Extended Gallai's Theorem

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Abstract

Let G and H be graphs. We say G is H-critical, if every proper subgraph of G except G itself is homomorphic to H. This generalizes the widely known concept of k-color-critical graphs, as they are the case $H = K_{k-1}$. In 1963 [2], Gallai proved that the vertices of degree k in a K_k-critical graph induce a subgraph whose blocks are either odd cycles or complete graphs. We generalize Gallai's Theorem for every H-critical graph, where $H = K_{k-2}+H'$, (the join of a complete graph K_{k-2} with any graph H'). This answers one of the two unknown cases of a problem given in [7]. We also propose an open question, which may be a characterization of all graphs for which Gallai's Theorem holds.

1 Introduction

Gallai's Theorem [2] is powerful as it implies the classical Brook's Theorem [1] and also has been used to solve other coloring problems [9], [5]. We show here that its generalization yields more clues to the generalized form of coloring problem via graph homomorphisms.

Graphs in this note are finite, undirected and simple (with no loops and parallel edges). Let G, G' be graphs. A *homomorphism* from G to G' is a mapping $f{:}V(G) \to V(G')$ which preserves adjacency. That is, if $uv \in E(G)$ then $f(u)f(v) \in E(G')$. We write $G \leq G'$ if there is a homomorphism from G to G'. The notation $G \sim G'$ means $G \leq G' \leq G$. The smallest graph H for which $G \sim H$ is called the *core* of G. For finite graphs, the core is

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uniquely determined up to an isomorphism. It can also be seen that H is an induced subgraph of G. See [4] for introduction to graph homomorphisms.

Let H be a graph. We say G is H-critical if $G \not\leq H$ and $G' \leq H$ for every proper subgraph G' of G. The problem of deciding if $G \leq H$ is introduced as the *H-coloring* problem in [3], [4]. Several types of graph colorings that are refinements of the usual vertex-coloring such as the circular- and fractional-colorings are equivalent to homomorphisms to some special types of graphs. In this sense, H-coloring is the most general form of coloring, because we have no restriction to what H can be. In the next section we prove the main result of this note. We adopt a nice proof technique used in [5] for vertex-coloring, to fit to our graph-homomorphism problem.

2 Gallai's Theorem for H-critical graphs

As a notational convenience we assume K_0 is the *null-graph* with $V(K_0) = E(K_0) = \emptyset$. Thus, $K_0 + G = G$ for any graph G. We say a graph G is a *Gallai-graph* if every block of G is either an odd cycle or a clique. If $H = H_1 + H_2$, for the sake of distinguishing the two special subgraphs of H from other subgraphs of H, we use the notation $\underline{H_1}$ and $\underline{H_2}$, respectively. For example $K_1 + C_5$ is a wheel and $\underline{K_1}$ refers to the hub of the wheel and $\underline{C_5}$ refers to the pentagon induced by the rims of the wheel. We start by stating Gallai's Theorem:

Theorem 1 *(Gallai [2]) Let G be a K_k-critical graph. Then, the degree-k vertices of G induce a Gallai-graph.*

Below, we generalize the theorem as follows:

Theorem 2 *Let G be an H-critical graph, where $H = K_{k-2} + H'$, $k \geq 2$ and H' is any graph. Then, the degree-k vertices of G induce a Gallai-graph.*

Proof: Let G be an H-critical graph. Any map-critical graph is a core, and so G is a core. Also, G is connected, since otherwise each component maps to H, by definition, and we get $G \leq H$, a contradiction. We avoid redundant cases by assuming that H is a core. It is easy to see that H is a core if and only if H' is a core. Suppose first $H' \sim K_1$. If $k = 2$, then $H = K_1$ and the result holds trivially, since K_2 is the only K_1-critical graph. If $k > 2$, we regard $H = K_{(k-1)-2} + K_2$. We may assume H' has at least one edge. If H' is a non-trivial bipartite graph then $H' \sim K_2$ and so $H' = K_2$. Also, since H' is a core it contains no isolated vertex.

Now in general, the result holds for $k = 2$, because the degree-2 vertices of G induce a forest or just an odd cycle as G is a connected core. Assume $k \geq 3$. Let G'' be a block in the subgraph G' that is induced by degree-k vertices of G and let C be an even cycle in G''. If $v \in C$, then we claim that v has a chord in C.

Suppose $C = \{v_0, v_1, \ldots, v_{2p} = v_0\}, p \geq 2$. Let $G_j = G - v_j, 0 \leq j \leq 2p - 1$. For a contradiction, assume that v_1 does not have a chord. By definition, there exits a homomorphism f_1 obtaining $G_1 \leq H$. We note that $f_1(N(v_1))$ contains $\underline{K_{k-2}}$, since otherwise we find a vertex w in $\underline{K_{k-2}}$ where we can map v_1 and so extend f_1 to get $G \leq H$, a contradiction. Next observe that f_1 is injective on $N(v_1)$ since otherwise $f_1(N(v_1))$ will be properly contained in a K_k subgraph in H, and so extends to $G \leq H$. It follows that two neighbors of v_1 are mapped to two distinct vertices u and u' of $\underline{H'}$.

Recursively, for $j \geq 2$, we define a homomorphism f_j from G_j to H which agrees with f_{j-1} on $G_{j-1} \cap G_j$ and maps v_{j-1} by the following rule: If $f_{j-1}(v_j) = w \in V(\underline{K_{k-2}})$, we let $f_j(v_{j-1}) = w$. Otherwise, (assuming $f_{j-1}(v_j) = u'$), we let $f_j(v_{j-1}) = u''$, where $uu'' \in E(\underline{H'})$. Note that if $\underline{H'} = K_2$, then these two cases are the same. Once again f_j is injective on $N(v_j)$, and $f_j(N(v_j)) \supseteq \underline{K_{k-2}}$. After applying $f_1, f_2, \ldots, f_{2p}$, every vertex v_j that is mapped by f_1 to $\underline{K_{k-2}}$ is replaced by v_{j-1}. Call this process a reduction-round.

We claim that $f_1(N(v_1) - \{v_0, v_2\}) = \underline{K_{k-2}}$. Suppose not and assume without loss of generality $f_1(v_2) = w \in \underline{K_{k-2}}$. Then, $f_1(v_0) = w' \neq w$. Let r be minimal such that $f_1(v_{2r+2}) \neq w, 1 \leq r \leq p-1$. Then after $2r$ reduction-rounds, at w we have v_0 and perhaps other vertices v_{j-2r} of C. Since $j \neq 2r+2$, no neighbor of v_1 except v_0 is at w. Resuming the next round we stop the recursion after we get $f_0(v_{2p-1}) = w$ and $f_1(v_0) = w'' \neq w$. We have $f_1(N(v_1)) \not\supseteq \underline{K_{k-2}}$, a contradiction. Hence $f_1(N(v_1) - \{v_0, v_2\}) = \underline{K_{k-2}}$ as claimed.

We see now that $f_1(C - v_1)$ is contained in $\underline{H'}$. For if $f_1(v_j) = w \in V(\underline{K_{k-2}})$, then after a few reduction-rounds we have $w = f_3(v_2) = f_4(v_2) = \ldots = f_{2p}(v_2) = f_1(v_2)$, contrary to the previous paragraph result. Hence $f_1(C - v_1) \subseteq \underline{H'}$.

Each v_j in $C - v_1$ has at least $k - 2$ neighbors mapped to $\underline{K_{k-2}}$ by f_1, for otherwise we can change the image of v_j to be in $\underline{K_{k-2}}$, contrary to $f_1(C - v_1) \subseteq \underline{H'}$. Now, since $deg_G(v_j) = k$ for all j, we deduce that C has no chord at all. We also have $f_1(N(v_1) - \{v_0, v_2\}) = \underline{K_{k-2}}$. Hence, we may map C to an edge of $\underline{H'}$ and have $G \leq H$, a contradiction. Hence v_1 has a chord.

It is well known [5] (also easy to see) that any 2-connected graph with this property is either an odd cycle or a complete-graph. The result follows.

□

In [8], Thomassen proved that for every orientable surface S other than the sphere there are only finitely many K_k-critical graphs on S if and only if $k \geq 5$. In [7], it was shown that there are other types of graphs H for which this finiteness property holds if $\chi(H) \geq 8$. As an open question, it was asked if the same is true for $k = 6$ and 7.

The case $k = 7$ has now a positive answer, by Theorem 2, by the following observation of Thomassen: From Euler equation we can obtain the inequality $\sum_{v \in V}(deg_G(v) - 6) \leq 12(g-1)$ for any graph G embeddable in a surface S_g. Now if G has more than $96(g-1)$ vertices and if the minimum degree of G is at least six, then it has more than $84(g-1)$ vertices of degree 6. Any K_6-critical graph (for our case $(K_4 + H')$-critical graph, $H' \not\to K_1$) has minimum degree six. Then, it follows that G contains a degree-6 vertex v with all degree-6 neighbors, such that every face incident to v is a triangle. By Gallai's Theorem [2] (in our case by Theorem 2), we deduce that $N[v_1]$ induces a K_7, as it can not be an odd cycle. Since G is H-critical, and $K_7 \not\leq H$, we observe that G can not properly contain K_7.

Corollary 1 *For every graph $G \not\to K_1$ and $H = K_4 + G$ there are only finitely many H-critical graphs in the class of all graphs embeddable in a surface S, where H is embeddable in S.*

3 Conclusion

What is proved in this note is a sufficient structural property for a graph H. If we assume H' is triangle-free, where $H = K_{k-2} + H'$, then in Theorem 2 we can replace the degree k condition, by degree-$\omega(H)$ vertices of G. This suggests that Gallai's Theorem is likely related to the clique number of a graph than for instance its chromatic number. This can be seen by considering large chromatic triangle-free graphs. In this case, the clique number of H remains k, but the chromatic number can be arbitrarily large. It is interesting to characterize the class of all graphs for which Gallai's Theorem is true, using a graph parameter such as ω. There are several examples where the theorem does not hold.

Example: Let $H = H_7$, the graph that is obtained from two copies K and K' of K_4 by performing the Hajo-sum (delete an edge from each, glue a degree-2 vertex of K with a degree-2 vertex of K' and adding an edge e_0 between the other degree-2 vertices of K and K'). To construct an H_7-critical graph G, take a copy of H_7 and a copy $K'' = K_4 - e$: let a, b be the two degree-2 vertices of K''. Add an edge from a to one end and from b to the other end of e_0 in H_7. It is easy to check that G is H_7-critical and that the subgraph induced by degree-3 vertices of G is not a Gallai-graph. In fact it is easy to find infinitely many examples, by recursively Hajo-summing two copies of Hajo-sums.

This leads us to the following:

Problem 1 *Let H be a core. If for every H-critical graph G, the subgraph G' of G induced by the degree-k vertices of G is a Gallai-graph, then is it necessary that every $k-1$ vertices of H have a common neighbor in H? Is the converse true?*

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