On Nowhere Dense Graphs

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Abstract

A set A of vertices of a graph G is called d-scattered in G if no two d-neighborhoods of (distinct) vertices of A intersect. In other words, A is d-scattered if no two distinct vertices of A have distance at most $2d$. This notion was isolated in the context of finite model theory by Gurevich and recently it played a prominent role in the study of homomorphism preservation theorems for special classes of structures (such as minor closed families). This in turn led to the notions of wide, semiwide and quasi-wide classes of graphs. It has been proved previously that minor closed classes and classes of graphs with locally forbidden minors are examples of such classes and thus (relativised) homomorphism preservation theorem holds for them. In this paper we show that (more general) classes with bounded expansion and (newly defined) classes with bounded local expansion and even (very general) classes of nowhere dense graphs are quasi wide. This not only strictly generalizes the previous results and solves several open problems but it also provides new proofs. It appears that bounded expansion and nowhere dense classes are perhaps a proper setting for investigation of wide-type classes as in several instances we obtain a structural characterization. This also puts classes of bounded expansion in the new context and we are able to prove a trichotomy result which separates classes of graphs which are dense (somewhere dense), nowhere dense and finite. Our motivation stems from finite dualities. As a corollary we obtain that any homomorphism closed first order definable property restricted to a bounded expansion class is a duality.

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1 Introduction

For a (finite or infinite) family $\mathcal{F}$ of graphs we denote by $\mathrm{Forb_m}(\mathcal{F})$ the class of all finite graphs G not containing any $F \in \mathcal{F}$ as a subgraph. By a subgraph we mean here not necessarily induced subgraph (and "m" in $\mathrm{Forb_m}(\mathcal{F})$ stands for *monomorphism*). In this paper we study mainly asymptotic properties of such classes which are defined by means of edge densities. In order to digest this we list some examples:

- $\mathrm{Forb_m}(\mathcal{K}_2)$ is the class of all discrete graphs;
- $\mathrm{Forb_m}(\mathcal{P}_\mathrm{k})$ is the class of graphs with a bounded *tree depth* ([NOdM06b]; see Section 3);
- $\mathrm{Forb_m}(\mathcal{S}_{1,\mathrm{k}})$ is the class of all graphs with all degrees bounded by k;
- $\mathrm{Forb_m}(\mathcal{T})$ for any given tree T is a class of graphs with bounded degeneracy (or bounded maximal average degree; this will be denoted below as ∇_0).

On the other side of our spectrum is the class $\mathrm{Forb}_m(\mathcal{C}_\ell : \ell \leq g)$ of all graphs with girth $> g$. This class it is right to consider as a class of *random - like* graphs. In the next sextion we shall define the notions of *shallow minor* and *topological minor* and this will lead to much more general classes of type $\mathrm{Forb_m}(\mathcal{F})$ which are called *bounded expansion classes, bounded local expansion classes* and finally here newly defined classes of *nowhere dense graphs*. For all these classes we shall be able to prove characterizations and structural theorems.

Classes of nowhere dense structures are defined in this paper (in Section 2.4) and they have several interesting (and we believe surprising) properties and equivalent formulations, see Theorem 4.1. This not only relates classes of nowhere dense structures to characterizations of classes with bounded expansion but also alternatively defines them as classes of structures where all shallow minors have edge density $n^{1+o(1)}$. The interplay between dense classes (more precisely the classes of nsomewhere dense graphs) and classes of nowhere dense graphs is very interesting and it is expressed by the *trichotomy* Theorem 3.2.

Despite the generality of these classes we can deduce from the results of [NOdM08a] and [NOdM08b] several algorithmic consequences. For example any first-order sentence preserved under homomorphisms on a class $\mathcal{C}$ of structures may be decided in time $O(n)$ if $\mathcal{C}$ has bounded expansion and in time $n^{1+o(1)}$ if $\mathcal{C}$ is a class of nowhere dense structures.

These classes strictly contain all previously studied -in this context- classes of structures such as classes with bounded local tree width, locally excluded minors, etc, see [Cou90][KS99][ADK04][ADG05][ADK06] [DGK07]; see Fig 4 for the schema of inclusion of these classes. Yet we can prove for all these new classes that the relativized homomorphism preservation theorem holds even for them. Perhaps this also provides a proper setting for questions related to scattered sets in graphs. This leads to notions *wide, semi-wide* and *quasi-wide* classes and we obtain characterization theorems for these classes. This is mentioned in the last section devoted to applications.

In Section 3 we prove Theorem 4.1 which should be seen as the main result of this paper. It not only characterizes classes of nowhere dense graphs (and structures) but it also explains the pleasing interplay of various invariants defined for bounded expansion classes. This has several consequences for First Order definable classes, for dualities but also algorithmic consequences. Some of this is listed in the Section 4 devoted to applications.

2 Definitions

For graphs and, more generally, relational structures, we use standard notation and terminology. In this Section we give the key definitions of this paper.

2.1 Distances, Shallow Minors and Grads

The *distance* in a graph G between two vertices x and y is the minimum length of a path linking x and y (or ∞ if x and y do not belong to the same connected component of G) and is denoted by $\mathrm{dist}_G(x, y)$. Let $G = (V, E)$ be a graph and let d be an integer. The *d-neighborhood* $N_d^G(u)$ of a vertex $u \in V$ is the subset of vertices of G at distance at most d from u in G: $N_d^G(u) = \{v \in V : \mathrm{dist}_G(u, v) \leq d\}$.

For a graph $G = (V, E)$, we denote by $|G|$ the *order* of G (that is: $|V|$) and by $\|G\|$ the *size* of G (that is: $|E|$).

For any graphs H and G and any integer d, the graph H is said to be a *shallow minor* of G at *depth* d ([PRS94]) if there exists a subset $\{x_1, \ldots, x_p\}$ of G and a collection of disjoint subsets $V_1 \subseteq N_d^G(x_1), \ldots, V_p \subseteq N_d^G(x_p)$ such that H is a subgraph of the graph obtained from G by contracting each V_i into x_i and removing multiple edges (see Fig. 1). The set of all shallow minors of G at depth d is denoted by $G \,\triangledown\, i$. In particular, $G \,\triangledown\, 0$ is the set of all subgraphs of G.

The *greatest reduced average density* (shortly *grad*) with rank r of a graph G [NOdM08a] is defined by formula

$$\nabla_r(G) = \max\left\{\frac{\|H\|}{|H|} : H \in G \,\triangledown\, r\right\} \tag{1}$$

By extension, for a class of graphs $\mathcal{C}$, we denote by $\mathcal{C} \,\triangledown\, i$ the set of all shallow minors at depth i of graphs of $\mathcal{C}$, that is:

$$\mathcal{C} \,\triangledown\, i = \bigcup_{G \in \mathcal{C}} (G \,\triangledown\, i)$$

Hence we have

$$\mathcal{C} \subseteq \mathcal{C} \,\triangledown\, 0 \subseteq \mathcal{C} \,\triangledown\, 1 \subseteq \cdots \subseteq \mathcal{C} \,\triangledown\, i \subseteq \ldots$$

Also, for a class $\mathcal{C}$ of graphs we define the *expansion* of the class $\mathcal{C}$ as:

$$\nabla_i(\mathcal{C}) = \sup_{G \in \mathcal{C}} \nabla_i(G)$$

Notice that $\nabla_i(\mathcal{C}) = \nabla_0(\mathcal{C} \,\triangledown\, i)$.

These definitions may be carried over for any graph invariant. Explicitely, let us define

$$\omega_i(\mathcal{C}) = \sup_{G \in \mathcal{C} \,\triangledown\, i} \omega(G) = \omega(\mathcal{C} \,\triangledown\, i).$$

As we shall see (and a bit surprisingly) this parameter characterizes classes of nowhere dense graphs.

2.2 Shallow Topological Minors and Top-grads

For any (simple) graphs H and G and any integer d, the graph H is said to be a *shallow topological minor* of G at *depth* d if there exists a subset $\{x_1, \ldots, x_p\}$ of G and a collection of internaly vertex disjoint paths $P_1 \ldots P_q$ each of length at most $2d + 1$ of G with endpoints in $\{x_1, \ldots, x_p\}$ whose contraction into single edges define on $\{x_1, \ldots, x_p\}$ a graph isomorphic to H (see Fig. 2).

The set of all the shallow topological minors of G at depth d is denoted by $G \mathbin{\widetilde{\triangledown}} d$. In particular, $G \mathbin{\widetilde{\triangledown}} 0$ is the set of all the subgraphs of G. Notice that for every graph G and every integer i we clearly have $(G \mathbin{\widetilde{\triangledown}} i) \subseteq (G \mathbin{\triangledown} i)$.

The *topological greatest reduced average density* (*top-grad*) with rank r of a graph G is:

$$\widetilde{\nabla}_r(G) = \max\left\{\frac{\|H\|}{|H|} : H \in G \mathbin{\widetilde{\triangledown}} r\right\} \tag{2}$$

By extension, for a class of graphs $\mathcal{C}$, we denote by $\mathcal{C} \mathbin{\widetilde{\triangledown}} i$ the set of all shallow topological minors at depth i of graphs of $\mathcal{C}$, that is:

$$\mathcal{C} \mathbin{\widetilde{\triangledown}} i = \bigcup_{G \in \mathcal{C}} (G \mathbin{\widetilde{\triangledown}} i)$$

Hence we have

$$\mathcal{C} \subseteq \mathcal{C} \mathbin{\widetilde{\triangledown}} 0 \subseteq \mathcal{C} \mathbin{\widetilde{\triangledown}} 1 \subseteq \cdots \subseteq \mathcal{C} \mathbin{\widetilde{\triangledown}} i \subseteq \ldots$$

For a class $\mathcal{C}$ of graphs we define the *topological expansion* of $\mathcal{C}$ as:

$$\widetilde{\nabla}_i(\mathcal{C}) = \sup_{G \in \mathcal{C}} \widetilde{\nabla}_i(G)$$

Notice that $\widetilde{\nabla}_i(\mathcal{C}) = \widetilde{\nabla}_0(\mathcal{C} \mathbin{\widetilde{\triangledown}} i)$.

The top-grads and the grads have been related by Z. Dvořák:

Theorem 2.1 ([Dvo07]). *For every integer r and every graph G:*

$$\frac{1}{4}\left(\frac{\nabla_r(G)}{4}\right)^{\frac{1}{(r+1)^2}} \leq \widetilde{\nabla}_r(G) \leq \nabla_r(G)$$

2.3 Cliques Minors and Topological Cliques

We prove here that the clique size of shallow minors and the clique size of topological shallow minors are strongly related. Precisely, for any graph G and any integer r, $\omega(G \mathbin{\triangledown} r)$ lies between $\omega(G \mathbin{\widetilde{\triangledown}} r)$ and $P_r(\omega(G \mathbin{\widetilde{\triangledown}} g(r)))$ where P_r is a polynomial and $g(r)$ is an exponentialy growing function.

In order to prove this result, we will need the following slight modification of the Lemma 9.8 of [NOdM06a].

Lemma 2.2. *Let G be a graph and let $H \in G \mathbin{\triangledown} 1$. Assume $K_{p'} \in H \mathbin{\widetilde{\triangledown}} k$. Then $K_p \in G \mathbin{\widetilde{\triangledown}} (9k+10)$ if $p' \geq 2p^2 - 6p + 8$.*

Proof. If $p = 1, 2$ or 3 the result is obvious as $p' \geq p$ and G will obviously include a vertex, an edge or a cycle of length at most $9k + 9$ (respectively). Thus we may assume $p \geq 4$ hence $p' - p(p-1) \geq \max(p, (p-2)(p-3)+2)$.

By considering a subgraph of G if necessary, we may assume that $V(G)$ is partitioned into $A_1, \ldots, A_i, \ldots, A_{p'}$, $L_{1,1}, \ldots, L_{i,j}, \ldots, L_{p',p'}$ where:

- for $1 \leq i \leq p'$, $G[A_i]$ is a star (possibly reduced to a single vertex or a single edge);
- for $1 \leq i < j \leq p'$, there exists $v_{i,j} \in A_i$ and $v_{j,i} \in A_j$ such that $G[L_{i,j} \cup \{v_{i,j}, v_{j,i}\}]$ is a path of length at most $3k + 3$ with endpoint $v_{i,j}$ and $v_{j,i}$.

For sake of simplicity, we define $L_{j,i} = L_{i,j}$ and $L_{i,i} = \emptyset$. For a subset Y of $\{1, \ldots, p\}$ we also define G_Y has the subgraph of G induced by $\bigcup_{i \in Y} A_i \cup \bigcup_{i,j \in Y} L_{i,j}$.

We first claim the following result: Let N be a positive integer and let X be a subset of $\{1, \ldots, p'\}$ of cardinality at least $\max(N, (N-2)(N-3)+2)$. Then there exists a subset $X' = \{k_{a,1}, \ldots, k_{a,N}\}$ of X of cardinality $(N-1)$ such that there exists in $G_{X'}$ a spider (that is: a subdivision of a star) with center $r_a \in A_{k_{a,a}}$ and leaves $l_{a,1}, \ldots, l_{a,a-1}, l_{a,a+1}, \ldots, l_{a,N}$ with $l_{a,i} \in L_{a,k_{a,i}}$, such that the length of the paths from the center to the leaves is at most $3k + 4$ (see Fig 3). This claim is easily proved as follows: Assume no vertex of $A_{k_{a,a}}$ has degree at least $(N-1)$ in G_X. Then $|X| - 1 \leq (N-2)(N-3)$, a contradiction. Choose for r_a any vertex of $A_{k_{a,a}}$ with degree at least $(N-1)$ in G_X. Then there exists in G_X a spider with center r_a and at least $(N-1)$ leaves belonging to different $A_{k_{a,i}}$ linked to r_a by paths of length at most $3k + 4$.

Assume $p' - N(N-1) \geq (N-2)(N-3) + 2$, i.e. $p' \geq 2N^2 - 6N + 8$. Using the previous claim, we inductively define $Z_1, \ldots, Z_N$ with $Z_i = \{k_{i,1}, \ldots, k_{i,N}\}$ such that G_{Z_i} contains a spider with center $r_i \in A_{k_{i,i}}$ and leaves $l_{i,j} \in A_{k_{i,j}}$ connected by paths of length at most $3k+4$: to construct Z_i, we consider $X = \{1, \ldots, p'\} \setminus \bigcup_{1 \leq j < i} Z_j$. Then G includes a $\leq (9k+10)$-subdivision of K_N with principal vertices $r_1, \ldots, r_N$ as the union of all the spiders (and connections of length at most 2 within the $L_{i,j}$ if necessary). $\square$

Lemma 2.3. *Let $f : \mathbb{N} \to \mathbb{N}$ be defined by:*

$$f(p,r) = \begin{cases} 2p^2 - 6p + 8, & \text{if } r = 0 \\ 2f(p,r-1)^2 - 6f(p,r-1) + 8, & \text{otherwise} \end{cases}$$

Let G be a graph and let r be an integer. Then:

$$\omega(G \,\widetilde{\triangledown}\, r) \leq \omega(G \triangledown r) < f\left(\omega\left(G \,\widetilde{\triangledown}\, \left(\frac{5}{4}(9^r - 1)\right)\right) + 1, r\right)$$

Proof. Let $H \in G \triangledown r$. According to Lemma 2.2, we have that $K_{p'} \in H \,\widetilde{\triangledown}\, k$ implies $K_p \in G \,\widetilde{\triangledown}\, (9^r(k + 5/4) - 5/4)$ if $p' \geq f(p, r)$. In particular, if $K_{p'} \in G \triangledown r$ then $K_p \in \,\widetilde{\triangledown}\, (\frac{5}{4}(9^r - 1))$ if $p' \geq f(p, r)$. Hence $p > \omega\left(G \,\widetilde{\triangledown}\, \left(\frac{5}{4}(9^r - 1)\right)\right)$ implies $f(p, r) > \omega(G \triangledown r)$. Let $p = \omega\left(G \,\widetilde{\triangledown}\, \left(\frac{5}{4}(9^r - 1)\right)\right) + 1$, the lemma follows. $\square$

3 Classes of Sparse Graphs

Although almost all results of this paper can be formulated in the "local" form (for a single graph with special properties) we find it useful to formulate our results by means of properties of classes of graphs.

A class $\mathcal{C}$ of graphs is *hereditary* if every induced subgraph of a graph in $\mathcal{C}$ to $\mathcal{C}$, and it is *monotone* of every subgraph of a graph in $\mathcal{C}$ belongs to $\mathcal{C}$. For a class of graphs $\mathcal{C}$, we denote by $H(\mathcal{C})$ the class containing all the induced subgraphs of graphs in $\mathcal{C}$, that is the inclusion-minimal hereditary class of graphs including $\mathcal{C}$. For a class of graph $\mathcal{C}$ we define $\Delta(\mathcal{C}) = \sup_{G \in \mathcal{C}} \Delta(G)$ and $\omega(\mathcal{C}) = \sup_{G \in \mathcal{C}} \omega(G)$.

Let $\mathcal{C}$ be an infinite class of graphs and let $f : \mathcal{C} \to \mathbb{R}$ be a graph invariant. (By this we mean a function which is isomorphism invariant.) Let $\mathrm{Inj}(\mathbb{N}, \mathcal{C})$ be the set of all injective mappings from $\mathbb{N}$ to $\mathcal{C}$. Then we define:

$$\limsup_{G \in \mathcal{C}} f(G) = \sup_{\phi \in \mathrm{Inj}(\mathbb{N}, \mathcal{C})} \limsup_{i \to \infty} f(\phi(i))$$

Notice that $\limsup_{G \in \mathcal{C}} f(G)$ always exist and is either a real number or $\pm\infty$.

If $\limsup_{G \in \mathcal{C}} f(G) = \alpha \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ we have the following two properties:

- for every $\phi \in \mathrm{Inj}(\mathbb{N}, \mathcal{C})$, $\limsup_{i \to \infty} f(\phi(i)) \leq \alpha$;
- there exists $\phi \in \mathrm{Inj}(\mathbb{N}, \mathcal{C})$, $\limsup_{i \to \infty} f(\phi(i)) = \alpha$.

Note that the second property is easy to prove: consider a sequence $\phi_1, \dots, \phi_i, \dots$ such that $\lim_{i\to\infty} \limsup_{j\to\infty} f(\phi_i(j)) = \alpha$. For each i, let $s_i(1) < \dots < s_i(j) < \dots$ be such that $\limsup_{j\to\infty} f(\phi_i(j)) = \lim_{j\to\infty} f(\phi_i(s_i(j)))$. Then iteratively define $\phi \in \mathrm{Inj}(\mathbb{N}, \mathcal{C})$ by $\phi(1) = \phi_1(s_1(1))$ and $\phi(i) = \phi_i(s_i(j))$, where j is the minimal integer greater or equal to i such that $\phi_i(s_i(j))$ will be different from $\phi(1), \dots, \phi(i-1)$. Then $\limsup_{j\to\infty} f(\phi(j)) = \alpha$.

3.1 Trichotomy

Defining the boundary between sparse and dense classes is not an easy task. Several definitions have been given for "sparse graphs", which do not allow a dense/sparse dichotomy (for instance: a graph is sparse if it has a size which is linear with respect to its order, dense if it is quadratic). Instead of defining what is a "sparse graph" or a "dense graph", we define "sparse classes of graphs" and "dense classes of graphs" by the limit behaviour of the "biggest" graphs in the class when their order tends to infinity. Moreover, we will demand that our definition stays invariant in the context of derived classes, i.e. when we perform lexicographic products with small graphs, contractions of small balls, etc. It appears that the right measure of the growth of edge densities is the fraction of logarithms. This leads to a following *trichotomy* results (Theorem 3.2) which is the starting point of our classification.

This trichotomy result relies on Lemma 2.3 and on Z. Dvořák's study of clique subdivisions arising in graphs with large minimum degree. It is easy to check that $\omega(G \mathbin{\widetilde{\nabla}} 1)$ almost surely lies between $\Omega(\log n)$ and $O(\sqrt{n})$ for a random graph G of order n. The conjecture of Mader [Mad72] and Erdős and Hajnal [EH69] that there exists a constant c such that any graph with average degree cp^2 contains a subdivision of K_p has been proved by Komlós and Szemerédi [KS94, KS96] and Bollobás and Thomasson [BT98]. Note that a similar result holds for minors — a graph with average degree $\Omega(p\sqrt{\log p})$ contains K_p as a minor, by Kostochka [Kos82] and Thomasson [Tho84].

Consider a graph G of order n and minimum degree n^ϵ, for some constant $0 < \epsilon < 1$. If G is random, the expected diameter of G would be constant (dependent on ϵ) hence would contain a subdivision of a large clique where each edge would be subdivided by a constant number of vertices. The question studied by Dvořák [Dvo07] is whether such a result holds in general. The proof by Bollobás and Thomasson [BT98] uses an argument from which the lengths of the subdivision paths are difficult to derive. The proof of

Komlós and Szemerédi [KS96] finds a subdivision of the complete graph where each edge is subdvided polylogarithmic number of times, because of the use of an expander to boost the degree. For the sake of completeness we give here a short sketch of Dvořák's result and refer the reader to his thesis for further details.

Theorem 3.1 ([Dvo07]). *For each $\epsilon (0 < \epsilon \leq 1)$ there exist integers n_0 and c_0 and a real number $\mu > 0$ such that every graph G with $n \geq n_0$ vertices and minimum degree at least n^ϵ contains the c-subdivision of K_{n^μ} as a subgraph, for some $c \leq c_0$.*

Sketch of the Proof. The first ingredient is a lemma (denoted here as Claim 1) allowing to boost the exponent in the minimum degree.

Claim 1: For any ϵ $(0 < \epsilon < 1)$ there exists n_0 such that for every graph G of order $n \geq n_0$ and minimum degree at least n^ϵ, $G \,\widetilde{\triangledown}\, 1$ either contains $K_{n^{\epsilon^3}}$ or a graph G_1 with order $n_1 = \Omega(n^{\epsilon-\epsilon^3})$ and minimum degree at least $d = \Omega\left(n_1^{\epsilon+\epsilon^2 \frac{1-\epsilon-\epsilon^2}{1-\epsilon+\epsilon^3}}\right)$.

This claim is verified as follows: let A be a subset of vertices of G obtained by picking each vertex randomly independently with probability $p = 2n^{-\epsilon+\epsilon^3}$. Let B be the set of vertices not in A having at least n^{ϵ^3} neighbours in A. Using Chernoff and Markov inequalities, there is a non zero probability that $|A| \leq 4n^{1-\epsilon+\epsilon^3}$ and $|B| \geq \frac{n}{2}$ (if n is sufficiently large), so choose such a pair of subsets A and B. Then form a graph G' with vertex set A as follows: order (arbitrarily) the vertices of B and consider iteratively each vertex $v \in B$. If $N_G(v) \cap A$ is not a clique in G', join in G' some two non-adjacent vertices in $N_G(v) \cap A$. If, for some $v \in B$, no edge is added then $N_G(v) \cap A$ is a clique in G' thus $K_{n^{\epsilon^3}} \in G \,\widetilde{\triangledown}\, 1$. Otherwise, $G' \in \,\widetilde{\triangledown}\, 1$ and G' has average degree at least $\frac{1}{4}n^{\epsilon-\epsilon^3}$ hence contains a subgraph G_1 with minimum degree at least $d = \frac{1}{8}n^{\epsilon-\epsilon^3}$ and order $\frac{1}{8}n^{\epsilon-\epsilon^3} \leq n_1 \leq 4n^{1-\epsilon+\epsilon^3}$. Expressing d relatively to n_1, the Claim 1 follows.

This result allows to augment the exponent in the expression of the minimum degree when $\epsilon \leq \frac{\sqrt{5}-1}{2} \approx 0.618$. In a similar way we can verify

Calim 2: There exists n_0 such that for every graph G of order $n \geq n_0$ and minimum degree at least $4n^{0.6}$ holds $K_{n^{0.1}} \in G \,\widetilde{\triangledown}\, 1$.

By applying Claim 1 at most $\frac{10}{\epsilon^2}$ times and then applying Claim 2, it follows that some K_{n^μ} belongs to $G \,\widetilde{\triangledown}\, \left(2^{\frac{10}{\epsilon^2}+1} - 1\right)$ (if G is sufficiently big) for some constant μ dependent on ϵ. □

Theorem 3.2. *Let $\mathcal{C}$ be an infinite class of graphs. Then*

$$\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} = \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\triangledown}\, r} \frac{\log\|G\|}{\log|G|} \in \{0,1,2\}$$

The extremal values may be characterized as follows:

$$\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} = 0 \iff \limsup_{G\in\mathcal{C}} \|G\| < \infty,$$

and

$$\begin{aligned}\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} = 2 &\iff \exists r_0\in\mathbb{N} : \omega(\mathcal{C}\,\widetilde{\triangledown}\, r_0) = \infty \\ &\iff \exists r_0'\in\mathbb{N} : \omega(\mathcal{C}\,\triangledown\, r_0') = \infty\end{aligned}$$

Proof. First notice that for every integer $r \geq 0$ we have $\omega(\mathcal{C}\,\triangledown\, r) \geq \omega(\mathcal{C}\,\widetilde{\triangledown}\, r)$ and $2 \geq \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} \geq \limsup_{G\in\mathcal{C}\,\widetilde{\triangledown}\, r} \frac{\log\|G\|}{\log|G|} \geq 0$ as $\mathcal{C}\,\triangledown\, r \supseteq \mathcal{C}\,\widetilde{\triangledown}\, r$ and has a graph has at most a quadratic number of edges.

If $\mathcal{C}$ is a class of graphs $\mathcal{C}$ such that the number of edges of the graphs in $\mathcal{C}$ is bounded (that is: the graphs in $\mathcal{C}$ only contain isolated vertices with the exception of a bounded number of vertices) then obviously $\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} = 0$.

Otherwise, there is an infinite sequence $H_1, \dots, H_i, \dots$ of distinct graphs in $\mathcal{C}\,\widetilde{\triangledown}\, 0 = \mathcal{C}\,\triangledown\, 0$ which have no isolated vertices. As this sequence is infinite, we get $\lim_{i\to\infty} \log|H_i| = \infty$. Moreover, $\|H_i\| = \frac{1}{2}\sum_{v\in V(H_i)} d(v) \geq |H_i|/2$. Hence:

$$\begin{aligned}\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\triangledown}\, r} \frac{\log\|G\|}{\log|G|} &\geq \limsup_{G\in\mathcal{C}\,\widetilde{\triangledown}\, 0} \frac{\log\|G\|}{\log|G|} \\ &\geq \lim_{i\to\infty} \frac{\log\|H_i\|}{\log|H_i|} \\ &\geq \lim_{i\to\infty} \frac{\log|H_i| - 1}{\log|H_i|} \\ &\geq 1\end{aligned}$$

So we have:

$$\limsup_{G\in\mathcal{C}} \|G\| < \infty \Rightarrow 0 \leq \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\triangledown}\, r} \frac{\log\|G\|}{\log|G|} \leq \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} = 0$$

and

$$\limsup_{G\in\mathcal{C}} \|G\| = \infty \Rightarrow \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log \|G\|}{\log|G|} \geq \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\triangledown}\, r} \frac{\log \|G\|}{\log|G|} \geq 1.$$

Hence in particular:

$$\begin{aligned}\limsup_{G\in\mathcal{C}} \|G\| < \infty &\iff \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\triangledown}\, r} \frac{\log \|G\|}{\log|G|} = 0 \\ &\iff \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log \|G\|}{\log|G|} = 0\end{aligned}$$

Now assume that $\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\triangledown}\, r} \frac{\log \|G\|}{\log|G|} > 1$, i.e. that there exists r_1 and $\alpha > 0$ such that $\limsup_{G\in\mathcal{C}\,\widetilde{\triangledown}\, r_1} \frac{\log \|G\|}{\log|G|} \geq 1+\alpha$. Then there exists an infinite sequence $H_1, \dots, H_i \dots$ of distinct graphs in $\mathcal{C}\,\widetilde{\triangledown}\, r_1$ such that $\lim_{i\to\infty} \frac{\log \|H_i\|}{\log|H_i|} \geq 1+\alpha$. Moreover, each H_i has a subgraph H_i' of order at least $\sqrt{\|H_i\|}$, size at least $\|H_i\|/2$ and minimum degree at least $\frac{\|H_i\|}{2|H_i|}$: from H_i remove iteratively vertices with degree at most $\frac{\|H_i\|}{2|H_i|}$. When the process is finished, we have removed at most $|H_i|\frac{\|H_i\|}{2|H_i|} = \|H_i\|/2$ edges and hence the graph H_i' has order at least $\sqrt{\|H_i\|}$. As $\lim_{i\to\infty}|H_i'| = \infty$, we can extract a subsequence $G_i = H'_{f(i)}$ such that all the G_i are distinct and have increasing orders. We have:

$$\begin{aligned}\lim_{i\to\infty} \frac{\log \|G_i\| - 1}{\log|G_i|} &= \lim_{i\to\infty} \frac{\log \|G_i\|}{\log|G_i|} \\ &= \lim_{i\to\infty} \frac{\log \|H'_{f(i)}\|}{\log|H'_{f(i)}|} \\ &\geq \lim_{i\to\infty} \frac{\log \|H_{f(i)}\| - 1}{\log|H_{f(i)}|} \\ &= \lim_{i\to\infty} \frac{\log \|H_i\| - 1}{\log|H_i|} \\ &= \lim_{i\to\infty} \frac{\log \|H_i\|}{\log|H_i|} \\ &\geq 1+\alpha.\end{aligned}$$

It follows that there exists N such that $\frac{\log\|G_i\|-1}{\log|G_i|} \geq 1+\alpha/2$ for every $i \geq N$, thus G_i has minimum degree at least $\frac{\|G_i\|/2}{|G_i|} \geq |G_i|^{\alpha/2}$. According to Theorem 3.1, for every $\epsilon > 0$ there exist integers $n_0(\epsilon)$ and $c_0(\epsilon)$ and a real number $\mu(\epsilon) > 0$ such that every graph G with $n \geq n_0(\epsilon)$ vertices and minimum degree at least n^ϵ contains the c-subdivision of $K_{n^{\mu(\epsilon)}}$ as a subgraph, for some $c \leq c_0(\epsilon)$. Thus, for every $i \geq \max(N, n_0(\alpha/2))$, G_i contains the c-subdivision of $K_{|G_i|^{\mu(\alpha/2)}}$ as a subgraph, for some $c \leq c_0(\alpha/2)$. Let $r_0 = (2r_1+1)(2c_0(\alpha/2)+1)-1$. We have $K_{|G_i|^{\mu(\alpha/2)}} \in G_i \,\widetilde{\nabla}\, c_0(\alpha/2) \subseteq \mathcal{C} \,\widetilde{\nabla}\, r_0$. As $\lim_{i\to\infty}|G_i| = \infty$ we infer that $\omega(\mathcal{C} \,\widetilde{\nabla}\, r_0) = \infty$.

Moreover, if there exists $r_0 \in \mathbb{N}$ such that $\omega(\mathcal{C} \,\widetilde{\nabla}\, r_0) = \infty$ we have:

$$\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\, r} \frac{\log\|G\|}{\log|G|} \geq \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\, r_0} \frac{\log\|G\|}{\log|G|} \geq \sup_{i\to\infty} \frac{\log\|K_i\|}{\log|K_i|} = 2.$$

Hence:

$$\begin{aligned}\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\, r} \frac{\log\|G\|}{\log|G|} > 1 &\iff \exists r_0 \in \mathbb{N} : \omega(\mathcal{C} \,\widetilde{\nabla}\, r_0) = \infty \\ &\iff \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\, r} \frac{\log\|G\|}{\log|G|} = 2.\end{aligned}$$

Now assume that $\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} > 1$, i.e. that there exists r_1' and $\alpha > 0$ such that $\limsup_{G\in\mathcal{C}\,\triangledown\, r_1'} \frac{\log\|G\|}{\log|G|} \geq 1+\alpha$. As in the previous case, we infer that there exists r_0' such that $\omega(\mathcal{C} \,\triangledown\, r_0') = \infty$ thus

$$\begin{aligned}\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} > 1 &\iff \exists r_0' \in \mathbb{N} : \omega(\mathcal{C} \,\triangledown\, r_0') = \infty \\ &\iff \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} = 2.\end{aligned}$$

According to Lemma 2.3, we have:

$$\exists r_0 \in \mathbb{N} : \omega(\mathcal{C} \,\widetilde{\nabla}\, r_0) = \infty \iff \exists r_0' \in \mathbb{N} : \omega(\mathcal{C} \,\triangledown\, r_0') = \infty,$$

what completes the proof. □

The characterization of classes of graphs for which

$$\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log\|G\|}{\log|G|} \leq 1$$

is the main theme of this paper. We call such classes *classes of nowhere dense graphs*.

Remark 3.3. According to Theorem 3.2, a class $\mathcal{C}$ is a class of nowhere dense graphs if and only if

$$\forall r \in \mathbb{N} \quad \omega(\mathcal{C} \mathbin{\triangledown} r) < \infty.$$

In the other words: there exists no integer r_0 such that every graph is a shallow minor of depth r_0 of some graph in $\mathcal{C}$.

Trichotomy theorem can be expressed using the average degrees of the graphs in the class instead of the number of edges, as shown by the next lemma:

Lemma 3.4. *Let $\mathcal{C}$ be an infinite class of graphs. Then:*

$$\limsup_{G \in \mathcal{C}} \frac{\log \bar{\mathrm{d}}(G)}{\log |G|} = \limsup_{G \in \mathcal{C}} \frac{\log \|G\|}{\log |G|} - 1$$

Hence

$$\begin{aligned}\lim_{r \to \infty} \limsup_{G \in \mathcal{C} \mathbin{\triangledown} r} \frac{\log \bar{\mathrm{d}}(G)}{\log |G|} &= \lim_{r \to \infty} \limsup_{G \in \mathcal{C} \mathbin{\widetilde{\triangledown}} r} \frac{\log \bar{\mathrm{d}}(G)}{\log |G|} \\ &= \lim_{r \to \infty} \limsup_{G \in \mathcal{C} \mathbin{\triangledown} r} \frac{\log \|G\|}{\log |G|} - 1\end{aligned}$$

where $\bar{\mathrm{d}}(G) = 2\|G\|/|G|$ denotes the average degree of the graph G.

Proof.

$$\frac{\log \bar{\mathrm{d}}(G)}{\log |G|} = \frac{1 + \log \|G\| - \log |G|}{\log |G|} = \frac{1 + \log \|G\|}{\log |G|} - 1$$

□

3.2 Dichotomy

Here we shall consider the only dichotomy between classes of nowhere dense graphs, that is classes

$$\mathcal{C} \text{ such that } \lim_{r \to \infty} \limsup_{G \in \mathcal{C} \mathbin{\triangledown} r} \frac{\log \|G\|}{\log |G|} \leq 1$$

and *classes of somewhere dense graphs*, which are defined as classes $\mathcal{C}$ such that

$$\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log \|G\|}{\log |G|} = 2.$$

Instead of the average degree, one can consider the minimum degree $\delta(G)$ of graph G, as shown by the next lemma:

Lemma 3.5. *Let $\mathcal{C}$ be a hereditary infinite class of graphs. Then*

$$\begin{aligned}\limsup_{G\in\mathcal{C}} \frac{\log \bar{\mathrm{d}}(G)}{\log |G|} = 0 \quad &\iff \quad \limsup_{G\in\mathcal{C}} \frac{\log \delta(G)}{\log |G|} = 0\\ &\iff \quad \limsup_{G\in\mathcal{C}} \frac{\log \nabla_0(G)}{\log |G|} = 0\end{aligned}$$

Proof. Every graph G contains a subgraph H with mimum degree $\delta(H) \geq \bar{\mathrm{d}}(G)/4$ obtained by iteratively removing the vertices of degree at most $\bar{\mathrm{d}}(G)/4$ (the degrees take the previous deletions into account). During the process, at most $\bar{\mathrm{d}}(G)/4$ edges have been removed by vertex deletion, hence $\|H\| \geq \|G\|/2$ and $|H| \geq \sqrt{2\|H\|} \geq \sqrt{\|G\|}$. It follows that every graph G has a subgraph H such that:

$$\frac{\log \bar{\mathrm{d}}(G) - 2}{\log |G|} \leq \frac{\log \delta(H)}{\log |H|} \leq 2\frac{\log \bar{\mathrm{d}}(G)}{\log |G|}$$

Hence if $\mathcal{C}$ is a hereditary infinite class of graphs:

$$\limsup_{G\in\mathcal{C}} \frac{\log \bar{\mathrm{d}}(G)}{\log |G|} = 0 \quad \iff \quad \limsup_{G\in\mathcal{C}} \frac{\log \delta(G)}{\log |G|} = 0$$

and also

$$\limsup_{G\in\mathcal{C}} \frac{\log \bar{\mathrm{d}}(G)}{\log |G|} = 0 \quad \iff \quad \limsup_{G\in\mathcal{C}} \frac{\log \nabla_0(G)}{\log |G|} = 0$$

as $\delta(G) \leq \nabla_0(G) \leq \bar{\mathrm{d}}(G)$. □

Lemma 3.6. *Let $\mathcal{C}$ be a monotone infinite class of graphs and let r be an integer. Then*

$$\limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log \nabla_0(G)}{\log |G|} \geq \limsup_{H\in\mathcal{C}} \frac{\log \nabla_r(H)}{\log |H|} \geq \frac{1}{2} \limsup_{G\in\mathcal{C}\,\triangledown\, r} \frac{\log \nabla_0(G)}{\log |G|}$$

Proof. Let $G \in \mathcal{C} \,\triangledown\, r$ and let H be a minimal graph of $\mathcal{C}$ such that $G \in H \,\triangledown\, r$. Then, each vertex v of G corresponds to a tree Y_v of H with order at most $1 + d_G(v)$ (as $\mathcal{C}$ is monotone, we may delete any unnecessary vertices or edges of H). It follows that $|H| \leq 2\|G\|r + |G| \leq 2|G|^2 r$. Hence for every $G \in \mathcal{C} \,\triangledown\, r$ there exists $H \in \mathcal{C}$ such that $\frac{\log \nabla_r(H)}{\log|H|} \geq \frac{\log \nabla_0(G)}{2\log|G|+\log r}$. From this follows

$$\frac{1}{2} \limsup_{G \in \mathcal{C} \,\triangledown\, r} \frac{\log \nabla_0(G)}{\log|G|} \leq \limsup_{H \in \mathcal{C}} \frac{\log \nabla_r(H)}{\log|H|}$$

Now, let $H \in \mathcal{C}$ and let $G \in H \,\triangledown\, r$ be such that $\nabla_r(H) = \nabla_0(H)$. As $|G| \leq |H|$ we have

$$\frac{\log \nabla_r(H)}{\log|H|} \leq \frac{\log \nabla_0(G)}{\log|G|}$$

hence

$$\limsup_{H \in \mathcal{C}} \frac{\log \nabla_r(H)}{\log|H|} \leq \limsup_{G \in \mathcal{C} \,\triangledown\, r} \frac{\log \nabla_0(G)}{\log|G|}$$

what completes the proof. □

Lemma 3.7. *Let $\mathcal{C}$ be an infinite class of graphs and let r be an integer. Then*

$$\frac{1}{(r+1)^2} \limsup_{G \in \mathcal{C}} \frac{\log \nabla_r(G)}{\log|G|} \leq \limsup_{G \in \mathcal{C}} \frac{\log \widetilde{\nabla}_r(G)}{\log|G|} \leq \limsup_{G \in \mathcal{C}} \frac{\log \nabla_r(G)}{\log|G|}$$

Proof. This is a direct consequence of Theorem 2.1. □

3.3 Within the Nowhere Dense World

Class $\mathcal{C}$ has *bounded expansion* [NOdM08a] if each of the classes $\mathcal{C} \,\triangledown\, i$ has bounded density:

$$\begin{aligned} \mathcal{C} \text{ has bounded expansion } &\iff \forall i \geq 0: \ \sup\left\{\frac{\|G\|}{|G|} : G \in \mathcal{C} \,\triangledown\, i\right\} < \infty \\ &\iff \forall i \geq 0: \ \nabla_i(\mathcal{C}) < \infty \end{aligned}$$

It has been proved in [Dvo07] that bounded expansion classes may be also defined in terms of density of the shallow topological minors:

$$\begin{aligned}\mathcal{C} \text{ has bounded expansion } &\iff \forall i \geq 0 : \ \sup \left\{ \frac{\|G\|}{|G|} : G \in \mathcal{C} \,\widetilde{\triangledown}\, i \right\} < \infty \\ &\iff \forall i \geq 0 : \ \widetilde{\nabla}_i(\mathcal{C}) < \infty\end{aligned}$$

We shall add two more types of classes: bounded local expansion and class of nowhere dense graphs. The class $\mathcal{C}$ has *bounded local expansion* if the balls of bounded radius of graphs in $\mathcal{C}$ have bounded expansion:

$$\mathcal{C} \text{ has bounded local expansion } \iff \forall \rho, i \geq 0 : \sup_{v \in G \in \mathcal{C}} \nabla_i(G[N_\rho^G(v)]) < \infty$$

Bounded expansion classes strictly contain proper minor closed classes (as classes with constant expansion). Bounded local expansion classes generalize classes with locally forbidden a minor.

For an extensive studies of bounded expansion classes we refer the reader to [NOdM08a][NOdM08b][NOdM08c][Dvo07][Dvo08],[Zhu08]. Notice that a class $\mathcal{C}$ is a class of nowhere dense graphs if for every integer $i \geq 0$, the class $\mathcal{C} \triangledown i$ does not contain all finite simple graphs.

The inclusion of these classes and of several other types of classes of nowhere dense graphs is depicted Fig. 4.

Remark 3.8. Let us note (a pleasing fact) that the class of "locally nowhere dense" graphs coincides with the notion of nowhere dense graphs. If $\mathcal{C}$ is "locally nowhere dense", then for any r the subclass $\mathcal{C}_{2r}$ of the graphs in $\mathcal{C}$ with diameter at most $2r$ is nowhere dense, hence the clique number of the shallow topological minor class of $\mathcal{C}_{2r}$ with depth r is bounded. But a r-subdivision of a clique is in a graph of $\mathcal{C}_{2r}$ if and only if it is in a graph of $\mathcal{C}$. Hence the clique number of the shallow topological minors of depth r of $\mathcal{C}_{2r}$ is bounded. By definition, this means that $\mathcal{C}$ is a class of nowhere dense graphs.

Note that the same kind of argument does not apply to bounded expansion classes which are characterized by "dense" minors which may have unbounded diameter.

4 Classes of Nowhere Dense Graphs

In this section we give several class properties which provide equivalent characterizasion of classes of nowhere dense graphs. We first state the main result which will be proved in the following sections 4.1–4.2. This result combine virtually all concepts which were developed for the study of bounded expansion classes and exposes them in the new light. It also appears that classes of nowhere dense graphs are a quantitative generalization of bounded expansion classes and that these classes reach the limit for structural properties: Graphs with n vertices and $n^{1+\epsilon}$ edges have already typical properties of random graphs; see e.g. [Erd59][AS08]. This is yet another manifestation of the dichotomy "randomness vs structure", see [Tao07]. In its variety this also resembles the characterization of quasi-random structures [CGW89]. (Sparse quasi-random structures are more particularly studied in [CG02].)

The undefined notions in this characterizations will be defined below.