Preface

The workshop on Coverings and Colorings has been organized by DIMATIA, KAM and ITI on March 6th, 2009 in the lecture room S9 of School of Computer Science of Faculty of Mathematics and Physics, Charles University.

The choice of the date and the subject of the workshop has not been causeless — it has been assembled in the occasion of the 50th birthday of our colleague and friend professor Jan Kratochvíl, and has been known to participants under an informal name "Jan session" (although we all call him Honza).

The scientific program has been split in two parts. At first, former and current Jan's students presented lectures on subject related to their cooperation with their advisor:

- Petr Hliněný: *On a surprising fall of Fellows' conjecture*
- Marek Tesař: *Locally injective homomorphisms to* Θ*-graphs*
- Jiří Fiala: *On distant representatives*
- Jan Kára: *Fixed parameter tractability of independent set in segment intersection graphs*
- Dan Král': *Treewidth and homomorphisms*

The Jan session continued after a break with an agile problem session, chaired by Honza himself. In this booklet you can find texts of the problems presented there together with necessary background and references.

The workshop has been concluded with an informal banquet at the nearby restaurant Nebozízek.

Together with more than 30 participants of the workshop (in the alphabetical order: Josef Cibulka, Tomáš Chudlarský, Mike Fellows, Tomáš Gavenčiak, Peter Golovach, Pavol Hell, Petr Hliněný, Přemysl Holub, Stanislav Jendrol', Jan Kára, Martin Klazar, Petr Kolman, Daniel Král', Antonín Kučera, Bernard Lidický, Jana Maxová, Patrice Ossona de Mendez, Ondřej Pangrác, Daniël Paulusma, Hana Polišenská, Andrzej Proskurowski, Pavel Pudlák, Aleš Pultr, Frances Rosamond, Miklós Ruszinkó, Zdeněk Ryjáček, Jiří Sgall, Robert

Šámal, Martin Škoviera, Ondřej Suchý, Marek Tesař, Pavel Valtr and Margit Voigt) we wish Honza happy birthday and another half-century of success in mathematics.

Jaroslav Nešetřil, Jiří Fiala

The senior part of the authors of this preface would like to add a personal note: I know Jan Kratochvíl for many years from his early undergraduate studies. During the rapid expansion of our department KAM, in founding DIMATIA in getting large grants such as ITI, he has been always my closest collaborator. Not only that, without him some of our past adventures wouldn't be possible. For a teacher and administrator at any level it is a dream (and unfortunately mostly a wishful thinking) to have somebody who is able to continue and further develop. In this case, I have been very lucky.

Open problems and conjectures presented on the workshop

The Complexity of Making Parameterized Structural Connections Effective

Mike Fellows

Michael.Fellows@newcastle.edu.au
The University of Newcastle, Australia

There are many theorems of combinatorial mathematics (some quite famous) that have the form, for some sort of mathematical object X,

Theorem. *If* $\pi(X) \leq k$ *then* $\pi'(X) \leq g(k)$*, for some function* g*.*

For example, if the length of a longest cycle in a graph is at most k, then the treewidth of G is at most k. Determining if G has a cycle of length k is NP-hard, and determining if the treewidth of G is at most k is also NP-hard, nevertheless, the implication in the above theorem can be *effectivized* in polynomial time. There is an algorithm (based on depth-first search) that runs in time polynomial in both k and the size of the graph, that either: (1) determines that G has a cycle of length greater than k (thus negating the hypothesis bound of the implication), or else produces a tree-decomposition of width at most k (thus fulfilling the conclusion bound of the implication).

Given the prevalence and importance of these kinds of theorems in combinatorial mathematics (and mathematics generally), it seems entirely inevitable to be drawn to exploring the computational complexity of their effectivization. (As a side note, such P-time effectivizations also play an important role in kernelization algorithms for FPT parameterized problems, a matter of practical significance.)

For another example, if the bandwidth of a graph G is at most k, then the pathwidth of G is at most k. The best current P-time (only *approximate*) *effectivization* of this connection between these two structural parameters of graphs, is a P-time algorithm that either proves that the bandwidth of G is greater than k, or proves that the pathwidth of G is at most 2^k.

Is there any hope of a P-time approximate effectivization where the approximation function is polynomial in k? If not, why not? What sort of complexity-theoretic machinery running on what sort of reasonable conjectures might allow us to get at such issues?

Historical Note. I recently found a floppy disk containing (according to the label) an unfinished project with Jan, targeting some such issues that arise in the context of the Local Lemma, from around 1989. Maybe some even ask, 'What's a floppy disk?"

A partition problem

Pavol Hell

pavol@cs.sfu.ca
Simon Fraser University Burnaby, B.C., Canada

This problem is originally due to Peter Winkler; it would be nice to revive interest in it. A related problem (in which there is no restriction on the number of edges between A, B and B, C and C, D and D, A) has been proved NP-complete by N. Vikas.

Problem. *Can one decide in polynomial time whether the vertices of a given input graph G can be partitioned into four non-empty subsets A, B, C, D, so that there is at least one edge between A, B, between B, C, between C, D, and between D, A, but no edges between A, C and between B, D ?*

Planar Emulators

Petr Hliněný

hlineny@fi.muni.cz
Masaryk University, Brno, Czech Republic

A *cover* of a graph is a locally bijective homomorphism. Applications in topological graph theory led Seiya Negami to studying the problem of existence of a (finite) planar cover of a graph. Along results relating the existence of double or regular planar covers to embeddability in the projective plane, he formulated the following interesting hypothesis.

Conjecture (Negami, 1988). *A connected graph has a finite planar cover if and only if it embeds in the projective plane.*

Despite its easy formulation and existence of many partial results, the conjecture is still open, after more than 20 years of research. Actually, to finish the proof of this conjecture, it is enough to show that the graph $K_{1,2,2,2}$ (an octahedron with a central vertex) has no finite planar cover, but we have been stuck at this point since 1995.

Independently of Negami, Mike Fellows studied planar emulators of graphs in his 1985 thesis. An *emulator* of a graph is a locally *surjective* homomorphism. He formulated the following naturally looking hypothesis.

Conjecture (Fellows, 1988). *A connected graph has a finite planar emulator if and only if it has a finite planar cover.*

In view of this hypothesis, researchers have so far concentrated mostly on planar covers and Negami's conjecture. However, a surprising recent result has proved that Fellows' conjecture is false (while Negami's one remains open).

Theorem (Rieck and Yamashita, 2008). *The graphs* $K_{4,5} - 4e$ *and* $K_{1,2,2,2}$ *do have finite planar emulators.*

Subsequently to this result, Chimani and Hliněný have construc-ted planar emulators for another graphs which are known not to have finite planar covers.

In view of this recent development, it appears desirable to study the "minimal differences" between the classes of graphs having finite planar emulators and planar covers. We consider the following question particularly interesting.

Problem. *Are there finite planar emulators of the graphs* $K_{4,4} - e$ *and* $K_7 - C_4$*?*

Problem. *Is there an infinite nontrivial family of* non-projective *graphs having finite planar emulators?*

Nice Formula for the q-chromatic Function of a Complete Bipartite Graph

Martin Loebl

loebl@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Let $G = (V, E)$ be a graph and n a positive integer. Let $V = \{1, \ldots, n\}$ and for $k \in \mathbb{P} = \{1, 2, \ldots\}$ let $V(G, k)$ denote the set of all vertex colourings $s : V \to \{0, \ldots, k-1\}$ such that $s(u) \neq s(v)$ whenever $uv \in E$.

$$M_q(G, k) = \sum_{s \in V(G,k)} q^{\sum_{v \in V} s(v)}.$$

Note that $M_q(G, k)|_{q=1}$ is the classical chromatic polynomial of G.

We first recall some notation:

For $k \in \mathbb{P}$, let $(k)_q = q^{k-1} + \cdots + q + 1$ denote a *quantum integer*, with the convention that $(0)_q = 0$, and let $(k)!_q = \prod_{1 \leq n \leq k} (n)_q$, with the convention that $(0)!_q = 1$. For $0 \leq n \leq k$ the *quantum binomial coefficients* are defined by

$$\binom{k}{n}_q = \frac{(k)!_q}{(n)!_q (k-n)!_q}.$$

A simple quantum binomial formula

$$(a - z)(a - qz) \ldots (a - q^{k-1} z) = \sum_{i=0}^{k} (-1)^i \binom{k}{i}_q q^{i(i-1)/2} a^{k-i} z^i$$

leads to a well-known formula for the summation of the products of distinct powers. This gives the q-chromatic function for the complete graph.

For $k \in \mathbb{P}$, the q-chromatic function of the complete graph on $n \leq k$ vertices is given by

$$M_q(K_n, k) = n! \binom{k}{n}_q q^{n(n-1)/2}$$

and $M_q(K_n, k) = 0$ for $n > k$.

It may be interesting to find out if there are reasonable formulas for complete bipartite graphs as well. As far as I know, the only related result is a sub-exponential algorithm by Hliněný et.al.

Finding High Girth High Chromatic Subgraphs in High Chromatic Sparse Graphs

Jaroslav Nešetřil and Patrice Ossona de Mendez

nesetril@kam.mff.cuni.cz; pom@ehess.fr
Charles University, Prague, Czech Republic;
Centre d'Analyse et de Mathmatique Sociales, Paris, France

Erdős and Hajnal [2] conjectured that for all integers c, g there exists an integer $f(c, g)$ such that every graph G of chromatic number at least $f(c, g)$ contains a subgraph of chromatic number at least c and girth at least g (where the girth of G is the length of the shortest cycle in G).

The case $g = 4$ of the conjecture was proved by Rödl [7], while the general case is still open. Remark that the existence of graphs of both arbitrarily high chromatic number and high girth is a well known result of Erdős [1].

We propose the following weakening of this difficult problem: For all integers c, g there exist integers $f(c, g)$ and $s(g)$ such that every graph G of chromatic number at least $f(c, g)$ contains a subgraph H such that

- either H has chromatic number at least c and girth at least g,
- or H is a $\leq s(g)$-subdivision of a complete graph on c vertices (i.e. a shallow subdivision).

Notice that if we don't ask the subdivision to be shallow the conjecture of course holds.

This conjecture is motivated by the strong change of structural properties between monotone classes of graphs $\mathcal{C}$ for which for some integer d every complete graph appears as a $\leq d$-subdivision in $\mathcal{C}$ (the monotone *somewhere dense* classes) and those that do not (the monotone *nowhere dense* classes) [3, 4, 5, 6].

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Two Problems on Domination on Spanning Trees

Andrzej Proskurowski

andrzej@cs.uoregon.edu
University of Oregon, Eugene, USA

A *k-attack* A on a graph G consists of a selection of k distinct vertices of V, say $\{a_1, ...a_k\}$, that are said to be *under attack*. This k-attack can be *countered* by a multiset of vertices X iff there exists an assignment of k distinct members of X to the members of A, represented as $\{(a_1, x_1), ..., (a_k, x_k)\}$, such that either $a_i = x_i$ or (a_i, x_i) is an edge of G, for all i, $1 \leq i \leq k$. In other words, an attack at a vertex must be countered by a *defender* in the vertex itself or in a neighboring vertex.

Given a graph G, a subset D_k of V is a *k-defensive dominating set* of G if and only if it can counter any k-attack in G.

The motivation for the following problem is a message dissemination process called multicasting in which a message is broadcast to multiple receivers across a network. One possible paradigm of multicasting has several sources from a set of vertices transmit the data with every vertex in the network as a receiver. Multicast protocols often use a single routing tree which is shared by all transmissions. The goal of the tree construction may be, for instance, to minimize the time it takes to complete a message dissemination. The optimality of a tree is determined by minimizing some given cost function. Multiple cost metrics are considered because different applications may call for different requirements. All the metrics are combinations of distances between sources and vertices in the tree, $d_T(s, v)$, and the operations combining the distances $\bigoplus, \bigotimes$ vary over all $s \in S$ and $v \in V$.

k-Source Spanning Tree Problems

Instance: A graph G with k sources $S = \{s_1, \dots s_k\} \subseteq V(G)$, a positive integer K.

Question: Is there a spanning tree T of G such that $\bigoplus \bigotimes d_T(s, v) \leq K$?

The problem has been resolved for $\bigoplus, \bigotimes$ being $\sum$ and max. What are other meaningful and interesting operators $\bigoplus, \bigotimes$? Complexity of the corresponding problems?

Two problems in combinatorics

Miklós Ruszinkó

ruszinko@sztaki.hu
Hungarian Academy of Sciences, Budapest, Hungary

The following problem comes from an ongoing research with Zol-tán Füredi on a set Sidon type problem. We managed to improve the existing bounds on regular k-uniform union-free families for $k \geq 4$. In our proof we combine arguments of algebraic flavor and the famous construction of Behrend on large sets of integers having no three term arithmetic progression. On the other hand, we did not manage to use our approach for the case $k = 3$. The solution of the following problem, which may be interesting in its own, may help in this.

Problem. *Does there exist a system $\mathcal{F} \subseteq \binom{[n]}{3}$ of triples with the following properties?*

- $\mathcal{F}$ *is regular*
- $|A \cap B| \leq 1$, $\forall A, B \in \mathcal{F}$
- *There are no two pairwise disjoint collections* $\{A_1, A_2, A_3\}$, $\{B_1, B_2, B_3\}$ *which cover the same nine elements, i.e.,* $\cup_{i=1}^3 A_i = \cup_{i=1}^3 B_i$.
- $|\mathcal{F}| = \Theta(n^2)$

If not cn^2, then how large can be such a system?

Many investigations have been done in finding rainbow objects in a properly edge colored K_n. An old problem is to find a large rainbow cycle. Improving the existing $n/2$ lower bound we proved the following.

Theorem. *(Gyárfás, Ruszinkó, Sárközy, Schelp) For arbitrary $\varepsilon > 0$ $\exists n_0(\varepsilon)$ such that if $n \geq n_0$ then in any proper edge-coloring of K_n, there is a rainbow cycle with length at least $\left(\frac{4}{7} - \varepsilon\right) n$.*

It would be interesting to know how far can be pushed this bound.

Problem. *Is it true that in any proper edge-coloring of* K_n, *there is a rainbow cycle with length at least* $(1 - o(1))n$*?*

Does the Thomassen's conjecture imply P=NP?

Zdeněk Ryjáček

ryjacek@kma.zcu.cz

University of West Bohemia, Pilsen, Czech Republic

By a *graph* $G = (V(G), E(G))$ we mean a simple loopless finite undirected graph. Denote $E^+(G) = \{xy|\ x, y \in V(G)\}$, and for $X \subset E^+(G)$ set $G+X = (V(G), E(G) \cup X)$ (i.e., X is a set of "new" edges that are "added" to G; if $e_1 = \{x, y\} \in E(G)$ and $e_2 = \{x, y\} \in X$, we consider e_1 and e_2 as parallel edges of $G+X$). A graph G is said to be *k-edge-Hamilton-connected* if, for any $X \subset E^+(G)$ such that $|X| = k$ and the edges of X determine a path system, the graph $G+X$ has a Hamiltonian cycle containing all edges in X. The following facts are easy to observe.

1. A graph G is 1-edge-Hamilton-connected if and only if G is Hamilton-connected.
2. A graph G is 2-edge-Hamilton-connected if and only if
   - (i) G is 1-Hamilton-connected (i.e., $G - x$ is Hamilton-connected for any vertex $x \in V(G)$), and
   - (ii) for any four distinct vertices $x_1, x_2, x_3, x_4 \in V(G)$, G has a path factor consisting of 2 paths P_1, P_2 such that both P_1 and P_2 have one endvertex in $\{x_1, x_2\}$ and one endvertex in $\{x_3, x_4\}$.
3. If G is 2-edge-Hamilton-connected, then G is 4-connected.

Consider the following two decision problems.

k-E-HC
Instance: *A graph G.*
Question: *Is G k-edge-Hamilton-connected?*

k-E-HCL
Instance: *A line graph G.*
Question: *Is G k-edge-Hamilton-connected?*

(i.e., k-E-HCL is k-E-HC restricted to line graphs).

Question 1: *Determine the complexity of 2-E-HCL.*

The following facts are known:

HAM
Instance: *A graph G.*
Question: *Does G contain a hamiltonian cycle?*

HAM $\in$ NPC, even if restricted to line graphs.

H-PATH
Instance: *A graph G and distinct vertices $u, v \in V(G)$.*
Question: *Does G contain a hamiltonian (u, v)-path?*

H-PATH $\in$ NPC, even if restricted to line graphs [1].

H-CONN
Instance: *A graph G.*
Question: *Is G Hamilton-connected?*

H-CONN $\in$ NPC [3].

1-H-CONN
Instance: *A graph G.*
Question: *Is G 1-Hamilton-connected?*

1-H-CONN $\in$ NPC [6].

Thus, a common guess would be that probably 2-E-HCL $\in$ NPC.

Question 2: *Why is Question 1 interesting?*

The following conjecture was posed in [5].

Conjecture [Thomassen]. *Every 4-connected line graph is hamiltonian.*

There are many known equivalent versions of the the Thomassen's conjecture; among others, we mention the following.

Theorem. *The following statements are equivalent:*
(i) *Every 4-connected line graph is hamiltonian.*
(ii) *Every 4-connected line graph is 2-edge-Hamilton-connected* [4].
(iii) *Every snark has a dominating cycle* [2].

Thus, if the Thomassen's conjecture is true, then a line graph G is 2-edge-Hamilton-connected if and only if G is 4-connected, implying that 2-E-HCL is polynomial. Consequently, proving the "common guess" 2-E-HCL $\in$ NPC would mean

- disproving the Thomassen's conjecture,
- proving the existence of a snark with no dominating cycle,

unless P=NP.

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On b-colorings of graphs

Margit Voigt

mvoigt@informatik.htw-dresden.de
University of Applied Sciences, Dresden, Germany

A *b-coloring* of a graph G by k colors is a proper coloring of the vertices of G such that in each color class exists a vertex having neighbors in all other $k-1$ color classes. Such a vertex is called a *color dominating vertex*. The *b-chromatic number* $b(G)$ of a graph G is the maximum k for which G has a b-coloring by k colors.

Obviously, a coloring of a graph G with $\chi(G)$ colors where $\chi(G)$ is the chromatic number of G is a b-coloring. Thus $b(G) \geq \chi(G)$ for all G. To give an upper bound on $b(G)$ we assume that the vertices $v_1, \ldots, v_n$ of G are ordered such that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$. Let $t(G) := \max\{i \,;\, d(v_i) \geq i-1\}$ be the maximum number i such that G contains at least i vertices of degree $\geq i-1$. Obviously, $t(G) \leq \Delta(G)+1$ where $\Delta(G)$ denotes the maximum degree of G. It is observed in [5] and [6] that $t(G)$ is an upper bound for $b(G)$. Furthermore this upper bound is tight. On the other hand $b(G)$ may be arbitrarily far from $t(G)$ shown by the complete bipartite graph $K_{n,n}$ [5]. Moreover also the differences $t(G) - \chi(G)$ and $\Delta(G) - t(G)$ can be arbitrarily large as seen by $K_{n,n}$ and the star $K_{1,n}$ respectively. Some recent results and a list of citations on b-colorings can be found in [3].

Algorithmic aspects:

We are interested in the complexity of deciding whether a b-coloring by a given number of colors exists, or whether the b-chromatic number is at least a given number.

b-Coloring

Instance: Graph $G = (V, E)$, integer k

Question: Is there a b-coloring of G by k colors?

b-Chromatic Number

Instance: Graph $G = (V, E)$, integer k

Question: Is $b(G) \geq k$?

Irving and Manlove [5] proved that b-Chromatic Number is NP-complete in general. In fact they proved that it is NP-complete to decide whether there is a b-coloring by $t(G)$ colors. However, their construction does not answer what happens in the special case $t(G) = \Delta(G) + 1$.

In [7] it is proved that b-Coloring is NP-complete for $k = t(G)$ even for connected bipartite graphs and $t(G) = \Delta(G) + 1$.

On the other hand every tree T has b-chromatic number $b(T)$ equal to either $t(T)$ or $t(T) - 1$ as pointed out by Irving and Manlove. Their proof is a polynomial algorithm that computes the value of $b(T)$. Maffray [9] mentioned that the b-chromatic number can be computed in polynomial time also in the class of P_4-free graphs [1, 2] and more generally of P_4-sparse graphs [4].

Chordal graphs:

It is known [8] that every chordal graph G is b-continuous, that is there are b-colorings of G by k colors for all $\chi(G) \leq k \leq b(G)$. However the complexity of b-chromatic number for chordal graphs is not known so far as pointed out by Maffray [9].

Open Problem [9]: What is the complexity of b-chromatic number for chordal graphs?

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Two photos from the workshop:

Questions after Dan's talk.

Honza