[-1,1]$ for the sum of any subsequence), and it suffices to show that the sum of some two of the x-coordinates lies in $[-1,1]$. In other words, it now suffices to check that $\tau(1)=2$, which we leave to the reader.

An example showing $\tau(3) \geq 4$ is the sequence $(1,1,\frac{2}{3})$, $(1,-\frac{2}{3},-1)$, $(-\frac{2}{3},1,-1)$, $(-\frac{2}{3},-\frac{2}{3},\frac{2}{3})$.

The quantity $\tau(d)$ was introduced by Dash, Fukasawa, and Günlük [3] in the context of integer programming (we will discuss the motivation later). They found the values $\tau(2)=2$ and $\tau(3)=4$, and they asked whether $\tau(d)$ is finite for all d. We provide a positive answer, with the following upper bound:

Theorem 1.1. *For all $d \geq 1$ we have $\tau(d) < 4(2d)^d$.*

Our proof is based on the so-called Steinitz lemma, in a quantitative version due to Grinberg and Sevastyanov [4].

We also show that the upper bound is not far from the truth.

Theorem 1.2. *There is a constant $c>0$ such that*

$$\tau(d) \geq (cd)^{d/2}$$

for all d that are powers of 2.

The proof is based on a construction of very ill-conditioned square matrices with ± 1 entries, due to Alon and Vũ [1] (the basic idea going back to Håstad [5]). It seems likely that the lower bound could be extended to all d, instead of just powers of 2, but this might need a careful analysis of another construction from [1].

There is a natural and, in our opinion, interesting variant of the quantity $\tau(d)$, where one again considers sequences $\mathbf{v}_1,\ldots,\mathbf{v}_t \in [-1,1]^d$ satisfying (i) and (ii) above, but the $\mathbf{v}_i$ are restricted to only ± 1 vectors. Let $\tau_{\pm 1}(d)$ denote the corresponding maximum length of such a sequence; we have $\tau_{\pm 1}(d) \leq \tau(d)$ by definition. We obtain the following slightly weaker lower bound:

Theorem 1.3. *There is a constant $c>0$ such that $\tau_{\pm 1}(d) \geq (cd)^{d/4}$ for all d that are powers of 2.*

The IP connection. The quantity $\tau(d)$ has been motivated by a connection to an algorithm for integer programming.

Let us consider an integer program in the form

$$\min\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in \mathbb{Z}^\ell, A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}, \tag{1}$$

where A is an $m \times \ell$ integer matrix, $\mathbf{c} \in \mathbb{Z}^\ell$, and $\mathbf{b} \in \mathbb{Z}^m$. This optimization problem is well-known to be NP-hard even for $m = 1$, i.e., for a single equality constraint (this follows, e.g., from the hardness of the knapsack problem). On the other hand, Papadimitriou [6] proved that if m is fixed and the entries of A and $\mathbf{b}$ are small integers, bounded in absolute value by a parameter N, then the integer program can be solved in *pseudo-polynomial* time. That is, the running time can be bounded by a polynomial in ℓ and N (and the input size of $\mathbf{c}$); the polynomial depends on m.

Papadimitriou's algorithm is based on dynamic programming (also see Schrijver [7] for a description); it searches for a shortest path in an auxiliary graph. Dash et al. [3] provided a completely different algorithm for the same problem, which consists in solving a linear program over an auxiliary polyhedron (the so-called *polaroid*).[8] They obtained a pseudo-polynomial bound for the number of inequalities in the linear program, and thus also for the running time, but only for input integer programs with $m \leq 3$ constraints (actually, they handled the case of $m = 1$ constraint earlier in [2]). To get pseudo-polynomiality for larger m, they needed the finiteness of $\tau(m)$. Thus, combined with our Theorem 1.1, their algorithm provides an alternative to Papadimitriou's method.

Acknowledgment

We would like to thank Tibor Szabó for raising the problem at the GWOP'09 workshop, Sanjeeb Dash for prompt answers to our questions, and Patrick Traxler for useful discussions.

[8] More precisely, two linear programs are needed. Moreover, the basic algorithm discussed in [3] solves the *separation problem* for the set $Q := \mathrm{conv}\{\mathbf{x} \in \mathbb{Z}^\ell, A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, but a dual version of it can also be used for the optimization variant. But here we don't want to go into details.

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Spanning trees of 3-uniform hypergraphs

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Joint work with Andrew Goodall, Martin Loebl and Marc Noy.

1 Introduction

A 3-*graph* is a 3-uniform hypergraph $H = (V, \Delta)$, where $\Delta \subseteq \binom{V}{3}$; the elements of V are called *vertices* and those of Δ, *triples*. The *underlying graph* of H is the simple graph (V, E) where $\{a, b\} \in E$ if and only if there is some $c \in V$ such that $\{a, b, c\} \in \Delta$. A 3-graph T is a *tree* if its underlying graph is connected and has no cycles except for the triangles that correspond to triples. A *spanning tree* of a 3-graph $H = (V, \Delta)$ is a tree $T = (V, \Delta_T)$ such that $\Delta_T \subseteq \Delta$.

In this work we present several results concerning the existence and counting of spanning trees. We first review known results about the complexity of the problem of deciding whether a given 3-graph has a spanning tree and of that of counting spanning trees of a 3-graph. We give a sufficient condition for the existence of a spanning tree, and we show that Steiner Triple Systems, viewed as 3-graphs, have superexponentially many spanning trees.

In the second part we build on the theorem of Masbaum and Vaintrob [3] that expresses a generating function of signed spanning trees as a Pfaffian of a matrix depending on an orientation given to triples. The main question is whether triples can be oriented in such a way that all trees came out with the same sign, and hence spanning trees can be counted. We give some families of 3-graphs for which this is true, although the property seems in general to be quite rare.

2 Existence and counting of spanning trees

It is easy to verify that a tree (in the 3-graph sense) has an odd number of vertices, so not every 3-graph has a spanning tree. Moreover, it is also easy to find connected 3-graphs without spanning trees, so connectivity is not a necessary and sufficient condition as in the case of ordinary graphs. The problem of deciding whether a 3-graph has a spanning tree is in P, since it is a particular case of the matroid matching problem for linear matroids, for which Lovász [2] gave a polynomial time algorithm (see also [1]). The problem of counting spanning trees in a 3-graph is #P-complete by a reduction from the problem of counting perfect matchings in an ordinary graph. Indeed, given a graph $G = (V, E)$, let a be an element not in V and define a 3-graph H_G on $V \cup \{a\}$ with triples $\{e \cup \{a\} : e \in E\}$. Then spanning trees of H_G are in correspondence with perfect matchings of G.

If the 3-graph is sufficiently dense, we can guarantee the existence of a spanning tree.

Theorem 2.1. *Let $H = (V, \Delta)$ be a 3-graph with the property that for all $a, b \in V$ there is a triple $t \in \Delta$ such that $\{a, b\} \subset t$. Then H has a spanning tree.*

The 3-graphs obtained from Steiner Triple Systems have many spanning trees. (Note that the complete 3-graph on $2n + 1$ vertices has $(2n - 1)!!(2n + 1)^{n-1}$ spanning trees.)

Theorem 2.2. *Let $H = (V, \Delta)$ be a 3-graph on $2n + 1$ vertices with the property that for all $a, b \in V$ there is exactly one triple $t \in \Delta$ such that $\{a, b\} \subset t$. Then H has $\Omega((2n - o(n))!^{1/6})$ spanning trees.*

3 3-Pfaffian 3-graphs

Let $H = ([2n+1], \Delta)$ be a 3-graph. To each triple $t \in \Delta$ associate a variable y_t and an orientation $\omega(t) \in \{1, -1\}$. Masbaum and Vaintrob [3] construct a skew-symmetric matrix A such that its Pfaffian is the generating function for signed spanning trees, that is,

$$\mathrm{Pf}(A) = \sum_{T \in \mathcal{T}(H)} \mathrm{sign}(T, \omega) \prod_{t \in T} y_t,$$

where $\mathcal{T}(H)$ denotes the set of spanning trees of H and $\mathrm{sign}(T, w) \in \{1, -1\}$ (see [3] for details). Observe the similarity with the expression as a Pfaffian of the generating function for signed perfect matchings of an ordinary graph. In parallel with the rich theory of Pfaffian graphs (see for instance [4]), we introduce the following definitions.

An orientation $\omega : \Delta \to \{1, -1\}$ is 3-*Pfaffian* if $\mathrm{sign}(T, \omega)$ is constant for all $T \in \mathcal{T}(H)$. A 3-graph is 3-*Pfaffian* if it has a 3-Pfaffian orientation.

Our first result produces 3-Pfaffian graphs from Pfaffian graphs.

Theorem 3.1. *Let $H = (V, \Delta)$ be a 3-graph such that for each $a, b \in V$ there is at most one triple t such that $\{a, b\} \subset t$. Let $v \in V$ be such that $H - v$ has no cycle of triples and let G denote the underlying graph of H. Then spanning trees of H are in bijection with perfect matchings of $G - v$ and, moreover, H is 3-Pfaffian if and only if $G - v$ is Pfaffian.*

Next we give another construction and study when it gives rise to 3-Pfaffian graphs. For an ordinary graph $G = (V, E)$ and an integer $k \geq 1$, the *k-suspension* of G is the 3-graph on $V \cup \{u_1, \ldots, u_k\}$ with triples $\{a, b, u_i\}$ for $\{a, b\} \in E$ and $1 \leq i \leq k$.

Theorem 3.2.

1. *The 1-suspension of G is 3-Pfaffian if and only if G is Pfaffian.*
2. *The 2-suspension of G is 3-Pfaffian if and only if $G - v$ is Pfaffian for all vertices v and G contains no copy of C_3, C_5, P_6 or $K^-_{2,3}$ whose complement has a perfect matching.*
3. *For $k \geq 3$, the k-suspension of G is not 3-Pfaffian (except if it has no spanning tree).*

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A note on kernels and Sperner's lemma

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Joint work with Tamás Király [2].

In a directed graph $D = (V, A)$, a stable set $S \subseteq V$ is said to be a *kernel* if from every node of $V \setminus S$ there is an arc to S. Kernels have several applications in combinatorics and game theory, and there has been extensive work on the characterization of digraphs that have kernels. See [3] for a survey on the topic.

One approach to characterize the existence of kernels has been to identify undirected graphs for which every "nice" orientation has a kernel. This led to the introduction of the notion of kernel-solvability by Berge and Duchet.

Definition 0.1. Let $G = (V, E)$ be an undirected graph. A *superorientation* of G is a directed graph $\vec{G}$ obtained by replacing each edge uv of G by an arc uv or an arc vu or both. A *proper directed cycle* in a superorientation is a directed cycle consisting of arcs that are not present reversed in the digraph. In this article we define a *source node* of an induced subdigraph $\vec{G}[U]$ as a node in U from which there are arcs to all of its neighbours in $G[U]$. A superorientation is *clique-acyclic* if no clique contains a proper directed cycle (equivalently, if every clique contains a source node). A graph G is *kernel solvable* if every clique-acyclic superorientation of G has a kernel.

Berge and Duchet conjectured that the kernel solvable graphs are exactly the perfect graphs. The kernel-solvability of perfect graphs was proved by Boros and Gurvich [2], and later a shorter proof was given by Aharoni and Holzman [1]. A common feature of these proofs is that they use Scarf's Lemma [9], a result originating in game theory. We should also mention that the other direction of the Berge-Duchet conjecture follows from the Strong Perfect Graph Theorem, and no easier proof is known.

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Our contribution is a simple proof of the kernel-solvability of perfect graphs based on Sperner's Lemma instead of Scarf's Lemma. Since Sperner's Lemma is more widely known and conceptually simpler, this might have some interest. We note that the methods presented here can also be used to derive Scarf's Lemma from Sperner's Lemma, see [5]. We also present new results on kernels in superorientations of non-perfect graphs.

The results are described in more detail in [6].

1 Polyhedral versions of Sperner's lemma

In this section we present versions of Sperner's lemma that deal with colourings of vertices and facets of polytopes and polyhedra.

Definition 1.1. For a colouring of the vertices of a polytope P, a facet of P is *multi-coloured* if it contains vertices of every colour. For a colouring of the facets of P, a vertex of P is *multi-coloured* if it lies on facets of every colour.

The following theorem is a variant of Sperner's Lemma, and can be proved similarly.

Theorem 1.2. *Let P be an n-dimensional polytope, with a simplex facet F_0. Suppose we have a colouring of the vertices of P with n colours such that F_0 is multi-coloured. Then there is another multi-coloured facet.*

By polarity, the following theorem is also true.

Theorem 1.3. *Let P be an n-dimensional polytope, with a simplicial vertex v_0. Suppose we have a colouring of the facets of P with n colours such that v_0 is multi-coloured. Then there is another multi-coloured vertex.*

The above results can be generalized to unbounded pointed polyhedra, which is the form that we will use later. First, we have to extend the notion of vertices.

Definition 1.4. The *ends* of a pointed polyhedron P are its vertices and its extreme directions (an *extreme direction* of a polyhedron is an extreme ray of its characteristic cone).

We extend also the incidences between facets and vertices to ends in the natural way: a facet of P contains an extreme direction of P if it is also an extreme direction of the facet. In addition, if a pointed polyhedron's characteristic cone is full-dimensional, then we consider the extreme directions as being on a "facet at infinity".

Definition 1.5. Two polyhedra are called *combinatorially equivalent* if there is a bijection between their facets (including the "facet at infinity") and their ends which preserves the incidences. Two polyhedra are called *combinatorially polar* if there is a bijection between the facets of one and the ends of the other and vice versa which reverses the inclusion relation.

We claim that if P is a pointed full-dimensional polyhedron then there exists a polytope which is combinatorially equivalent to it. This is because if we move P so that the origin is in its interior and then take its polar, it will be a polytope which is combinatorially polar to P; moreover, it will be full-dimensional since P is pointed. If we do the same a second time, we get a polytope which is combinatorially equivalent to P. Now we can state the analogue of Theorem 1.3 for unbounded polyhedra.

Corollary 1.6. *Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If we colour the facets of the polyhedron by n colours such that facets containing the i-th extreme direction do not get colour i, then there is a multi-coloured vertex.*

We will apply this corollary to polyhedra of the form $P = Q - \mathbb{R}^n_+$ where Q is a bounded polytope. We use the notation $[n] = \{1, \dots, n\}$. If $a \in \mathbb{R}^n_+$ and $J \subseteq [n]$, then we denote by a^J the vector whose j-th coordinate is

$$a^J(j) = \begin{cases} a(j) & \text{if } j \in J \\ 0 & \text{if } j \notin J. \end{cases}$$

We will need the following lemma:

Lemma 1.7. *If $P = Q - \mathbb{R}^n_+$ where $Q = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ is a bounded polytope and A and b are nonnegative, then P is described by inequalities of the form $a_i^J x \leq b_i$, where a_i is the i-th row of A, b_i is the i-th coordinate of b, and $\emptyset \neq J \subseteq \operatorname{supp}(a_i)$. The extreme directions of P are $-e_j$ ($j = 1, \dots n$).*

2 Kernel solvability of perfect graphs

Boros and Gurvich [2] proved the following conjecture of Berge and Duchet, using game theoretic results that rely on Scarf's Lemma.

Theorem 2.1 ([2]). *Every perfect graph is kernel solvable.*

Aharoni and Holzman [1] gave a shorter proof of Theorem 2.1 using Scarf's Lemma directly. In this section we present a similarly short proof that relies on the more familiar Sperner's Lemma instead.

Proof of Theorem 2.1. Let $G = (V, E)$ be a perfect graph, with $V = [n]$, and let $\vec{G}$ be a clique-acyclic superorientation of G. Let $\mathcal{C}$ denote the set of all (not necessarily maximal) cliques of G. We consider the polyhedron $P := \mathrm{STAB}(G) - \mathbb{R}^n_+$. By the well known results of Fulkerson [4] and Lovász [7], STAB(G) is described by the clique-inequalities, so Lemma 0.1 implies that

$$P = \{x \in \mathbb{R}^n : x(C) \leq 1 \text{ for every } C \in \mathcal{C}\},$$

and the extreme directions of P are $-e_j$ $(j = 1, \ldots n)$.

Let the colour of a facet $\{x \in P : x(C) = 1\}$ be a source node j of clique C. Clearly the extreme direction $-e_j$ does not belong to a facet of colour j, so applying Corollary 1.6 we get that there exists a multi-coloured vertex x^* of P. By the definition of P, $x^* = \chi_S$ for a maximal stable set S.

Since x^* is multi-coloured, for each node j of V, there is a clique C such that the facet $\{x \in P : x(C) = 1\}$ contains x^* and it has colour j. This means that $|C \cap S| = 1$ and j is a source node of C. Thus from each node $j \notin S$ there is an arc to S, so S is a kernel. □

3 If we know STAB(G)

In this section we extend Theorem 2.1 to arbitrary undirected graphs, provided some conditions hold that depend on the facets of STAB(G). We say that a superorientation $\vec{G}$ of $G = (V, E)$ is *acyclic* in $U \subseteq V$ if there is no proper directed cycle in $\vec{G}[U]$.

Theorem 3.1. *If* $\mathrm{STAB}(G) = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ *(where A and b are nonnegative) and $\vec{G}$ is a superorientation of G which is acyclic in* supp(a) *for every row a of A, then there is a kernel in $\vec{G}$.*

Proof. Let $P = \mathrm{STAB}(G) - \mathbb{R}^n_+$. By Lemma 0.1, P is described by the inequalities of the form $a_i^J x \leq b_i$, where a_i is the i-th row of A, b_i is the i-th coordinate of b, and $\emptyset \neq J \subseteq \mathrm{supp}(a_i)$.

Let the colour of a facet of the form $P \cap \{x : a_i^J x = b_i\}$ be a source node of the subdigraph of $\vec{G}$ induced by J. Such a source node exists because $\vec{G}$

is acyclic in $\text{supp}(a_i)$. In order to apply Corollary 1.6, we have to show that a facet containing the j-th extreme direction does not have colour j. This is true because in this case $j \notin J$. Thus Corollary 1.6 implies that P has a multi-colured vertex $x^* = \chi_S$ for a maximal stable set S.

For every node j, there is a facet F of colour j containing x^*. Let F be $P \cap \{x : a_i^J x = b_i\}$. Then $S \cap J$ is a maximal stable set in $G[J]$ because $a_i^J x^* = b_i$ and $\text{supp}(a_i^J) = J$. This and $j \in J$ imply that either $j \in S$ or j has a neighbour in $S \cap J$, in which case j has an out-neighbour in $S \cap J$, since j is a source node of J. We proved that S is a kernel of $\vec{G}$. □

4 Kernels in h-perfect graphs

Sbihi and Uhri [8] introduced the class of *h-perfect graphs* as the graphs for which the stable set polytope is described by the following set of inequalities:

$$x_v \geq 0 \qquad \text{for every } v \in V, \tag{1}$$

$$x(C) \leq 1 \qquad \text{for every maximal clique } C, \tag{2}$$

$$x(Z) \leq \frac{|Z|-1}{2} \qquad \text{for every odd hole } Z. \tag{3}$$

To apply Theorem 3.1 to h-perfect graphs, let us call a superorientation of a graph *odd-hole-acyclic* if no oriented odd hole is a proper directed cycle. Theorem 3.1 implies the following result.

Theorem 4.1. *If G is an h-perfect graph then every clique-acyclic and odd-hole-acyclic superorientation of G has a kernel.*

Obviously a superorientation of a perfect graph is always odd-hole-acyclic, thus Theorem 4.1 is an extension of Theorem 2.1. We have mentioned that the reverse direction of Theorem 2.1 is also true, due to the Strong Perfect Graph Theorem. The same does not hold for Theorem 4.1. Nevertheless, one may hope for a stronger theorem where the reverse direction also holds. We give here a less elegant but stronger theorem for which we conjecture that this is the case.

Let G be an h-perfect graph, and let $\vec{G}$ be a clique-acyclic superorientation of G. Some odd holes of G may become proper directed cycles; let us denote these by $Z_1, \ldots, Z_k$. Let us select nodes $v_1, \ldots, v_k$ such that $v_i \in Z_i$ for $i = 1, \ldots, k$ (the selected nodes need not be distinct). We call this a *superorientation with special nodes*. An *almost-kernel* for a superorientation with special nodes is a stable set S with the following property:

If a node $v \notin S$ has no outgoing arc into S, then $v = v_i$ for some i and $|Z_i \cap S| = (|Z_i| - 1)/2$.

Theorem 4.2. *If G is an h-perfect graph then every clique-acyclic superorientation with special nodes has an almost-kernel.*

Note that this theorem is stronger than Theorem 4.1 since every almost-kernel in a clique-acyclic and odd-hole-acyclic orientation is a kernel. We conjecture that here the converse also holds:

Conjecture 4.3. *A graph G is h-perfect if and only if every clique-acyclic superorientation with special nodes has an almost-kernel.*

This would be a consequence of the following stronger conjecture:

Conjecture 4.4. *Let P be a polyhedron of the form $\{x \in \mathbb{R}^n : 0 \leq x \leq 1,\ Ax \geq b\}$ where A is a 0–1 matrix, and b is integer. If P is not an integer polyhedron, then we can choose an index $f(i) \in \operatorname{supp}(a_i)$ for every row a_i of A, and an index $g(x^*) \in \operatorname{supp}(x^*)$ for every integer vertex x^* of P, so that if $f(i) = g(x^*)$, then $a_i x^* > b_i$.*

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Maximizing the number of q-colorings

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Joint work with Po-Shen Loh and Benny Sudakov.

The problem of counting the number $P_G(q)$ of q-colorings of a given graph G has been the focus of much research over the past century. Although it is already NP-hard even to determine whether this number is nonzero, the chromatic polynomial $P_G(q)$ itself has very interesting properties.

Independently, Linial [6] and Wilf (unpublished, 1980s) posed the problem of maximizing $P_G(q)$ over all graphs with n vertices and m edges. Linial's motivation came from the worst-case computational complexity of determining whether a particular function $f : V(G) \to \mathbb{R}$ is a proper coloring. Wilf [9] was analyzing the so-called *backtrack* algorithm for finding a proper q-coloring of a graph. Although this generated much interest in the problem, it was only solved in sporadic cases. The special case $q = 2$ was completely solved for all m, n, by Lazebnik [2]. Very little was known for general m, n when $q \geq 3$. Although many upper and lower bounds for $P_G(q)$ were proved by various researchers (Byer [1], Lazebnik [3, 4], Lazebnik, Pikhurko, and Woldar [5], Liu [7], Simonelli [8], and others), these bounds were widely separated. Even the $q = 3$ case resisted solution: twenty years ago, Lazebnik conjectured that when $m \leq n^2/4$, the n-vertex graphs with m edges which maximized the number of 3-colorings were complete bipartite graphs minus the edges of a star, plus isolated vertices.

We propose an approach that one might be able to use to determine the maximal graphs in many nontrivial ranges of m, n. Our methodology can be roughly outlined as follows. We show, via Szemerédi's Regularity Lemma, that the asymptotic solution to the problem reduces to a certain quadratically-constrained linear program in 2^q variables. For any given q, this task can in principle be automated by a computer code that symbolically solves the optimization problem, although a more sophisticated approach would be required to solve this for all q. Our solutions to the optimization problem then give us the approximate structure of the maximal graphs. Finally, we use various local arguments, such as the so-called "stability" approach introduced by Simonovits, to refine their structure into precise results.

We successfully applied our machinery to solve the Linial-Wilf problem for many nontrivial ranges of m, n, and $q \geq 3$. In particular, for $q = 3$, our results confirm a stronger form of Lazebnik's conjecture when m is large. In addition, for each $q \geq 4$ we show that for all densities $\frac{m}{n^2}$ up to approximately $\frac{1}{q \log q}$, the extremal graphs are also complete bipartite graphs minus a star.

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On the number of simple arrangements of five double pseudolines

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Joint work with Julien Ferté [1] and Vincent Pilaud [2].

Abstract

We present an incremental algorithm to enumerate arrangements of double pseudolines.

1 Introduction

Pseudoline arrangements (or in higher dimension, pseudohyperplane arrangements) have been extensively studied in the last decades as a useful combinatorial abstraction of real two-dimensional projective configurations of points [2, 3, 8]. Recently, Habert and the third author of this paper [9] introduced double pseudoline arrangements as a combinatorial abstraction of real two-dimensional projective configurations of pairwise disjoint convex bodies. The first main structural properties of pseudoline arrangements (connectedness of their mutation graphs, duality principle, axiomatic characterization of their isomorphism classes in terms of chirotopes) extend to double pseudoline arrangements. We recall some definitions and results of [9] in Section 2.

To help our understanding of double pseudoline arrangements, and in order to carry out computer experiments, it is interesting to develop algorithms to enumerate isomorphism classes of arrangements with few double pseudolines. For pseudolines arrangements, different enumeration algorithms have been implemented [4, 5, 1]. In particuler it is known that the number p_n of isomorphism classes of simple pseudoline arrangements of order n is 1 for $n \leq 5$ and then follows the table:

n	6	7	8	9	10	11
p_n	4	11	135	4 382	312 356	41 848 591.

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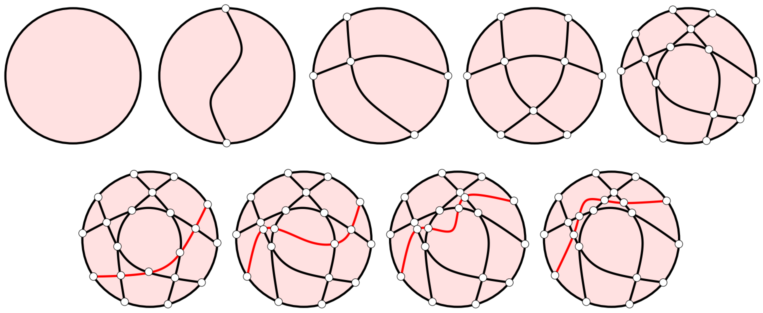


Figure 1: Representatives of the isomorphism classes of simple arrangements of one, two, three, four, five and six pseudolines.

Representatives of the isomorphism classes of simple arrangements of order at most six are depicted in Figure 1.

In this paper we prove that the one-extension spaces of arrangements of double pseudolines are connected under mutation. From this result we derive a simple incremental enumeration algorithm consisting of traversing the graphs of mutations of the one-extension spaces of double pseudoline arrangements of given order. It turns out that this relatively naive algorithm is sufficient in practice for the enumeration of simple arrangements of five double pseudolines using a relatively modest amount of CPU time. In particular we derive from our implementation of this incremental algorithm that the numbers of isomorphism classes of simple arrangements of order 3, 4 and 5 are 13, 6570, and 181 403 533, respectively.

2 Preliminaries

2.1 Arrangements of double pseudolines

We denote the (real two-dimensional) *projective plane* by $\mathcal{P}$ and represent it as a disk with antipodal boundary points identified.

A simple closed curve of $\mathcal{P}$ is a *pseudoline* if it is not contractible, and a *double pseudoline* otherwise (Fig. 2). The complement of a double pseudoline γ has two connected components: a Möbius strip $\mathcal{M}_\gamma$ and a topological disk $\mathcal{D}_\gamma$.

An *arrangement of double pseudolines* is a finite set of double pseudolines

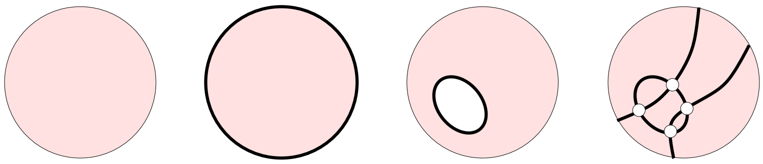


Figure 2: A projective plane represented by a circular diagram with antipodal points identified, a pseudoline, a double pseudoline, and an arrangement of two double pseudolines.

such that any two double pseudolines have exactly four intersection points, cross transversally at these points, and induce a cell decomposition of $\mathcal{P}$ (Fig. 2). The *order* of an arrangement is its number of double pseudolines. As usual a *simple* arrangement is an arrangement where no three curves meet at the same point.

Two arrangements A and B are *isomorphic* if there is an homeomorphism of the projective plane that sends A on B (or equivalently, if there is an isotopy joining A to B). In this paper, we are interested in enumerating isomorphism classes of arrangements. It is routine to see that there is a unique isomorphism class of arrangements of order two.

2.2 Chirotopes

The *chirotope* of an indexed oriented arrangement is the application that assigns to each triple of indices the isomorphism class of the subarrangement indexed by this triple.

As for pseudoline arrangements, an isomorphism class of an indexed oriented arrangement of double pseudolines only depends on its chirotope. Furthermore, given an application χ that assigns to each triple of indices an isomorphism class of an oriented arrangement of double pseudolines indexed by this triple, the following properties are equivalent [9]:

(1) χ is the chirotope of an indexed oriented arrangement,

(2) the restriction of χ to the set of triples of any subset of at most five indices is the chirotope of an indexed oriented arrangement.

This result provides a motivation for enumerating arrangements of at most five double pseudolines.

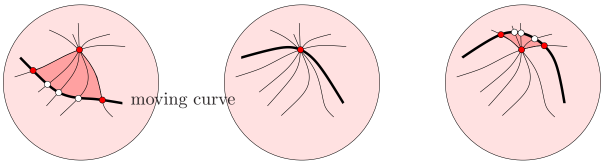


Figure 3: A mutation destroys, creates, or inverts a fan supported by the moving curve

2.3 Mutations

A *mutation* is a local transformation of an arrangement Γ that only destroys, or creates, or inverts a fan of Γ. More precisely, it is a homotopy of arrangements in which only one curve γ moves, sweeping (reaching or leaving or first reaching and then leaving) a single vertex of the remaining arrangement $\Gamma \setminus \{\gamma\}$ (cf. Fig. 3). It is known that any two arrangements of the same order are homotopic via a finite sequence of mutations followed by an isotopy; in other terms the graph of mutations on (isomorphism classes of) arrangements of given order is connected [9].

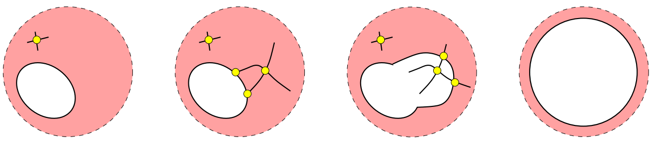


Figure 4: Pumping vertices out of the Möbius strip of a double pseudoline

The main idea of the proof of the connectedness of mutation graphs presented in [9] is to come down to the case of pseudoline arrangements by pumping the vertices of the arrangement out of its Möbius strips (cf. Fig. 4). This is indeed possible thanks to the so-called Pumping Lemma:

Lemma 2.1 (Pumping Lemma [9])**.** *Let Γ be a simple arrangement of double pseudolines and let γ be a distinguished double pseudoline of Γ. Assume that there is a vertex of Γ lying in the interior of the Möbius strip $\mathcal{M}_\gamma$ bounded by γ. Then there is a triangular face of Γ supported by γ and included in $\mathcal{M}_\gamma$.* □

In the next section we will slightly improve this lemma by showing that the number of triangular faces supported by γ and included in $\mathcal{M}_\gamma$ is actually at least two.

From the connectedness property of mutation graphs we may derive a simple enumeration algorithm consisting of traversing the graph of mutations starting from any given arrangement. This naive algorithm is sufficient for the enumeration of simple arrangements of three or four double pseudolines but already fails (because of RAM memory limitations) for simple arrangements of five double pseudolines. The graph of mutations on simple arrangements of three double pseudolines is depicted in Figure 5.

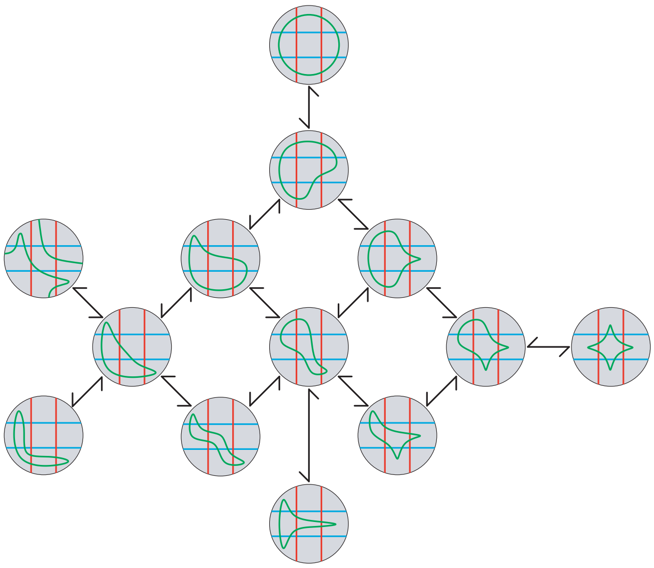


Figure 5: The graph of mutations on simple arrangements of three double pseudolines

In order to go a little bit further in practice (and particularly, to enumerate arrangements of five double pseudolines), we show that the mutation graph on the space of one-extensions of any given arrangement is connected. From this result we derive a simple incremental enumeration algorithm consisting of traversing for each arrangement of given order the graph of mutations of its space of one-extensions. It turns out that this relatively naive algorithm is sufficient in practice for the enumeration of simple arrangements of five double pseudolines.

2.4 Duality principle

A (real two-dimensional) *projective geometry* is a topological point-line incidence geometry $(\mathcal{P}, \mathcal{L})$ whose point space $\mathcal{P}$ is a projective plane and whose line space $\mathcal{L}$ is a subspace of the space of pseudolines of $\mathcal{P}$. It is known that $\mathcal{L}$ is a projective plane, that the dual of a point—defined as its set of incident lines—is a pseudoline in $\mathcal{L}$, and that the set of dual pseudolines of the points of any finite set of points is an arrangement of pseudolines in $\mathcal{L}$ [12, 11]. The duality principle for pseudoline arrangements of Goodman [6, 2] asserts that the converse is true: any arrangement of pseudolines is isomorphic to the dual arrangement of a finite set of points of a projective geometry. (This formulation of the duality principle of Goodman is a reformulation of the original one of Goodman, taking advantage of the embedabbility of any arrangement of pseudolines in the line space of a projective geometry [7].)

The duality principle for pseudoline arrangements has the following extension for arrangements of double pseudolines: first, the dual of a convex body—defined as its set of tangent lines, i.e., the set of lines touching the body but not its interior—is a double pseudoline in $\mathcal{L}$; second, the set of dual double pseudolines of the convex bodies of any finite set of pairwise disjoint convex bodies is an arrangement of double pseudolines in $\mathcal{L}$, and, third, any arrangement of double pseudolines is isomorphic to the dual arrangement of a finite set of pairwise disjoint convex bodies of a projective geometry [9].

From this result we derive easily the following useful extension of the Pumping Lemma.

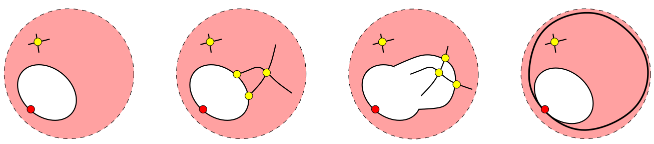


Figure 6: Pumping vertices out of the Möbius strip of a double pseudoline while keeping fixed one of the points of the double pseudoline

Lemma 2.2. *Let Γ be an arrangement of double pseudolines and let $\gamma \in \Gamma$. Assume that there is a vertex of Γ lying in the interior of the Möbius strip*

$\mathcal{M}_\gamma$ bounded by γ and that the vertices of Γ supported by γ are simple (i.e., have degree four). Then there is at least two fans of Γ included in $\mathcal{M}_\gamma$ and supported by γ. □

This enhanced version of the Pumping Lemma allows to pump the vertices out of the Möbius strip of a double pseudoline while keeping fixed one of the points of the double pseudoline (cf. Fig. 6).

3 Connectedness of the spaces of one-extensions

A one-extension of an arrangement of n double pseudolines Γ is an arrangement of $n+1$ double pseudolines Γ' of which Γ is a subarrangement. The double pseudoline of Γ' not in Γ is called the one-extension element.

Theorem 3.1. *Let Γ' and Γ'' be two one-extensions of an arrangement of double pseudolines Γ. Then Γ' and Γ'' are homotopic via a finite sequence of mutations followed by an isotopy during which the only moving curves are the one-extension elements of Γ' and Γ''.*

Proof. Let γ be a double pseudoline of Γ and let γ' and γ'' be the one-extension elements of Γ' and Γ'', respectively. Using continuous motions—thanks to the duality principle—one can easily reduce the analysis to the following case

(1) γ' is a *thin* double pseudoline in Γ', i.e., a double pseudoline whose Möbius strip is free of vertices of Γ;

(2) γ and γ' are *touching* at σ': by this we mean that σ' is one the two 2-cells of size two of the subarrangement $\{\gamma, \gamma'\}$ and that σ' is also a 2-cell of the whole arrangement Γ';

(3) similarly γ'' is a thin double pseudoline in Γ'', and γ and γ' are touching at σ'';

(4) γ' and γ'' coincide in the disk $\mathcal{D}_\gamma$, and $\sigma' = \sigma''$;

(5) $\mathcal{M}_{\gamma'}$ is a tubular neighbourhood of a pseudoline γ'_* with the property that γ'_* intersects any double pseudoline of Γ in exactly two points, the intersection points being transversal; similarly $\mathcal{M}_{\gamma''}$ is a tubular neighbourhood of a pseudoline γ''_* with the property that γ''_* intersects

any double pseudoline of Γ in exactly two points, the intersection points being transversal;

(6) γ'_* and γ''_* coincide in $\mathcal{D}_\gamma$, and intersect finitely many in $\mathcal{M}_\gamma$, the intersection points being transversal;

(7) $\mathcal{M}_{\gamma'} \cup \mathcal{M}_{\gamma''}$ is a tubular neighbourhood of $\gamma'_* \cup \gamma''_*$;

Let $\Gamma'_* = \Gamma \cup \{\gamma'_*\}$ and, similarly, let $\Gamma''_* = \Gamma \cup \{\gamma''_*\}$. It should be clear to the reader that the proof of our theorem boils down to show that the mixed arrangements Γ'_* and Γ''_* are homotopic via a finite sequence of mutations during which the only moving curves are the pseudolines γ'_* and γ''_*. This is precisely here that we are going to use Lemma 2.2. But before using this lemma we define a γ-curve as a connected component of the trace on $\mathcal{M}_\gamma$ of a double pseudoline of $\Gamma \setminus \{\gamma\}$, we observe that the pseudoline γ'_* intersects a γ-curve in at most one point (necessarily a transversal intersection point), and similarly that the pseudoline γ''_* intersects a γ-curve in at most one point. Let now τ be a copy of γ and let Γ_τ be the symmetric difference between Γ and $\{\gamma, \tau\}$. If τ is thin in Γ_τ we are done modulo an isotopy. Otherwise Lemma 2.2 asserts that there exists a fan Δ of the arrangement Γ_τ supported by τ, included in $\mathcal{M}_\tau$, and missing the 2-cell σ'. Modulo a finite sequence of mutations in Γ'_τ or in Γ''_τ or in both Γ'_τ and in Γ''_τ during which the only moving curves are γ'_* or γ''_* or both γ'_* and γ''_* one can assume that Δ misses γ'_* and γ''_*. We then perform a mutation of Δ in Γ_τ with τ in the role of the moving curve—during this process we do not touch at σ'. By repeated application of this process we arrive at the situation where τ is thin in Γ_τ and where γ'_* and γ''_* are included in $\mathcal{M}_\tau$, excepted their common part in $\mathcal{D}_\gamma$—which is included in $\mathcal{D}_\tau$. At this point we are done modulo an isotopy. □

4 The incremental algorithm

4.1 Description

A *pointed* arrangement is an arrangement with a distinguished double pseudoline. We always use the notation $A^\bullet$ for a pointed arrangement and A for its non-pointed version (and similarly for sets of pointed arrangements). We also use the notation $\mathrm{Sub}(A)$ for the set of subarrangements of an arrangement A.

Let $\mathcal{A}_n$ denote the set of isomorphism classes of simple arrangements of n double pseudolines, and p_n be its cardinality. Our algorithm enumerates $\mathcal{A}_n$ from $\mathcal{A}_{n-1}$, by mutating an added distinguished double pseudoline.

Algorithm 1 Incremental enumeration

Require: $\mathcal{A}_{n-1} = \{a_1, \ldots, a_{n-1}\}$.
Ensure: $\mathcal{A}_n$.

```
for i from 1 to p_n do
  A• ← add a pointed double pseudoline to a_i.
  if Sub(A) ∩ {a_1, ..., a_{i-1}} = ∅ then
    write A.
  end if
  Q• ← [A•]. S• ← {A•}.
  while Q• ≠ ∅ do
    A• ← pop Q•.
    T ← list the triangles of A• adjacent to its pointed double pseudoline.
    for t ∈ T do
      B• ← mutate the triangle t in A•.
      if B• ∉ S• then
        if Sub(B) ∩ {a_1, ..., a_{i-1}} = ∅ and B ∉ S then
          write B.
        end if
        push(B•, Q•). S• ← S• ∪ {B•}.
      end if
    end for
  end while
end for
```

For each $i \in \{1, \ldots, p_{n-1}\}$, the algorithm enumerates the subset $S_i^\bullet$ of arrangements of $\mathcal{A}_n^\bullet$ containing a_i as a subarrangement, by mutations of a distinguished added double pseudoline. From the set S_i, it selects the subset R_i of arrangements with no subarrangements in $\{a_1, \ldots, a_{i-1}\}$. Finally it computes the set T_i of non-pointed isomorphism classes of arrangements of R_i. In other words, R_i is the subset of arrangements of $\mathcal{A}_n$ whose first subarrangement, among $\{a_1, \ldots, a_{p_{n-1}}\}$, is a_i. Thus, $\mathcal{A}_n$ is the union of the R_i.

An alternative approach[3] for counting arrangements is to enumerate the

[3] We thank Luc Habert for this suggestion.

subsets $S_i^\bullet$ and to compute, for each arrangement $A^\bullet$ of $S_i^\bullet$, the number $\sigma(A^\bullet)$ of double pseudolines α of A such that A pointed at α is isomorphic to $A^\bullet$. Then

$$p_n = \frac{1}{m} \sum_{i=1}^{p_{n-1}} \sum_{A^\bullet \in S_i^\bullet} \sigma(A^\bullet).$$

The main advantage of this version is to avoid the storage of $\mathcal{A}_{n-1}$. However, it only counts p_n and can not provide a data base for $\mathcal{A}_n$.

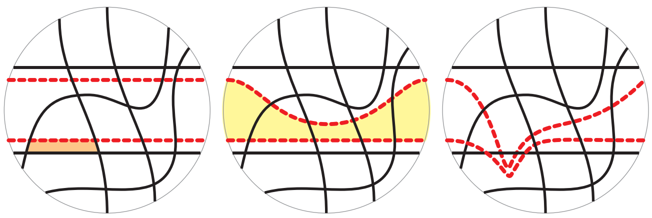
4.2 Adding a double pseudoline

One of the important steps of the incremental method is to add a double pseudoline to an initial arrangement. Our method uses three steps (see Fig. 7):

(1) *duplicate a double pseudoline*: we choose one arbitrary double pseudoline ℓ, duplicate it, drawing a new double pseudoline ℓ' completely included in the Möbius strip $\mathcal{M}_\ell$ and we denote R any rectangle delimited by ℓ and ℓ'.

(2) *flatten*: we pump the double pseudoline ℓ' such that no vertex of the arrangement lies in the Möbius strip $\mathcal{M}_{\ell'}$. During this process, we do not touch the rectangle R.

(3) *add four crossings*: we replace the rectangle R by four crossings between ℓ and ℓ'.



(a) (b) (c)

Figure 7: Three steps to insert a double pseudoline in a double pseudoline arrangement: duplicate a double pseudoline (a), flatten it (b) and add four crossings (c).

If we think of our double pseudoline arrangement as the dual of a configuration of convex bodies, this method corresponds to: (1) choosing one convex C and drawing a new convex C' inside C; (2) flattening the convex C' until it becomes almost a single point, maintaining it almost in contact with the boundary of C; and (3) moving C' outside C.

4.3 Encoding an arrangement

In order to manipulate arrangements, one can encode it in several different ways. We used two different encodings (one for an easy manipulation of it and one for short storage):

(1) *flag representation*: We call a *flag* of an arrangement A any triple (v, e, f) consisting of one vertex v, one edge e and one face f (of the cell complex defined by A), such that $v \in e \subset f$. The three involutions $\sigma_0, \sigma_1, \sigma_2$, that change the vertex, the edge and the face of a flag (Fig. 8), completely determine A.

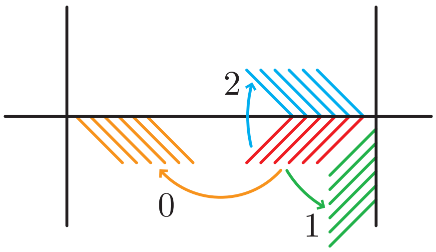


Figure 8: The three involutions σ_0, σ_1 and σ_2.

(2) *encoding*: We associate to each flag $\phi = (v, e, f)$ of the arrangement A a word w_ϕ constructed as follows. Let ℓ_1 be the simple or double pseudoline containing e, and, for all $2 \leq p \leq n+m$, let ℓ_p be the pth simple or double pseudoline crossed by ℓ_1 on a walk starting at ϕ and oriented by $\sigma_0\sigma_1\sigma_2\sigma_1$. This walk also defines a starting flag ϕ_i for each ℓ_i. We walk successively on $\ell_1, \ldots, \ell_{n+m}$, starting from ϕ_i and in the direction given by $\sigma_0\sigma_1\sigma_2\sigma_1$, and index the vertices by $1, 2, \ldots, V$ in the order of appearance. For all i, let w_i denote the word formed by reading the indices of the vertices of ℓ_i starting from ϕ_i. The word w_ϕ is the concatenation of $w_1, w_2, \ldots, w_{n+m}$. Finally, we associate to

the arrangement A the lexicographically smallest word among the w_ϕ where ϕ ranges over all the flags of A.

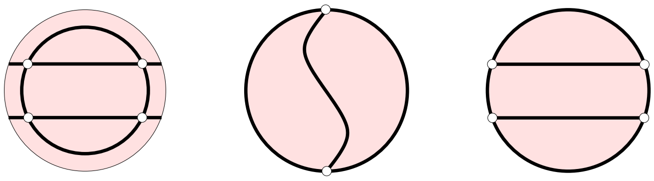


Figure 9: The three mixed arrangements of size two.

4.4 Extension to simple and non-simple mixed arrangements

A *mixed arrangement* is a finite set of pseudolines and double pseudolines such that (1) any two pseudolines have a unique intersection point; (2) a pseudoline and a double pseudoline have exactly two intersection points and cross transversally at these points; and (3) any two double pseudolines have exactly four intersection points, cross transversally at these points, and induce a cell decomposition of $\mathcal{P}$ (Fig. 9). The *order* of a mixed arrangement is the pair (n, m) where n and m are its numbers of pseudolines and double pseudolines, respectively. We denote by $\mathcal{A}_{n,m}$ the set of isomorphism classes of simple mixed arrangements of order (n, m), and denote $p_{n,m}$ the cardinality of $\mathcal{A}_{n,m}$. Our incremental enumeration algorithm for simple double pseudoline arrangements extends easily to simple and non-simple mixed arrangements.

5 Results

A C++ implementation of this algorithm for simple mixed arrangements is available upon request to the authors.

This implementation provided us with the following values of $p_{n,m}$.

$n \backslash m$	0	1	2	3	4	5
0	1	1	1	13	6 570	181 403 533
1	1	1	4	626	4 822 394	
2	1	2	48	86 715		
3	1	5	1 329			
4	1	25	80 253			
5	1	302				
6	4	9 194				
7	11	556 298				
8	135					
9	4 382					
10	312 356					

In particular, the number of mixed arrangements on at most five curves is 186 321 272.

Let us briefly comment on running time. Observe that our algorithm can be parallelized very easily (separating each enumeration of S_i and R_i, for $i \in \{1, \ldots, p_{n,m-1}\}$). In order to obtain the last column, we used four processors of 2GHz for almost 3 weeks. The working space is bounded by $\max |S_i^\bullet| = 279\,882$ (times the space of the encoding of a single configuration, i.e., about 80 characters).

Finally, Fig. 10 shows the evolution of the ratio between the sizes of the sets R_i and $S_i^\bullet$ (during the enumeration of arrangements of five double pseudolines). We have also observed that $\sum |S_i^\bullet| / \sum |R_i| \simeq 5$, which confirms that hardly any configurations of five convex bodies have symmetries.

6 Further developments

The following questions and developments may be treated in a subsequent paper:

(1) *Developing further implementation*: we will soon extend the implementation to the enumeration of arrangements in the Möbius strip, to indexed arrangements, and to non-simple arrangements.

(2) *Drawing an arrangement*: we have seen a method to add a pseudoline in an arrangement. Combined with a planar-graph-drawing algorithm, this provides an algorithm to draw an arrangement in the unit disk.

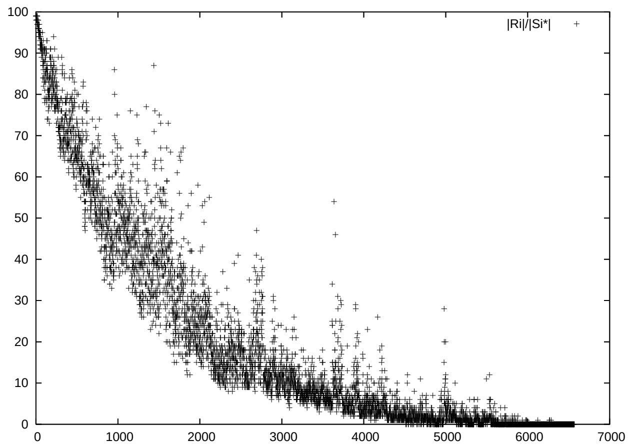


Figure 10: Percentage of new configurations.

For example, $p_{1,m}$ can be interpreted as the number of drawings of the arrangements of $\mathcal{A}_{0,m}$.

(3) *Axiomatization*: pseudoline arrangements admit simple axiomatizations [2, 10], with few axioms dealing with configurations of at most five pseudolines. The *Axiomatization Theorem* of [9] affirms that the complete list of arrangements of at most five double pseudolines is an axiomatization of the class of double pseudoline arrangements. Is there any simpler axiomatization? Is it possible to algorithmically reduce this axiomatization?

(4) *Realizability*: it is well-known that certain pseudoline arrangements are not realizable in the standard projective geometry $\mathcal{P}^2(\mathbb{R})$. Inflating pseudolines into thin double pseudolines in such an arrangement give rise to non-realizable double pseudoline arrangements. Are there smaller examples? Are all arrangements of at most five double pseudolines realizable?

Acknowledgments

We thank Luc Habert, Éric Colin de Verdière and Francisco Santos for interesting discussions on the subject (and technical support).

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Three Ramsey type problems

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Abstract

We pose three problems on partitioning and covering of an r-edge colored complete graph with monochromatic paths and cycles.

1 The problems

Assume that K_n is a complete graph on n vertices whose edges are colored with r colors ($n >> r \geq 2$). A classical question is the following: How many monochromatic cycles are needed to partition the vertex set of K_n? Throughout this note single vertices and edges are considered to be cycles. Maybe surprisingly for the first view, Erdős, Gyárfás and Pyber [2] showed that such a partition can be always done with a number of cycles which does not depend on n. According to this let $p(r)$ denote the minimum number of monochromatic cycles needed to partition the vertex set of any r-colored K_n. In [2] (see also [3]) the authors conjectured the following.

Conjecture 1.1. $p(r) = r$.

This conjecture was proved for $r = 2$ and $n \geq n_0$ by Łuczak, Rödl and Szemerédi [5]. For general r Erdős, Gyárfás and Pyber [2] proved the following.

Theorem 1.2. *There exists a constant c such that* $p(r) \leq cr^2 \log r$.

Gyárfás, Ruszinkó, Sárközy and Szemerédi [4] improved this as follows.

Theorem 1.3. *For every integer* $r \geq 2$ *there exists a constant* $n_0 = n_0(r)$ *such that if* $n \geq n_0$ *and the edges of a finite complete graph* K_n *are colored with* r *colors then the vertex set of* K_n *can be partitioned into at most* $100r \log r$ *vertex disjoint monochromatic cycles.*

In this improvement the authors use the Szemerédi Lemma and basically all but one of the steps in this proof are tight. This particular step is that at some moment the authors use consecutively greedily the Erdős-Gallai theorem [1] to obtain a partition a big portion of vertices an r colored K_n with $cr \log r$ monochromatic cycles. A solution of the following problems may help to obtain a linear bound for $p(r)$ (if such exists at all). In the following, c_i-s are absolute constants.

Problem 1.4. *Is it possible to partition all but $c_1 n/r$ vertices of an r edge colored K_n into $c_2 n$ monochromatic cycles?*

A slight weakening of this problem is the following.

Problem 1.5. *Is it possible to partition all but $c_1 n/r$ vertices of an r edge colored K_n into $c_2 n$ monochromatic paths?*

We were also not able to solve this even in case when we do not insist on partition, just on cover.

Problem 1.6. *Is it possible to cover all but $c_1 n/r$ vertices of an r edge colored K_n into $c_2 n$ monochromatic cycles?*

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Some Unexpected(ly) Open Problems

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Abstract

This note collects open questions and conjectures related to graph drawing research. In spite of their topological provenance, the first three problems are purely combinatorial problems on digraphs and words; only the last problem is a straight-forward crossing number problem.

1 Tournaments and Linear Arrangements

A *tournament* is a directed graph containing exactly one arc between any two vertices. For a tournament T on a set V of vertices let $T(u,v)$ be the number of directed paths of length 2 from u to v. A *tournament arrangement* of a graph $G = (V,E)$ is a tournament T on V, the *cost* of the arrangement is defined as

$$|E| + \sum_{uv \in E} T(u,v).$$

Tournament arrangements generalize *linear arrangements*, in which the tournaments are restricted to be linear orders. We can also view a linear arrangement as a permutation φ mapping V to $\{1,\ldots,|V|\}$; then the value of the linear arrangement is $|E| + \sum_{uv\in E} T(u,v) = \sum_{uv\in E}(T(u,v)+1) = \sum_{uv\in E} |\varphi(u) - \varphi(v)|$, which is the usual way to define the value of a linear arrangement of the vertices of G, so our definition agrees with the standard definition for tournaments which are linear orders.

Conjecture 1.1 (Pelsmajer, Schaefer, Štefankovič [5]). *The minimum cost of a tournament arrangement of a graph is achieved by a linear arrangement.*

This seems a natural conjecture to make, and its truth would have simplified the proof of **N**P-hardness for deciding the independent odd crossing number (a variant of the standard crossing number). For that proof it turned out to be sufficient to show that the conjecture holds for complete graphs:

Theorem 1.2 (Pelsmajer, Schaefer, Štefankovič [5]). *The minimum cost of a tournament arrangement of K_n is $\binom{n+1}{3}$.*

For a minimum linear arrangement of a complete graph, the order of the vertices does not matter, which makes it easy to see that the cost of a minimum linear arrangement is $\binom{n+1}{3}$, verifying the conjecture for complete graphs. The natural next step would be to verify the conjecture for simple families of graphs such as paths, cycles, and trees.

There is a very tempting avenue of attack on the conjecture: Suppose we have a tournament arrangement of a graph G which is not a linear arrangement; then the tournament must contain a directed cycle; find an arc in the tournament so that reversing the arc brings us closer to a linear arrangement (by reducing the number of directed cycles) without increasing the cost of the arrangement. This approach leads into dangerous territory: Ádám conjectured that any directed graph containing a directed cycle contains an edge that can be reversed so as to decrease the number of directed cycles; the conjecture fails for multigraphs, but its status is open for simple graphs. The attack we outlined would require settling Ádám's conjecture for tournaments: any tournament which is not a linear order contains an arc that can be reversed so as to decrease the number of directed cycles; even this variant is still open [1].

2 Edit Distance of Cyclic Words

Given two words, the *swapping distance* between the two words is the smallest number of transpositions of adjacent letters (*swaps*) that turn one word into the other. The swapping distance is a special case of the edit distance problem in which other operations (replace, insert, delete, swap) are allowed for various costs, and it is known to be solvable in time $O(n \log n)$. However, what happens if we ask for the swapping distance of cyclic words, that is words in which the first and last letter are adjacent?

The cyclic variant of the swapping distance models computing the crossing number of a multigraph on two vertices with a given rotation system (a cyclic order of the edges leaving each vertex): the two cyclic words are the orders in which the edges leave the vertices [5]. At this point one would expect to find a (dynamic programming?) algorithm to solve the cyclic swapping distance problem to calculate the crossing number for 2-vertex multigraphs. Instead the reverse is true: we know how to calculate the crossing number in polynomial time using integer programming using relaxation [4], thus giving us an algorithm for computing the swapping distance for cyclic words in which each letter occurs only once, that is, permutations. It seems rather hard to believe that an edit distance problem should require integer programming.

Conjecture 2.1. *The swapping distance of cyclic words can be computed in polynomial time.*

3 Binary Square-Free Words

Thue showed that it is possible to construct a word that does not contain any squares, that is subwords of the form ww. He even showed that an infinite square-free word can be built over the alphabet $\{0, 1, 2\}$. While binary words can be square-free, any binary word of length at least 4 must contain a square, so in particular there are no infinite binary square-free words. However, there are infinite binary words that do not contain any odd squares, that is squares ww where $|w|$ is odd. For example, $010101\ldots$ is such a word. So what about avoiding even squares, squares for which $|w|$ is even? While this problem is harder, one can define such a word $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ explicitly via

$$a_n = \lfloor n\phi \mod 2\rfloor,$$

where $\phi = (\sqrt{5} + 1)/2$ is the golden ratio [6]. Then

$$a_1 a_2 \ldots = 110001100011\ldots,$$

a sequence that is listed in Sloane's Encyclopedia of Integer Sequences [2, Sequence A085002].

What is unusual about here is that traditionally square-free words are constructed by repeatedly applying square-free morphisms (morphisms that map square-free words to square-free words) to an initial square-free word.

Question 3.1. *Is there a construction of a square-free word over a three-letter alphabet that is not based on recursively applying square-free morphisms?*

It is easy to build a square-free word over a four-letter alphabet by combining $\mathbf{a}$ and $010101\ldots$ bit by bit. There are other aspects of the sequence **a** which are intriguing; for example, the lengths of the squares that do occur are $1, 5, 21, 89, \ldots$, that is, Fibonacci numbers of the form f_{3n+2}. Numerical evidence seems to suggest that there is a relationship between the length of squares in a sequence $b_n = \lfloor n\alpha \mod 2\rfloor$, for irrational α and the numerators and denominators of the convergents of the continued fraction of α.

4 The Triviality of Adjacent Crossings

In his famous paper on algebraic aspects of the crossing number, Tutte wrote "We are taking the view that crossings of adjacent edges are trivial, and easily got rid of" [7]. Taking Tutte at his word, the following question should be trivial:

> If a graph can be drawn in the plane so that no two non-adjacent edges cross, is the graph planar?

While the answer to the question is yes, the easiest proof seems to involve the Hanani-Tutte theorem which is the stronger statement that a graph is planar if it can be drawn in the plane so that no two non-adjacent edges cross an odd number of times. Hence we ask whether our question above can be answered directly without using the Hanani-Tutte theorem. To emphasize that crossings of adjacent edges are hardly trivial, I propose the following conjecture:

Conjecture 4.1. *If a graph can be drawn in a surface so that no two non-adjacent edges cross, then the graph can be embedded in that surface.*

By the recently established Hanani-Tutte theorem for the projective plane, we know that the conjecture is true for the projective plane, but even the case of the torus remains open [3].

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Point sets with planar embeddings of cubic, connected graphs

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Abstract

Let P be a set of $n \geq 3$ points in the plane in general position with n even and let h being the number of points on the convex hull of P. We want to know in which cases P admits a planar embedding of a cubic, connected graph (a "cubic embedding")? Garca et al. proved in [1] that cubic embeddings always exists for $h \leq \frac{3}{4}n$, but for higher values of h the question remained open. We present a characterization for $h > \frac{3}{4}n$ under the assumption that cubic embeddings have to contain the cycle through the points of the convex hull.

Let $ch(P)$ denote the convex hull of P (i. e., the boundary of the minimal convex set containing P) and let a (combinatorial) edge $\in \binom{P}{2}$ be a *diagonal* of P if it joins two non-consecutive points in $ch(P)$.

We focus on the case $h > \frac{3}{4}n$ for a given point set P. If a cubic embedding on P exists, let D be the set of its diagonals. Then $D \neq \emptyset$ and diagonals in D can neither cross each other nor share an endpoint due to planarity and cubicness. We call the connected, bounded regions in which the plane is subdivided by $ch(P)$ and D *faces induced by* D.

For every face f induced by D let I_f be the set of of points strictly inside f, H_f the set of points on the boundary of f that are not endpoints of a diagonal and $N_f = H_f \cup I_f$. Let lower case letters denote the cardinality of that sets. We want to find a certain embedding on the points N_f for each induced face f such that adding the convec hull cycle and D yields a cubic embedding of P.

A planar embedding on the points N_f of a face with n_f even is called a *cage embedding*, if

- it contains no edge in $H_f \times H_f$,
- every point in I_f has degree 3,
- every point in H_f has degree 1,

[1]This research was supported by the Deutsche Forschungsgemeinschaft within the research training group "Methods for Discrete Structures" (GRK 1408).

– it contains a path from each point in I_P to some point in H_f.

Lemma 0.1 (without proof)**.** *For every face f induced by D with n_f even holds:*

$$\begin{aligned} \textit{for } h_f = 0: &\qquad i_f = 0 \Leftrightarrow \exists \textit{ cage embedding on } N_f && (1)\\ \textit{for } \frac{n_f}{2} + 1 \leq h_f: &\qquad h_f \leq \frac{3}{4} n_f \Leftrightarrow \exists \textit{ cage embedding on } N_f && (2)\\ \textit{always}: &\qquad h_f \leq \frac{3}{4} n_f \Leftarrow \exists \textit{ cage embedding on } N_f && (3) \end{aligned}$$

(This is also true in the more general context of H_f being a subset of the convex hull of a point set.)

Now we state how point sets can be characterized that admit a cubic embedding containing the convex hull cycle.

Theorem 0.2. *The following statements are equivalent (for arbitrary h):*

- *There is a cubic, connected, planar embedding on P containing $ch(P)$.*
- *$h \leq \frac{n}{2} + 1$ or there is a set D of diagonals of P not crossing each other nor sharing a point such that for every face f induced by D holds:*
  - *n_f even*
  - *Either $h_f = n_f = 0$,*
    $h_f = 3$ and $n_f = 4$ or
    $\frac{n_f}{2} + 1 < h_f \leq \frac{3}{4} n_f$.

The set D is obtained by every cubic embedding on P that contains $ch(P)$ and has the minimal number of diagonals. Moreover, if such a cubic embedding exists, there is even a 2-connected one.

Proof. We focus on the case $h > \frac{3}{4} n$ for the proof, as the other case was proved in [1].

$\Leftarrow$: For each face f we apply Lemma 0.1.(1) to N_f if $h_f = 0$ and Lemma 0.1.(2) otherwise, obtaining a cage embedding in both cases. With the addition of $ch(P)$ this yields a cubic embedding on P.

$\Rightarrow$: We choose the cubic embedding with a set D of diagonals that is minimal in size. Once again, no diagonals cross or share a point. For every face f induced by D we look at the geometric graph G_f induced by N_f without edges on the face boundary. Then G_f is planar, vertices in H_f

have degree 1, vertices in I_f degree 3 and there exists a path from every vertex in I_f to a vertex in H_f if $n_f > 0$. In particular, n_f is even, since G_f has only odd vertices and every graph has an even number of odd vertices. That concludes G_f to be a cage embedding on N_f.

Moreover, for every face holds

- $h_f \leq \frac{3}{4}n_f$: Otherwise, there would be an edge joining two non-consecutive points in H_f, forming an additional diagonal of P.

- $h_f = 0 \Rightarrow n_f = 0$: Using Lemma 0.1.(1).

- $h_f > 0 \Rightarrow h_f = 3$ and $n_f = 4$ or $\frac{n_f}{2} + 1 < h_f$: Suppose to the contrary that ($h_f \neq 3$ or $n_f \neq 4$) and $h_f \leq \frac{n_f}{2} + 1$. There must be at least one further internal face and we glue f with an arbitrary adjacent face g by deleting the separating diagonal. Then for the newly created face z holds $h_z = h_f + h_g + 2 \leq \frac{n_f}{2} + 3 + \frac{3}{4}n_g$ and $\frac{3}{4}n_z = \frac{3}{4}(n_f + n_g + 2)$.

  We want the new face to preserve the property $h_z \leq \frac{3}{4}n_z$, which by inserting both terms is true if $n_f \geq 6$. From $h_f > 0$ follows that $i_f > 0$, since otherwise one vertex in H_f has degree 2. In addition, n_f must be even, hence the only remaining cases in which the property might not hold are

  - $n_f = 2$: This implies with $i_f > 0$ that $h_f = i_f = 1$ with the vertex in I_f having degree ≤ 1, contradicting the cubicity of the embedding.
  - $n_f = 4$: Then $h_f \neq 3$ implies $i_f \geq 2$ and $h_f \leq 2$. In that case all vertices in I_f except possibly one with degree 3 have degree ≤ 2, contradicting again the cubicity.

  Thus, in all cases $h_z \leq \frac{3}{4}n_z$ holds and n_z is even, since we added exactly 2 vertices. If for the new face $h_z > \frac{n_h}{2} + 1$ holds, Lemma 0.1.(2) gives a cage embedding on N_z which contradicts the minimality of D. Otherwise, we iterate the procedure by merging z to an adjacent face until we eventually either find again a contradiction to the minimality or no adjacent faces are left. In the latter case we end up with the face f of the convex hull cycle which by assumption has $h > \frac{3}{4}n$. But we proved that merged faces f always have $h_f \leq \frac{3}{4}n_f$, yielding a contradiction.

□

I would like to thank Pavel Valtr for introducing me to that problem and raising the right questions at the right time.

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Semilattice Polymorphisms on Reflexive Graphs

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Joint work with Pavol Hell.

A graph is *reflexive* if it has a loop at each vertex. We denote reflexive graphs with bold letters: $\mathbf{G}$, and let G be the (simple) irreflexive graph which we get from removing all loops.

A binary operation

$$\wedge : V \times V \to V : (u, v) \mapsto u \wedge v$$

on a set V is called *semilattice* if it is idempotent ($v \wedge v = v$ for all $v \in V$), commutative, and associative. It is called 'semilattice' because it is equivalently defined by the semilattice ordering $\leq$ of V given by

$$u \leq v \iff u \wedge v = u.$$

A semilattice operation $\wedge$ on the vertex set $V(\mathbf{G})$ of a (reflexive) graph $\mathbf{G}$ is a *semilattice polymorphism* of $\mathbf{G}$ if it further satisfies

$$u \sim u', v \sim v' \Rightarrow u \wedge v \sim u' \wedge v', \tag{1}$$

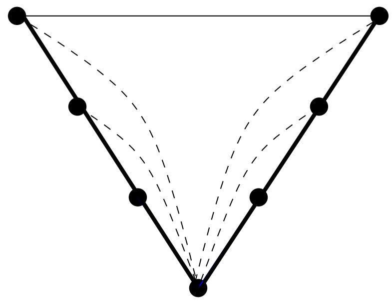
where '$\sim$' denotes adjacency in $\mathbf{G}$.

A semilattice polymorphism is an example of a totally symmetric idempotent polymorphism, a central object in the study of the complexity of constraint satisfaction problems. It is a rather useful example, as it 'can be drawn' via its ordering $\leq$: the *Hasse diagram* of $\leq$ is the digraph H on $V(\mathbf{G})$ defined by letting $u \to v$ if u covers v:

$$v \leq u, v \neq u, \text{ and } x \leq u \Rightarrow x \leq v.$$

A Hasse diagram of a partial ordering is usually drawn with all arcs pointing down on the paper.

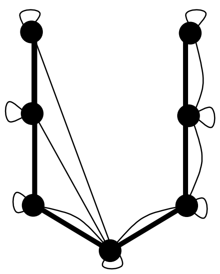
Given a reflexive graph $\mathbf{G}$ with a semilattice polymorphism, there is some interaction between the edges of $\mathbf{G}$ and the arcs of its Hasse diagram H. For example, in the following diagram, where the dark edges are Hasse edges. if the light solid edge is in $\mathbf{G}$, then it follows from (1) that the dotted edges must also be in $\mathbf{G}$.



The Hasse diagram, and its interaction with the graph **G**, gives natural specialisations of semilattice polymorphisms. A semilattice polymorphism of a reflexive graph **G** is called

- *embedded* if every arc of the Hasse diagram is an edge of **G**,
- *skeletal* if every edge of **G** is between vertices comparible by $\leq$, (connected by a directed path in the Hasse diagram,) and
- *a tree polymorphism* if the Hasse diagram is a tree.

For example, the figure below, where the light edges are the edges of a graph, and the dark edges make up the Hasse diagram of a semilattice ordering of the vertices, represents a non-embedded skeletal tree polymorphism of a reflexive graph.



An embedded skeletal semilattice polymorphism of a reflexive graph is necessarily a tree polymorphism, but apart from that, the above specialisations are independent for reflexive graphs.

However when we look at the classes of graphs admitting the various semilattice polymorphisms, many of these specialisations become redundant, and we get a proper heirarchy:

$$\begin{array}{c}\text{Graphs admitting semilattice polymophisms}\\ \bigcap\\ \text{Graphs admitting embedded semilattice polymorphisms}\\ \bigcap \neq\\ \text{Graphs admitting (embedded) tree polymorphisms}\\ \bigcap \neq\\ \text{Graphs admitting skeletal semilattice polymorphisms}\\ =\\ \text{Graphs admitting skeletal embedded tree polymorphisms}\end{array}$$

We refine this heirarchy and get relations to known classes of graphs. For example, it turns out that $\mathbf{G}$ admits a skeletal embedded tree polymorphism if and only if G is chordal, and the leafage of a chordal graph is the minimum number of leafs in the Hasse diagram of a (not necessarily embedded) skeletal tree polymorphism of $\mathbf{G}$.

Edge reconstruction of hypergraphs and a result of Berge and Rado

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Abstract

Berge and Rado (1972) constructed pairs of non-isomorphic hypergraphs that have the same collection of edge-deleted sub-hypergraphs, and characterized such pairs. In this talk, I will consider a generalization of hypergraph edge reconstruction under the action of a group. The question of interest is: what part of the characterization given by Berge and Rado generalizes to reconstruction under group action? I will present a conjecture and a linear algebraic approach to it. The work was motivated by a problem in population genetics.

1 A hypergraph reconstruction problem

We first define hypergraphs and their isomorphisms slightly differently for our purpose. Here we would like to consider *edge-labelled* hypergraphs, i.e., hypergraphs as tuples of subsets of a ground set rather than families of subsets of a ground set. We will then require an isomorphism between two hypergraphs to preserve the edge order. We will call it a *strong isomorphism*.

In the following, we denotes $\{1, 2, \ldots, m\}$ by $[m]$, the power set of a set X by 2^X, and the set of all subsets of X of cardinality k by $\binom{X}{k}$.

Definition 1.1 (Hypergraphs). Let X be a finite set and k, m non-negative integers. Then an *m-edge hypergraph* is a map $H : [m] \to 2^X$. A *simple m-edge hypergraph* is a one-one map $H : [m] \to 2^X$. A *k-uniform, m-edge hypergraph*, referred to as an (m, k)-hypergraph, is a map $H : [m] \to \binom{X}{k}$.

Next we consider a group G acting on the ground set X, and define G-equivalence (or G-isomorphism) between hypergraphs defined on X.

[1]Supported by the EPSRC grant "From Population Genomes to Global Pedigrees"

Definition 1.2 (**G-equivalence of hypergraphs**)**.** Let H_1 and H_2 be two m-edge hypergraphs on the same ground set X. Let G be a group acting on X from the left. We say that H_1 and H_2 are *G-equivalent* (and write it as $H_1 \sim_G H_2$) if there is permutation π of $[m]$ and there exists $g \in G$ such that $H_2(\pi(i)) = g(H_1(i)) := \{gv|v \in H_1(i)\}$ for all $i \in [m]$. We say that H_1 and H_2 are *strongly G-equivalent* (and write it as $H_1 \cong_G H_2$) if there is $g \in G$ such that $H_2(i) = g(H_1(i)) := \{gv|v \in H_1(i)\}$ for all $i \in [m]$. In this case, we write $H_2 = gH_1$, and $gH_1(i)$ instead of $(gH_1)(i)$ or $g(H_1(i))$.

Observe that when G is the symmetric group of permutations of X, a (strong) G-equivalence between hypergraphs is the standard (strong) isomorphism between them.

Let H be an m-edge hypergraph. Let $I \subseteq [m]$. We call the restriction of H to I the sub-hypergraph induced by I, and denote it by $H[I]$. When $I = [m]\backslash\{i\}$, we sometimes write $H\backslash\{i\}$ instead of $H[I]$. When $[m] = I|J$ is a partition of $[m]$, we write $H[\Diamond I] := \cap_{i\in I} H(i) \backslash \cup_{j\in J} H(j)$ (which is called the atom of I).

In [1] and [8] the following result was proved.

Proposition 1.3. *For each $m \geq 3$, there exist (m,k)-hypergraphs H_1 and H_2 on the vertex set X such that $H_1\backslash\{i\} \cong H_2\backslash\{i\}$ for all $i \in [m]$, but $H_1 \ncong H_2$, (in fact $H_1 \not\sim H_2$). In this case, $|X| \geq 2^{m-1}$ and $k \geq 2^{m-2}$.*

Proof. (Sketch) Let $X := 2^{[m]}$. Let $X_1 := \{A \subseteq [m]||A| \equiv 0 \pmod m\}$ and $X_2 := \{A \subseteq [m]||A| \equiv 1 \pmod m\}$ be the vertex sets of H_1 and H_2, respectively. Let $H_1(i) := \{A \subseteq [m]|i \in A, |A| \equiv 0 \pmod m\}$, and $H_2(i) := \{A \subseteq [m]|i \in A, |A| \equiv 1 \pmod m\}$. It can be verified that $H_1\backslash\{i\} \cong H_2\backslash\{i\}$ for all $i \in [m]$, but $H_1 \ncong H_2$. Further it can be shown that the graphs H_1 and H_2 are extremal, and that any two hypergraphs H_1' and H_2' on vertex sets X_1' and X_2', respectively, that satisfy the conditions of the proposition arise from the above construction, i.e., there exist $X_1 \subseteq X_1'$ and $X_2 \subseteq X_2'$ such that the restrictions of H_1' to X_1 and H_2' to X_2 are the graphs H_1 and H_2 (or H_2 and H_1) described above. □

In [8] we used the above hypergraphs to construct pedigrees (family histories of populations represented as directed graphs) that cannot be reconstructed from their proper sub-pedigrees (restrictions of family histories to subsets of the living population). The following conjecture and the reconstruction problem that it addresses were motivated by conjectures about a characterization of non-reconstructible family histories (see Conjectures 1 and 2 in [8]).

Conjecture 1.4. *Let X be a finite set and let G be a finite group acting on X. Let H_1 and H_2 be two (m,k)-hypergraphs on the ground set X. Suppose that $H_1\backslash\{i\} \cong_G H_2\backslash\{i\}$ for all $i \in [m]$, but $H_1 \not\cong_G H_2$. Then $k \geq 2^{m-2}$.*

When G is the symmetric group on X, the conjecture states the result of Berge and Rado (the Corollary on page 236 in [1]) and Proposition 1.3, and the constructions follow simply from the observation that two hypergraphs H_1 and H_2 are strongly isomorphic if and only if for all $I \subseteq [m]$, we have $|H_1(\Diamond I)| = |H_2(\Diamond I)|$. But for other groups, the above conjecture is probably significantly more difficult to prove than the results of Berge and Rado and the constructions in [8].

One can verify that pedigrees and their associated reconstruction problems can be viewed as suitably defined hypergraphs and an appropriately chosen group acting on them.

2 A linear algebraic approach to Conjecture 1.4

We define a matrix Δ_m^k as follows. The rows and the columns of Δ_m^k are indexed by (m,k)-hypergraphs H_i. The (i,j)-th entry of Δ_m^k is defined as follows.

$$\Delta_m^k[H_i, H_j] = \begin{cases} m & : \text{ if } \quad i = j, \\ 1 & : \text{ if } \quad (i \neq j) \wedge (\exists p | H_i\backslash\{p\} = H_j\backslash\{p\}), \\ 0 & : \text{ otherwise.} \end{cases}$$

In other words, $\Delta_m^k[H_i, H_j]$ counts the number of ways to redefine H_i at exactly one point so as to obtain H_j. We emphasize that there is an equality $(H_i\backslash\{p\}) = H_j\backslash\{p\})$, and not just an isomorphism $(H_i\backslash\{p\}) \cong_G H_j\backslash\{p\})$ in the second line above.

Now consider an (m,k)-hypergraph H. Let $\texttt{Aut}(H)$ denote its automorphism group (in the sense of G-equivalence), that is,

$$\texttt{Aut}(H) := \{g \in G \,|\, gH = H\}.$$

Now define a vector Y_H defined as follows. As with the matrix Δ_m^k, the entries of Y_H are indexed by the (m,k)-hypergraphs H_i (in the same order as the (m,k)-hypergraphs that index the columns and rows of Δ_m^k). The entries of Y_H are defined as follows.

$$Y_H[H_i] = \begin{cases} |\mathtt{Aut}(H)| & : \text{ if } \quad H_i \cong_G H, \\ 0 & : \text{ otherwise.} \end{cases}$$

Proposition 2.1. *Let H_1 and H_2 be two (m,k)-hypergraphs such that $H_1\backslash\{i\} \cong_G H_2\backslash\{i\}$ for all $i \in [m]$. Then $\Delta_m^k Y_{H_1} = \Delta_m^k Y_{H_2}$. Moreover, if $H_1 \ncong_G H_2$, then we have $\Delta_m^k Y = 0$ for the non-zero vector $Y := Y_{H_1} - Y_{H_2}$.*

Proof. The j'th entry of $\Delta_m^k Y_{H_1}$ is given by

$$\begin{aligned} (\Delta_m^k Y_{H_1})[H_j] &= m|\{g \in G \,|\, gH_1 = H_j\}| \\ &\quad + |\{g \in G \,|\, (\exists i|H_j\backslash\{i\} = (gH_1)\backslash\{i\}) \wedge (H_j \neq gH_1)\}| \\ &= \sum_{i=1}^{m} |\{g \in G \,|\, (gH_1)\backslash\{i\}) = H_j\backslash\{i\})\}| \\ &= \sum_{i=1}^{m} |\{g \in G \,|\, g(H_1\backslash\{i\}) = H_j\backslash\{i\})\}| \\ &= \sum_{i=1}^{m} |\mathtt{Aut}\,(H_1\backslash\{i\})\,| \end{aligned}$$

Therefore, if $H_1\backslash\{i\} \cong_G H_2\backslash\{i\}$ for all $i \in [m]$, then $\Delta_m^k Y_{H_1} = \Delta_m^k Y_{H_2}$. Then the second part of the proposition is only a re-statement of $H_1 \ncong_G H_2$ (implying $Y_{H_1} \neq Y_{H_2}$) and $\Delta_m^k Y_{H_1} = \Delta_m^k Y_{H_2}$. ☐

Conjecture 2.2. *If $k < 2^{m-2}$, then Δ_m^k has full rank. In particular, in this case, there are no hypergraphs H_1 and H_2 such that $H_1 \ncong_G H_2$ but $H_1\backslash\{i\} \cong_G H_2\backslash\{i\}$ for all $i \in [m]$.*

Remark 2.3. Observe that the group G under consideration does not play any role in the definitions of Δ_m^k, so the rank or other properties of Δ_m^k do not depend on G (but depend only on $|X|$, m and k). Therefore, studying Δ_m^k would prove part of Conjecture 1.4 in a group-independent manner. Matrices that define incidence relations between subsets of a set have been studied in design theory, and have also been used in graph reconstruction theory, and they do yield certain results that may be interpreted in a group-independent manner. In particular, the well known results of Lovász [3] and Nash-Williams [4] on the edge reconstruction conjecture have been proved in similar frameworks. See for example [2, 5, 6, 7]. In fact, Lovász's edge reconstruction result (that graphs on n vertices having more than $\frac{1}{2}\binom{n}{2}$ edges are edge reconstructible) does not depend on the group, and can be proved to follow from the invertibility of a suitably defined incidence matrix [6]. On the other hand, the result of Nash-Williams (which is somewhat technical to state here) may be proved as a characterization of non-zero vectors in the null space of the incidence matrix, when Lovász's $\frac{1}{2}\binom{n}{2}$ bound is not satisfied [5]. We would like to approach Conjecture 2.2 either by a direct proof or via a characterization of vectors in the null space of Δ_m^k (i.e. by proving a result analogous to the result of Nash-Williams on edge reconstruction).

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On triconnected and cubic plane graphs

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Joint work with Alfredo García, Ferran Hurtado, Clemens Huemer, and Javier Tejel.

Abstract

Let S be a set of $n \geq 4$ points in general position in the plane, and let $h < n$ be the number of extreme points of S. We construct a 3-connected plane graph with vertex set S, having $\max\{\lceil 3n/2 \rceil, n + h - 1\}$ edges, and we prove that there is no 3-connected plane graph on top of S with a smaller number of edges. In particular, this implies that S admits a 3-connected cubic plane graph if and only if $n \geq 4$ is even and $h \leq n/2 + 1$. The same bounds also hold when 3-edge-connectivity is considered. We also give a partial characterization of the point sets in the plane that can be the vertex set of a cubic plane graph. A polynomial-time algorithm for deciding the existence of a cubic palne graph on a given point set has been recently discovered by J. M. Schmidt and P. Valtr. In the workshop talk we also discussed the extensions of the results to 4-connectivity (work in progress).

1 Introduction

1.1 Preliminaries and previous work

A *geometric graph* G is a a simple finite graph whose vertex set $V(G)$ is a finite set of points in general position in the plane (i.e., no three of them are collinear), and each edge in $E(G)$ is a closed segment whose endpoints belong to $V(G)$. If $V(G) = S$ we also say that the geometric graph G *is on top of* S, or simply that G *is on* S. A geometric graph is a *plane graph* if no two edges cross. That is, two edges in a plane graph may intersect only at a common endpoint. It is also usual to use the expressions *non-crossing geometric graph* or *crossing-free geometric graph* as synonymous for *plane graph*. A (geometric) graph is *cubic*, if the degree of every vertex is three. A

[1]This abstract is based on the joint paper [9].

(geometric) graph on at least $k+1$ vertices is *k-connected* if it is connected and it remains connected whenever $k-1$ vertices are removed.

Problems on geometrically embedding planar graphs on given point sets have been attracting attention for almost two decades. Ikebe et al. [12] proved that a tree can always be drawn with a prescribed root, culminating previous results by Perles and by Pach and Törőcsik [18], and a similar result with prescribed degrees is given in [21]. Kaneko and Kano [15] obtained an extension to two trees. The fact that outerplanar graphs are the largest graph class always admitting embeddings was proven by Gritzmann et al. [11]. Several papers have been devoted to the algorithmic counterpart of these results [3, 2, 14, 19]. We refer the reader to the books [5, 4] for more details on geometric graphs and on graph drawing algorithms.

For any set S of n points in general position in the plane it is easy to construct a connected plane graph on top of S, even with the additional requirement that it has minimum possible number of edges, $n-1$. For example, we may take the minimum spanning tree of S or we may connect the points by a path visiting the points of S in lexicographically increasing order, say, of their coordinates. Similarly, it is also not difficult to construct a 2-connected plane graph on top of S with the minimum number, n, of edges: We can construct a *polygonization of S*, i.e., a simple polygon whose vertex set is S. Methods yielding polygonizations were described by Steinhaus and by Gemignani [20, 10] and later systematically studied in the field of computational geometry.

On the opposite direction, there are point sets that do not admit any 4-connected plane graph on top of them. Some examples are given by Dey et al. in the paper [7], where they also provide a necessary and sufficient condition for point sets whose convex hull consists of exactly three vertices. However, a general characterization of the sets of points admitting a 4- or 5-connected plane graph is not known [7].

For the case of 3-connectivity this characterization is quite obvious and was described in [7] as well. Let us recall that we say that a point set S is *in convex position* if each point of S is *extreme* (a vertex of the convex hull of S). If S is in convex position, then every plane triangulation of S contains vertices of degree two, therefore it is impossible to get any 3-connected plane graph on top of S. On the contrary, when S is not in convex position, it is easy to check that the following method produces a 3-connected plane graph on S: Let C be the cycle formed by the edges connecting consecutive vertices of the convex hull of S and let $v \in S$ be any point interior to the convex hull; join v to all the vertices in C and then insert iteratively the remaining points. At each step the point being inserted is connected to the three vertices of the triangular face it falls into.

Notice that in general this algorithm does not produce a 3-connected plane graph using as few edges as possible (see Figure 1, left). In fact, it always produces a *triangulation of* S, i.e., a plane graph on S with the maximum number of edges, in which all faces are triangles with the only possible exception of the outer face.

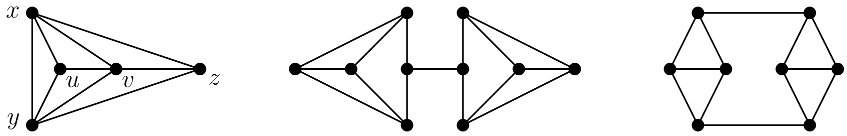

Figure 1: Left: Starting with xy, yz, zx, vx, vy, vz, vertex u is inserted and joined to x, y and v. The resulting graph remains 3-connected if the edge xy is suppressed. Middle and Right: Connected cubic plane graphs which are not 3-connected.

In the paper [9] we aim to the minimality of the construction, as was already known for 1- and 2-connectivity, and we describe a polynomial algorithm which, given a point set S not in convex position, finds a 3-connected plane graph on S with the minimum number of edges. Achieving good connectivity by adding as few edges as possible is a classic family of problems in graph theory; we refer the interested reader to the large literature on augmentation problems [1, 6, 8, 13, 16, 17, 23, 24].

Another natural and related problem that we consider here is that of characterizing the point sets that admit a cubic plane graph. Observe that a connected cubic graph on top of S is not necessarily 3-connected (see Figure 1, middle and right); therefore, a specific approach is required. The analogous problem of constructing 1- or 2-regular plane graphs is easily solved using a polygonization on S mentioned above — the edges of a simple polygon P on S form a 2-regular plane graph and, if n is even, taking every second segment in P (or in any plane Hamiltonian path on S) gives a 1-regular plane graph on S.

1.2 Results

Throughout this abstract, S denotes a set of $n \geq 4$ points in general position in the plane, $H = H(S)$ denotes the set of vertices of the convex hull of S,

$h = h(S)$ denotes the size of H, and $I = I(S) = S \setminus H$ denotes the set of interior points of S.

Here is our main result:

Theorem 1.1. *Let S be a set of n points in general position in the plane. Suppose that S is not in convex position. Then there is a 3-connected plane graph on S with $\max\{\lceil 3n/2 \rceil, n + h(S) - 1\}$ edges, and it can be found in polynomial time. Moreover, there is no 3-connected plane graph on S with a smaller number of edges.*

Theorem 1.1 immediately gives the following characterization of sets admitting 3-connected cubic plane graphs:

Corollary 1.2. *Let S be a set of $n \geq 4$ points in general position in the plane. Then there is a 3-connected cubic plane graph on S if and only if n is even and $h(S) \leq n/2 + 1$.*

A (geometric) graph on at least $k + 1$ vertices is *k-edge-connected* if it is connected and it remains connected whenever $k - 1$ edges are removed. The above results hold also for 3-edge-connectivity:

Theorem 1.3. *The statements of Theorem 1.1 and Corollary 1.2 also hold when 3-edge-connectivity is considered instead of 3-connectivity.*

If we focus on connecting the points of the set S by a cubic plane graph, without the additional requirement of 3-connectivity, the situation changes substantially. Of course, we need that n, the number of points of S, is even. Our main result in this topic is as follows:

Theorem 1.4. *Let $n \geq 4$ be an even integer. Then, we have:*

(i) Any set S of n points in general position in the plane satisfying $h(S) \leq 3n/4$ admits a cubic 2-connected plane graph on S.

(ii) If h is an integer such that $3n/4 < h < n - 1$, then among sets S of n points in general position with $h(S) = h$, at least one set admits a cubic 2-connected plane graph on S and at least one set admits no cubic plane graph on S.

(iii) Sets S of n points with $h(S) \geq n - 1$ admit no cubic plane graph on S, with the only exception the case $|S| = n = 4$ with $h(S) = n - 1 = 3$.

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Augmenting undirected node-connectivity by one

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Abstract

We present a min-max formula for the problem of augmenting the node-connectivity of a graph by one and give a polynomial time algorithm for finding an optimal solution.

An undirected graph $G = (V, E)$ is **k-node-connected**, or shortly, **k-connected** if $|V| \geq k + 1$ and after the deletion of any set of at most $k-1$ nodes, the remaining graph is still connected. By Menger's well-known theorem, a graph is k-connected if and only if it contains k openly disjoint paths between any two nodes. The node connectivity augmentation problem consists of finding a minimum number of edges whose addition to a given graph G results in a k-connected graph. The complexity of this problem is a longstanding open question. In this talk we give a min-max formula and a polynomial time algorithm for augmenting connectivity by one, the special case when the input graph G is already $(k-1)$-connected.

Besides node-connectivity, one may study edge-connectivity as well, and both augmentation problems may also be asked for directed graphs. The other three among these four basic connectivity augmentation problems were solved beforehand: undirected edge-connectivity by Watanabe and Nakamura [6], directed edge-connectivity by Frank [2], and directed node-connectivity by Frank and Jordán [3].

For the undirected node-connectivity version, the best previously known result is due to Jackson and Jordán [5]. They gave a polynomial time algorithm for finding an optimal augmentation for any fixed k. The running time is bounded by $O(|V|^5+f(k)|V|^3)$, where $f(k)$ is an exponential function of k.

Let us now formulate our theorem, conjectured by Frank and Jordán in 1994. In the $(k-1)$-connected graph $G = (V, E)$, a subpartition

[1]Supported by the Hungarian National Foundation for Scientific Research (OTKA), grants K60802 and NK67867.

$X = (X_1, \ldots, X_t)$ of V with $t \geq 2$ is called a **clump** if $|V - \cup X_i| = k - 1$ and $d(X_i, X_j) = 0$ for any $i \neq j$. The sets X_i are called the **pieces** of X while $|X|$ is used to denote t, the number of pieces. If $t = 2$ then X is a **small clump**, while for $t \geq 3$ it is a **large clump**. An edge $uv \in V^2$ **connects** X if u and v lie in different pieces of X. Two clumps are said to be **independent** if there is no edge $uv \in V^2$ connecting both.

A **bush** $\mathcal{B}$ is a set of pairwise different small clumps, so that each edge in V^2 connects at most two of them. A **shrub** is a set consisting of pairwise independent (possibly large) clumps. For a bush $\mathcal{B}$ let $def(\mathcal{B}) = \left\lceil \frac{|\mathcal{B}|}{2} \right\rceil$, and for a shrub $\mathcal{S}$ let $def(\mathcal{S}) = \sum_{K \in \mathcal{S}} (|K| - 1)$.

A **grove** is a set consisting of some (possibly zero) bushes and one (possibly empty) shrub, so that the clumps belonging to different bushes are independent, and a clump belonging to a bush is independent from all clumps belonging to the shrub. For a grove Π consisting of the shrub $\mathcal{B}_0$ and bushes $\mathcal{B}_1, \ldots, \mathcal{B}_\ell$, let $def(\Pi) = \sum_i def(\mathcal{B}_i)$. For a $(k-1)$-connected graph $G = (V, E)$, let $\tau(G)$ denote the minimum number of edges whose addition makes G k-connected, and let $\nu(G)$ denote the maximum value of $def(\Pi)$ over all groves Π.

Theorem 1.1. *For a $(k-1)$-connected graph $G = (V, E)$ with $|V| \geq k+1$, $\nu(G) = \tau(G)$.*

Both the proof and the algorithm are motivated by the algorithm given by Frank and the author [4] for augmenting directed node-connectivity by one. Let us now state the min-max formula for this problem. In a digraph $D = (V, A)$, an ordered pair (X^-, X^+) of disjoint non-empty subsets of V is called a **one-way pair** if $|V - (X^- \cup X^+)| = k - 1$ and there is no arc in A from X^- to X^+. $uv \in V^2$ **covers** (X^-, X^+) if $u \in X^-$, $v \in X^+$ and two pairs are **independent** if they cannot be covered by the same arc.

Theorem 1.2 ([3])**.** *The minimum number of arcs whose addition to the $(k-1)$-connected digraph $D = (V, A)$ with $|V| \geq k+1$ results in a k-connected digraph equals the maximum number of pairwise independent one-way pairs.*

We give a brief outline of the argument of [4]. A natural partial order $\preceq$ may be defined on the set of one-way pairs. A subset $\mathcal{K}$ of one-way pairs is called **cross-free** if any two non-independent pairs in $\mathcal{K}$ are comparable with respect to $\preceq$; such a $\mathcal{K}$ maximal for inclusion is called a **skeleton**. The two main ingredients of the proof are as follows: (i) for a cross-free $\mathcal{K}$, the maximum number of pairwise independent one-way pairs in $\mathcal{K}$ and an arc set F of the same cardinality covering all pairs in $\mathcal{K}$ may be determined using Dilworth's theorem on finding a maximum antichain and a minimum chain cover of a poset; (ii) an arc set F covering all one-way pairs in a skeleton $\mathcal{K}$ may be transformed to an arc set F' of the same cardinality covering every one-way pair in D.

Our proof for Theorem 1.1 will roughly follow the same lines. Although no natural partial order can be defined on the set of clumps, nestedness may be defined as a natural notion analogous to comparability: a cross-free system will be a set of clumps so that any two non-independent clumps are nested. For a cross-free $\mathcal{K}$ we will be able to determine an edge set F covering all clumps in $\mathcal{K}$ and a grove consisting of a shrub and bushes of clumps in $\mathcal{K}$ with deficiency $|F|$. Instead of Dilworth's theorem, we apply a reduction to Fleiner's theorem [1] on covering a symmetric poset by symmetric chains. For part (ii), the argument of [4] may be adapted with minor modifications.

While Dilworth's theorem can be derived from the Kőnig-Hall theorem on finding a maximum matching in bipartite graphs, Fleiner's theorem may be deduced from the Berge-Tutte theorem on the size of a maximum matching in general graphs. The relation between the directed and undirected connectivity augmentation problems is somewhat analogous: for example, the formula in Theorem 1.1 involves parity. This is the reason why the strikingly simple proof of Frank and Jordán for Theorem 1.2 cannot be adapted for the undirected case.

Another difficulty is that in contrast to one-way pairs, clumps may have more than two pieces. Fortunately, it turns out that the clumps of size at least three are nested with every other clump they are dependent with. Therefore, although such clumps will cause certain difficulties in the first part of the proof, they play only little role in the second part.

For the algorithm, we are going to construct a subroutine determining the dual optimum value $\nu(G)$ for a $(k-1)$-connected graph G. This gives rise to the following simple algorithm for finding an optimal augmenting edge set based on Theorem 1.1. First compute $\nu(G)$, and let $J = V^2 - E$ be the complement of E. In each step choose an edge $e \in J$, compute $\nu(G+e)$, and remove e from J. If $\nu(G+e) = \nu(G) - 1$ then add e to E, otherwise keep the same G. Note that Theorem 1.1 ensures the existence of an edge e with $\nu(G+e) = \nu(G) - 1$.

The full version of the paper is available in as EGRES Technical Report 2009-10 at http://www.cs.elte.hu/egres.

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Domination number of cubic graphs with large girth

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For a graph $G = (V, E)$ we say a set of vertices D is *dominating* if every vertex which is not contained in D has a neighbor in D. The *domination number* $\gamma(G)$ is the smallest size of a dominating set of G.

Löwenstein and Rautenbach [1] showed that every n-vertex cubic graph with girth at least $g \geq 5$ contains a dominating set of size at most

$$\left(\frac{44}{135} + \frac{82}{135g}\right) n \approx 0.325926n + O\left(n/g\right).$$

The bound was further improved by Rautenbach and Reed [2] to $0.321216n + O\left(n/g\right)$. We improve these results by establishing the following.

Theorem 1.1. *Every n-vertex cubic graph with girth at least g has the domination number $\gamma(G)$ at most $0.299871n + O(n/g) < 3n/10 + O(n/g)$.*

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