On necessary efficient solutions in interval multiobjective linear programming

Milan Hladík

Charles University, Faculty of Mathematics and Physics,

Malostranské nám. 25, 118 00, Prague, Czech Republic,

e-mail: `milan.hladik@matfyz.cz`.

Abstract

We investigate multiobjective linear programming problems with objective coefficients varying inside given intervals. A feasible solution x^* is called necessary efficient if it is efficient for all realizations of the interval coefficients. We show that the problem of testing necessary efficiency is NP-hard even in the case when x^* is a non-degenerate basic solution. If we are given only one objective then the problem is polynomially solvable as long as x^* is a non-degenerate basic solution, but it is NP-hard in the general case. Since the problem considered is computationally expensive, we propose one sufficient and also one necessary condition for necessary efficiency; this may significantly speed up the algorithms for testing necessary efficiency. We demonstrate usage of both conditions on illustrative examples.

Keywords: *Multiobjective linear programming, interval matrix, efficient solution, NP-completeness.*

1 Introduction

In many real-life situations we come across problems with imprecise input values. Imprecisions are dealt with by various ways. One of them is interval based approach in which we model imprecise quantities by intervals, and suppose that the quantities may vary independently and simultaneously within their intervals.

In this paper, we investigate multiobjective linear programming (MOLP) problems in which objective function coefficients perturb within prescribed intervals. Interval MOLP was investigated by many authors using different approaches. An overview on interval MOLP was given by Oliveira & Antunes [15]. Solving interval MOLP via preference ordering between intervals for was considered e.g. in [5, 20]. Possible and/or necessary efficiency was studied e.g. in [4, 9, 10, 11, 12, 13, 22]. More general approach involving uncertainties not only in the objective function coefficients but also in the constraints was considered in [16].

There are problems closely related to interval MOLP. Wang & Wang [21] reduced fuzzy MOLP problems to parametric interval MOLP ones. Benjamin [3] employed interval multi-objective programming for solving real-valued multi-objective decision problems by a branch & bound method; an application in robot controlling is studied in [2].

Notion of necessary efficiency is probably the most important concept of solution to interval MOLP since it ensures that a feasible point considered is efficient for all realizations of interval data. Some basic properties and theoretical foundations for necessary efficiency were discussed in [4, 13, 15]. A branch & bound implicit enumeration algorithm for testing necessary efficiency of a non-degenerate basic solution was proposed by Bitran [4], and later improved by Ida [10]. An application to portfolio selection problem can be found in Ida [11, 12].

The paper is organized as follows. In Section 3 we give a novel characterization of necessary efficiency, and in Section 4 we present an extension of sufficient condition by Bitran [4] and a new necessary condition. Complexity of testing necessary efficiency is discussed in Section 5; we show that—in essence—testing necessary efficiency is a hard problem and cannot be solved effectively in polynomial time in general.

Throughout the paper, $A_{i\bullet}$ denotes the i-th row of a matrix A, and e a vector of ones (with convenient dimension). A diagonal matrix with entries $z_1, \dots, z_n$ is written as $\mathrm{diag}(z)$.

2 Preliminaries

A multiobjective linear programming (MOLP) problem reads

$$\max_{x \in \mathcal{M}} \; Cx, \tag{1}$$

where the feasible set $\mathcal{M} := \{x \in \mathbb{R}^n \mid Ax \le b\}$, $C \in \mathbb{R}^{s \times n}$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. A feasible solution x^* to (1) is called *efficient* if there is no $x \in \mathcal{M}$ such that $Cx \ge Cx^*$ with at least one strict inequality; we denote it briefly $Cx \gneqq Cx^*$.

Efficiency of points may be characterized by tangent and normal cones [14, 18]. The tangent cone of $\mathcal{M}$ at the point x^* is defined

$$\mathcal{T}(x^*) := \{x \in \mathbb{R}^n \mid A_P x \le 0\},$$

where $P := \{i \mid A_{i\bullet} x^* = b_i\}$ and A_P denotes the submatrix of A consisting of the rows indexed by P. The normal (polar) cone [14, 18] of $\mathcal{M}$ at the point x is defined as

$$\begin{aligned} \mathcal{N}(x^*) :&= \{x \in \mathbb{R}^n \mid x^T y \le 0 \;\forall y \in \mathcal{T}(x^*)\} \\ &= \{A_P^T u \in \mathbb{R}^n \mid u \in \mathbb{R}^{|P|},\; u \ge 0\}. \end{aligned}$$

The extremal directions of the cone $\mathcal{N}(x^*)$ are constituted by the rows of the matrix A_P. Since $\mathcal{N}(x^*)$ is a convex polyhedral cone, it can by described by means of linear inequalities

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n \mid Dx \le 0\},$$

where $D \in \mathbb{R}^{r \times n}$ is an appropriate matrix. To determine such a description is an expensive task in general [17]. One way is to compute all extremal directions h_i, $i \in I$, of $\mathcal{T}(x^*)$ to obtain the desired description

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n \mid h_i^T x \le 0 \;\forall i \in I\}.$$

Nevertheless, as long as x^* is a non-degenerate basic solution corresponding to a basis $B \subseteq \{1, \ldots, m\}$ then the normal cone reads

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n \mid (A_B^T)^{-1} x \ge 0\},$$

that is, D is effectively computable and $D = -(A_B^T)^{-1}$.

Normal and tangent cones relate to efficiency in the following way. A point $x^* \in \mathcal{M}$ is efficient if and only if there is some positive combination of objectives lying inside $\mathcal{N}(x^*)$. In other words, if and only if $DC^T\lambda \leq 0$ for some $\lambda \in \mathbb{R}^s$, $\lambda > 0$. A point $x^* \in \mathcal{M}$ is not efficient if and only if there is $y \in \mathcal{T}(x^*)$ such that $Cy \gneqq 0$.

In this paper, we suppose that the objective functions coefficients are not known precisely. We are given only some lower and upper bounds as follows $\underline{c}_{ij} \leq c_{ij} \leq \overline{c}_{ij}$, $i = 1, \ldots, s,\ j = 1, \ldots, n$. Define an interval matrix

$$\boldsymbol{C} := [\underline{C}, \overline{C}] = \{C \in \mathbb{R}^{s \times n} \mid \underline{c}_{ij} \leq c_{ij} \leq \overline{c}_{ij},\ i = 1, \ldots, s,\ j = 1, \ldots, n\}.$$

The corresponding midpoint matrix and radius matrix are denoted respectively by $C^c := \frac{1}{2}.(\overline{C} + \underline{C})$ and $C^\Delta := \frac{1}{2}.(\overline{C} - \underline{C})$. By an interval MOLP problem we understood a family of problems

$$\max_{x \in \mathcal{M}} \ Cx, \quad \text{where} \ \ C \in \boldsymbol{C}. \tag{2}$$

A feasible solution x^* is called *necessary efficient* if it is efficient to (1) for every $C \in \boldsymbol{C}$.

3 Necessary efficiency

Lemma 1. *The inequality $Cx \gneqq 0$ is true for some $C \in \boldsymbol{C}$ if and only if $C^c x + C^\Delta |x| \gneqq 0$.*

Proof. It is a slight modification of Gerlach theorem [7, 8]. If $Cx \gneqq 0$, then

$$0 \lneqq C^c x + (C - C^c)x \leq C^c x + |C - C^c||x| \leq C^c x + C^\Delta |x|.$$

Conversely, suppose that $C^c x + C^\Delta |x| \gneqq 0$. Define $z = sgn(x)$. Then $|x| = \text{diag}(z)x$ and

$$0 \lneqq C^c x + C^\Delta \text{diag}(z)x = (C^c + C^\Delta \text{diag}(z))x.$$

Since $C := C^c + C^\Delta \text{diag}(z) \in \boldsymbol{C}$, the proof is completed. □

Remind that the tangent cone to $\mathcal{M}$ at the point x^* is described by the inequality system $A_P x \leq 0$. Below, we present a characterization of necessary efficiency.

Theorem 1. *The vector x^* is necessary efficient if and only if the system*

$$C^c x + C^\Delta |x| \gneqq 0,\ A_P x \le 0,\ e^T|x| = 1 \tag{3}$$

has no solution.

Proof. The vector x^* is efficient to (1) with fixed $C \in \boldsymbol{C}$ iff

$$Cx \gneqq 0,\ A_P x \le 0 \tag{4}$$

has no solution [6]. Thus x^* is necessary efficient iff there is no $C \in \boldsymbol{C}$ such that (4) is solvable. By Lemma 1, this is true iff

$$C^c x + C^\Delta |x| \gneqq 0,\ A_P x \le 0$$

is not solvable. Using L^1-norm to normalize x we obtain the final form of (3). $\square$

Remark 1. By using another normalization of the vector x we can equivalently characterize necessary efficiency by inequality system

$$C^c x + C^\Delta |x| \gneqq 0,\ A_P x \le 0,\ e^T|x| = 2 \tag{5}$$

By will employ this matter later on in proofs of Theorem 4 and 5.

Observe that Theorem 1 gives rise to a simple but expensive algorithm for testing necessary efficiency. Decomposing (3) according to signs of particular x_i-s we can provide the testing by solving 2^n linear programs. System (3) is solvable iff the systems

$$C^c x + C^\Delta |x| + y \ge 0,\ A_P x \le 0,\ e^T y \ge 1$$

is solvable, or equivalently, iff the linear system

$$C^c x + C^\Delta \operatorname{diag}(s) x + y \ge 0,\ A_P x \le 0,\ \operatorname{diag}(s) x \ge 0,\ e^T y \ge 1 \tag{6}$$

is solvable for all $s \in \{\pm 1\}^n$. Herein, $|x_i|$ was linearized by $s_i x_i$, where s_i is a sign of x_i. The number may be sometimes decreased when we employ sign restriction on variables x_i-s. Suppose that the system $A_P x \le 0$ contains some non-negativity constraint $x_i \ge 0$, $i \in I$ for certain $I \subseteq \{1, \ldots, n\}$; non-positive variables are handled in a similar manner. Then we fix $s_i := 1$ for each $i \in I$, and hence it suffices to check solvability of (6) for all $s \in \{\pm 1\}^n$ such that $s_i = 1$, $i \in I$ and $s_i = \pm 1$, $i \notin I$. We reduced the number of possibilities to $2^{n-|I|}$, which can still be very high.

4 Necessary efficiency: sufficient and necessary conditions

Testing necessary efficiency is a bit costly (cf. Section 5). That is why exploiting necessary or sufficient conditions may speed up significantly the decision process. We present a sufficient condition first, and then also a necessary condition, both illustrated on examples.

Theorem 2 (sufficient condition)**.** *Define the matrix $M \in \mathbb{R}^{r\times s}$ componentwise as*

$$m_{ij} := \sum_{k=1}^{n} d_{ik} c_{kj}(d_{ik}),$$

where

$$c_{kj}(d_{ik}) := \begin{cases} \overline{c}_{kj} & \textit{if } d_{ik} \geq 0, \\ \underline{c}_{kj} & \textit{if } d_{ik} < 0. \end{cases}$$

If the linear system

$$M\lambda \leq 0, \ \ \lambda \geq e \tag{7}$$

is solvable then x^* is necessary efficient.

Proof. Let λ be a solution to (7) and $C \in \boldsymbol{C}$. It suffices to show that $DC^T\lambda \leq 0$ holds true. Since $d_{ik}c_{kj} \leq d_{ik}c_{kj}(d_{ik})$ we get

$$\sum_{k=1}^{n} d_{ik}c_{kj} \leq \sum_{k=1}^{n} d_{ik}c_{kj}(d_{ik}).$$

Therefore $DC^T \leq M$, whence $DC^T\lambda \leq M\lambda \leq 0$ follows. □

Note that (7) can be equivalently formulated as $\overline{D\boldsymbol{C}^T}\lambda \leq 0, \ \lambda \geq e$ by using interval arithmetic [1].

Generally, the proposed sufficient condition is not the necessary one. The reason is that M is entrywise the best upper bound for DC^T, $C \in \boldsymbol{C}$, but the particular maximizers are attained for different matrices $C \in \boldsymbol{C}$.

To use the sufficient condition presented in Theorem 2 we have to check solvability of a linear system of inequalities. This is an easy task for a linear programming solver. Moreover, we accelerate the decision process when we check a promising candidate for a solution to (7). For instance, such a candidate may be a vector of weights proving efficiency of x^* for some $C \in \boldsymbol{C}$ (typically the midpoint matrix).

As long as x^* is non-degenerate, the proposed sufficient condition is very cheap; it requires just to solve one linear program to check solvability of a linear system (7). If it is not the case, calculation of D might be computationally expensive. We can overcome this drawback by computing only a subset of $\mathcal{N}(x^*)$ that correspond to any feasible basis. Particularly, take any feasible basis $B \subseteq \{1, \ldots, m\}$ corresponding to x^* and put $D := -(A_B^T)^{-1}$. Then the inequality system $Dx \leq 0$ determines a part of the normal cone $\mathcal{N}(x^*)$. The method remains still valid, but the sufficient condition will be weaker.

Notice that for a non-degenerate basic solution x^* our condition coincides with the stopping criterion in the branch & bound method used by Bitran [4]. Thus our approach generalizes Bitran's results to possibly degenerate point. In this manner we can extend Bitran implicit enumeration method to an arbitrary feasible point.

Example 1. Let us consider an example by Inuiguchi & Sakawa [13] with two objectives:

$$\boldsymbol{C} = \begin{pmatrix} [2,3] & [1.5, 2.5] \\ [3,4] & [0.5, 0, 8] \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 4 \\ 3 & 1 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 42 \\ 24 \\ 9 \\ 0 \\ 0 \end{pmatrix}.$$

We want to check whether a feasible solution $x^* = (6,6)^T$ is necessary efficient. Since x^* is a non-degenerate basic solution corresponding to the basis $B = \{1, 2\}$ we compute the normal cone at x^* as follows

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n \mid -(A_B^T)^{-1} x \leq 0\} = \{x \in \mathbb{R}^2 \mid x_1 - 3x_2 \leq 0,\ -4x_1 + 3x_2 \leq 0\}.$$

Now, the linear system (7) reads

$$-1.5\lambda_1 + 2.5\lambda_2 \leq 0,\ -0.5\lambda_1 - 9.6\lambda_2 \leq 0,\ \lambda_1, \lambda_2 \geq 1.$$

Obviously, this system has a solution, e.g. take $\lambda_1 = 2$, $\lambda_2 = 1$. Thus $(6,6)^T$ is necessary efficient.

In the following we are concerned with a necessary condition for necessary efficiency of x^*. Necessary conditions were not thoroughly studied even though its importance was observed already by Bitran [4]. The idea behind our approach is to generate a point that—in the case x^* is not necessary efficient—dominates x^* for some realization of interval data. Let the tangent cone to $\mathcal{M}$ at x^* de described by $A_P x \leq 0$.

Theorem 3 (necessary condition)**.** *Let $\widetilde{x} \in \mathbb{R}^n$, $\widetilde{z} \in \mathbb{R}$ be an optimal solution to the linear program*

$$\max\ z \quad \textit{subject to} \quad C^c x - ez \geq 0,\ A_P x \leq 0,\ -e \leq x \leq e. \tag{8}$$

If $\widetilde{z} > 0$ or $C^c\widetilde{x} + C^\Delta|\widetilde{x}| \gneqq 0$ then x^* is not necessary efficient.

Proof. If $\widetilde{z} > 0$ then $C^c\widetilde{x} > 0$ and x^* is dominated by another point in $\mathcal{M}$ in the direction of $\widetilde{x}$. Hence $\widetilde{x}$ is not efficient for the objective matrix $C^c \in \boldsymbol{C}$.

If $\widetilde{z} \leq 0$ then x^* is possibly efficient for the objective matrix C^c, but condition $C^c\widetilde{x} + C^\Delta|\widetilde{x}| \gneqq 0$ implies (according to Theorem 1 with another kind of normalization) that x^* is not efficient for another $C \in \boldsymbol{C}$. □

Remark 2. One can consider another variants of the linear program (8). For instance, we can analogously use

$$\max\ e^T y \quad \text{subject to} \quad C^c x - y \geq 0,\ y \geq 0,\ A_P x \leq 0,\ -e \leq x \leq e.$$

Let $\widetilde{x} \in \mathbb{R}^n$, $\widetilde{y} \in \mathbb{R}^s$ be its solution. Then x^* is not necessary efficient as long as $e^T\widetilde{y} > 0$ or $C^c\widetilde{x} + C^\Delta|\widetilde{x}| \gneqq 0$. This variant is stronger then that one based on (8), but the benefit is small and the linear program involves $(s-1)$ more variables.

Remark 3. The linear programming problem (8) finds a promising point $\widetilde{x}$ for testing $C^c\widetilde{x} + C^\Delta|\widetilde{x}| \gneqq 0$. However, when we have a good candidate in advance, we may check this inequality directly. This candidate may be e.g. a vector in direction to a neighboring vertex to x^*. That is, if x^0 is a neighbor to x^* then check the vector $\widetilde{x} := x^0 - x^*$. This is particularly promising direction if x^0 was previously recognized as necessary efficient solution.

Example 2. Consider the interval MOLP problem

$$\max\ \boldsymbol{C}x \ \text{ subject to } \ Ax \le b,\ x \ge 0$$

with data from [10, 15]

$$\boldsymbol{C} = \begin{pmatrix} [1,2] & [2,3] & [-2,-1] & [3,4] & [2,3] & [0,1] & [1,2] \\ [-1,0] & [1,2] & [1,2] & [2,3] & [3,4] & [1,2] & [0,1] \\ [3,4] & [0,1] & [1,2] & [1,2] & [0,1] & [-2,-1] & [-2,-1] \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & -1 & 1 & -2 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 16 \\ 16 \\ 16 \\ 16 \end{pmatrix}.$$

We want to test necessary efficiency of the point $x^* = \left(0, 0, \frac{32}{3}, \frac{16}{3}, 0, 0, 0\right)^T$. Determine $P = \{1, 4, 5, 6, 9, 10, 11\}$, where indices greater then four stand for non-negativity constraints. The linear program (8) reads

$$\begin{aligned} &\max\ z \ \text{ subject to} \\ &\quad \begin{pmatrix} 1.5 & 2.5 & -1.5 & 3.5 & 2.5 & 0.5 & 1.5 \\ -0.5 & 1.5 & 1.5 & 2.5 & 3.5 & 1.5 & 0.5 \\ 3.5 & 0.5 & 1.5 & 1.5 & 0.5 & -1.5 & -1.5 \end{pmatrix} x - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} z \ge \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ &\quad \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & -1 & 1 & -2 & -1 \end{pmatrix} x \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ &\quad x_i \ge 0, \quad i = 1, 2, 5, 6, 7, \\ &\quad -1 \le x_i \le 1, \quad i = 1, \dots, 7. \end{aligned}$$

The optimal solution is $\widetilde{x} = (0.2, 0, -1, 0.8, 0, 0, 0)^T$ and $\widetilde{z} = 0.4$. Since $\widetilde{z} > 0$, the point x^* is not necessary efficient.

Necessary efficiency of x^* can be also disproved along Remark 3. The vertex x^* is adjacent to the vertex $x^0 = (0, 0, 0, 16, 0, 0, 0)^T$, and for some $C \in \boldsymbol{C}$ the vertex x^0 dominates to x^* since

$$C^c(x^0 - x^*) + C^\Delta |x^0 - x^*| = (64, \tfrac{64}{3}, \tfrac{32}{3})^T \gneqq 0.$$

Notice that x^0 is necessary efficient, which may be shown by Theorem 2.

5 Complexity

In this section we discuss complexity of testing necessary efficiency. First, we consider one-objective interval MOLP, that is, $\boldsymbol{C} = \boldsymbol{c}$ is an interval $1 \times n$

matrix. Even though this task belongs to the interval linear programming rather then interval MOLP, this issue—to the best of our knowledge—has been studied in neither discipline.

Lemma 2. *Let $M \in \mathbb{Q}^{n\times n}$ be a non-negative positive definite matrix. Checking the solvability of the system*

$$|Mx| \le e,\ e^T|x| > 1 \tag{9}$$

is an NP-complete problem.

Proof. It is a slight modification of the proof of Theorem 2.3 from [7] where the NP-completeness of testing the solvability of a system $|Mx| \le e,\ e^T|x| \ge 1$ was proved. $\square$

Theorem 4. *Testing necessary efficiency is co-NP-complete problem on a class of problems* (2) *with one objective function and rational inputs.*

Proof. We have to show that the problem belong to co-NP class and that it is NP-hard. The former is easily seen as any certificate x that x^* is not necessary efficient must satisfy (3) (up to the normalization).

To prove NP-hardness we first show that (9) is solvable iff

$$|My| \le ez,\ e^T|y| > z,\ e^T|y| + |z| = 2 \tag{10}$$

is solvable. Let $x \in \mathbb{Q}^n$ be a solution to (9). Put $y := \frac{1+\varepsilon}{e^T|x|}x$ and $z := 1-\varepsilon$, where $\varepsilon > 0$ is sufficiently small. Then $e^T|y| + |z| = 1+\varepsilon+1-\varepsilon = 2$ and $e^T|y| > z$. Eventually,

$$|My| = \frac{1+\varepsilon}{e^T|x|}|Mx| \le \frac{1+\varepsilon}{e^T|x|}e \le (1-\varepsilon)e$$

since $\frac{1+\varepsilon}{1-\varepsilon} \le e^T|x|$. Conversely, let $y \in \mathbb{Q}^n$ and $z \in \mathbb{Q}$ be a solution to (10). Put $x := y$. From the second and third condition in (10) it follows that $e^T|x| > 1$ and $z < 1$. Then also $|Mx| \le ez < e$.

Now, we rewrite (10) in the form of (5). Note that $|My| \le ez$ is equivalent to $-ez \le My \le ez$. Substitute $x^T := (y^T, z)^T$ and put

$$A_P := \begin{pmatrix} M & -e \\ -M & -e \end{pmatrix}, \quad C^c := \begin{pmatrix} 0^T & -1 \end{pmatrix}, \quad C^\Delta := \begin{pmatrix} e^T & 0 \end{pmatrix}.$$

Therefore we reduced a problem of checking solvability of (9) to the problem of testing necessary efficiency. $\square$

Theorem 4 says that checking necessary efficiency is computationally expensive in general. However, when we restrict the problem to non-degenerate basic solutions then it becomes polynomially solvable.

Let x^* be a non-degenerate basic solution corresponding to a basis $B \subseteq \{1, \ldots, m\}$. We know that the normal cone to $\mathcal{M}$ at x^* reads

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n \mid Dx \leq 0\},$$

where $D = -(A_B^T)^{-1}$. The vector x^* is necessary efficient iff each objective function vector $c \in \boldsymbol{c}$ lies within $\mathcal{N}(x^*)$. This is easily checked. For each $i = 1, \ldots, n$ do the following. Put $c_j := \overline{c}_j$ if $d_{ij} \geq 0$ and $c_j := \underline{c}_j$ otherwise. If $\sum_{j=1}^n d_{ij} c_j > 0$ then x^* is not necessary efficient. Otherwise, if the condition holds true for all $i = 1, \ldots, n$, then x^* is necessary efficient. The readers familiar with interval computations know that this testing can be equivalently done by using interval arithmetic as $\overline{D\boldsymbol{c}^T} \leq 0$.

When we admit more then one criteria then the problem becomes NP-hard even when restricted to non-degenerate solutions.

Theorem 5. *Testing necessary efficiency of a non-degenerate basic solution is co-NP-complete problem on a class of problems* (2) *with rational inputs.*

Proof. We adapt the proof of Theorem 4 rewriting the system (10) as follows

$$My - ez \leq 0,\ M_{1,\bullet}y + z \geq 0,\ M_{2:n,\bullet}y + ez \geq 0,\ e^T|y| > z,\ e^T|y| + |z| = 2 \tag{11}$$

Herein, $M_{2:n,\bullet}$ denotes the matrix M after removing the first row. The constraint $e^T|y| > z$ may be replaced by $e^T|y| \geq (1+\varepsilon)z$, where $\varepsilon > 0$ is sufficiently small. According to [19] this value has a polynomially large size which can be calculated from the system coefficients. Substituting $x^T := (y^T, z)^T$, the system now takes the form of (5) with

$$A_P := \begin{pmatrix} M & -e \\ -M_{1,\bullet} & -1 \end{pmatrix}, \quad C^c := \begin{pmatrix} M_{2:n,\bullet} & e \\ 0^T & -1-\varepsilon \end{pmatrix}, \quad C^\Delta := \begin{pmatrix} 0 & 0 \\ e^T & 0 \end{pmatrix}.$$

If (11) is solvable then (5) has a solution y, z such that $e^T|y| > (1+\varepsilon)z$, hence $C^c x + C^\Delta |x| \gneqq 0$ holds. Conversely, if $x^T = (y^T, z)^T$ is a solution to (5) then $e^T|x| \geq (1+\varepsilon)z > z$. Note that z cannot be zero since otherwise $0 \leq My \leq 0$ and due to regularity of M also y is zero, which contradicts $e^T|y| + |z| = 2$.

Notice that A_P is nonsingular, so the proof is completed. ∎

Theorem 5 states that testing necessary efficiency is NP-hard for non-degenerate basic solution. From the proof we see that it remains true when we restrict considerations on n objective functions and only one of them is affected by intervals.

Corollary 1. *Testing whether there exist a necessary efficient point is NP-hard problem on a class of problems* (2) *with rational inputs.*

Proof. By Theorem 5, testing necessary efficiency of a non-degenerate basic solution x^* is NP-hard. Its tangent cone

$$\mathcal{T}(x^*) = \{x \in \mathbb{R}^n \mid A_P x \le 0\}$$

has just one vertex x^*. If there is some necessary efficient solution on $\mathcal{T}(x^*)$ then it must be x^* [4], and vice versa. Therefore we reduced the problem of testing necessary efficiency of x^* to the desired existence problem. $\square$

There are still some open questions remaining. For example, what is complexity of the problem of testing necessary efficiency of a non-degenerate basic solution when we have just two objective functions? Another open problem asks for complexity of testing existence of a necessary efficient solution on a bounded convex polyhedron.

6 Conclusion

We have shown that testing necessary efficiency of x^* is NP-hard problem, and it remains valid even when we consider only non-degenerate basic solutions. It means, branch & bound methods used so far for the testing may be very slow it the worst case. To accelerate such methods we proposed one sufficient condition and one necessary condition. The former is effective provided x^* is non-degenerate, otherwise it may be costly. The latter is effective in any way.

References

[1] G. Alefeld and J. Herzberger. *Introduction to interval computations.* Computer Science and Applied Mathematics. Academic Press, New York, 1983.

[2] M. R. Benjamin. *Interval Programming: A Multi-Objective Optimization Model for Autonomous Vehicle Control.* PhD thesis, Brown University, Providence, RI, USA, May 2002.

[3] M. R. Benjamin. The interval programming model for multi-objective decision making. technical report AI Lab Technical Memo AIM-2004-021, Massachusetts Institute of Technology, Cambridge, MA, USA, 2004.

[4] G. R. Bitran. Linear multiple objective problems with interval coefficients. *Manage. Sci.*, 26:694–706, 1980.

[5] S. Chanas and D. Kuchta. Multiobjective programming in optimization of interval objective functions – a generalized approach. *Eur. J. Oper. Res.*, 94(3):594–598, 1996.

[6] M. Ehrgott. *Multicriteria optimization. 2nd ed.* Springer, Berlin, 2005.

[7] M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, and K. Zimmermann. *Linear optimization problems with inexact data.* Springer, New York, 2006.

[8] W. Gerlach. Zur Lösung linearer Ungleichungssysteme bei Störung der rechten Seite und der Koeffizientenmatrix. *Math. Operationsforsch. Stat., Ser. Optimization*, 12:41–43, 1981.

[9] M. Hladík. Computing the tolerances in multiobjective linear programming. *Optim. Methods Softw.*, 23(5):731–739, 2008.

[10] M. Ida. Necessary efficient test in interval multiobjective linear programming. In *Proceedings of the Eighth International Fuzzy Systems Association World Congress*, pages 500–504, Taipei, Taiwan, August 1999.

[11] M. Ida. Portfolio selection problem with interval coefficients. *Appl. Math. Lett.*, 16(5):709–713, 2003.

[12] M. Ida. Solutions for the portfolio selection problem with interval and fuzzy coefficients. *Reliab. Comput.*, 10(5):389–400, 2004.

[13] M. Inuiguchi and M. Sakawa. Possible and necessary efficiency in possibilistic multiobjective linear programming problems and possible efficiency test. *Fuzzy Sets Syst.*, 78(2):231–241, 1996.

[14] F. Nožička, L. Grygarová, and K. Lommatzsch. *Geometrie konvexer Mengen und konvexe Analysis.* Akademie-Verlag, Berlin, 1988.

[15] C. Oliveira and C. H. Antunes. Multiple objective linear programming models with interval coefficients – an illustrated overview. *Eur. J. Oper. Res.*, 181(3):1434–1463, 2007.

[16] C. Oliveira and C. H. Antunes. An interactive method of tackling uncertainty in interval multiple objective linear programming. *J. Math. Sci.*, 161(6):854–866, 2009.

[17] M. Padberg. *Linear optimization and extensions. 2nd ed.* Springer, Berlin, 1999.

[18] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis. corr. 2nd print.* Springer, Berlin, 2004.

[19] A. Schrijver. *Theory of linear and integer programming. Repr.* Wiley, Chichester, 1998.

[20] A. Sengupta and T. K. Pal. *Fuzzy preference ordering of interval numbers in decision problems.* Springer, Berlin, 2009.

[21] H.-F. Wang and M.-L. Wang. A fuzzy multiobjective linear programming. *Fuzzy Sets Syst.*, 86(1):61–72, 1997.

[22] H.-F. Wang and M.-L. Wang. Decision analysis of the interval-valued multiobjective linear programming problems. In M. Köksalan and S. Zionts, editors, *Multiple criteria decision making in the new millennium.*, volume 507 of *Lect. Notes Econ. Math. Syst.*, pages 210–218. Springer, Berlin, 2001.