Midsummer Combinatorial Workshop 2008

Jana Maxová (ed.)

Preface

The 14. Prague Midsummer Combinatorial Workshop was held from July 28th to August 1st 2008 in our beautiful building Malostranské náměstí 25. This of course contributed to the comfort of the participants as all the activities (including the lunches) could be taken on the same site. Besides, as it was expressed by several participants, the renovated faculty building surely belongs to the most beautiful math and CS departments in the world! The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with the DIMATIA center. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have André Raspaud, Marco Pellegrini, Michael Stiebitz and Miklós Ruszinkó among the participants. The list of speakers is included in this booklet.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students. For example five undergraduate students from the USA and three undergraduate students from Charles University, together with their mentors Martin Bálek and Lara Pudwell, took part in the workshop, within a joint DIMATIA-DIMACS program International REU (supported jointly by NSF and Czech Ministry of Education).

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest some of the atmosphere at the workshop from these proceedings, and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

This volume was edited by Jana Maxová. Most of the contributions were supplied by the authors in an electronic form. In a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

The 14. Midsummer Combinatorial Workshops was supported by by our institute ITI (financed by the Ministry of Education of the Czech Republic as project 1M0545) DIMATIA was the main organizer.

We hope to meet again in 2009 the same midsummer week!

Jaroslav Nešetřil

The conference photo from the boat trip.

Forbidding induced subgraphs

Julia Böttcher

1 The Erdős-Hajnal conjecture

For a graph $G = (V, E)$ we let $\alpha(G)$ and $\omega(G)$ denote the size of a largest stable set and a largest clique in G, respectively. Further, $S \subset V$ is a *homogeneous set* in G if S is a stable set or a clique and we define $\hom(G) := \max\{\alpha(G), \omega(G)\}$.

Ramsey's theorem states that any graph on n vertices has $\hom(G) \geqslant \frac{1}{2}\log n$. The random graph $\mathcal{G}(n, \frac{1}{2})$ on the other hand illustrates that there are graphs G on n vertices with $\hom(G) \leqslant 2\log n$. But the random graph contains all fixed graphs H as induced subgraphs. In contrast a famous conjecture of Erdős and Hajnal states that excluding only one graph H as induced subgraph suffices to force a dramatically different edge distribution in G and $\hom(G)$ to jump from logarithmic to polynomial in n.

Conjecture 1 (Erdős-Hajnal [3]) *For all graphs H there is a positive constant ε and an integer n_0 such that for all graphs G on $n \geqslant n_0$ vertices that do not contain H as an induced subgraph we have* $\hom(G) \geqslant n^{\varepsilon}$.

One simple observation is that graphs G with chromatic number $\chi(G)$ at most n^{β} for any $0 \leqslant \beta < 1$ have stable sets of size $n^{1-\beta}$. This suggests the question whether a corresponding result is true for graphs of very high chromatic number. Indeed one can show the following result (see [5]).

Proposition 2 *Let k be fixed. For all $\varepsilon > 0$ there is an integer n_0 such that for all G on $n \geqslant n_0$ vertices with $\chi(G) \geqslant \frac{n}{k}$ we have $\omega(G) \geqslant n^{1/k-\varepsilon}$.*

This means that the Erdős-Hajnal conjecture gets interesting for graphs G with $\chi(G)$ somewhere between n^{β} and n/k, e.g. for $\chi(G) = n/\log(n)$.

If H is a path P_4 on 4 vertices then G is perfect and thus, clearly,

$$\alpha(G) \cdot \omega(G) = \alpha(G) \cdot \chi(G) \geqslant n$$

and therefore $\hom(G) \geqslant \sqrt{n}$. Erdös and Hajnal [3] showed moreover that their conjecture is true for all *very simple* graphs H: the graphs P_4 and K_1 are very simple and so are all the graphs that can be obtained from the disjoint union of two very simple graphs H_1 and H_2 by adding none or all the edges between H_1

and H_2. A much more difficult result was obtained by Chudnovsky and Safra [2] who proved that the Erdős-Hajnal conjecture is true when H is the bull graph. Not much more is known however and already for $H = C_5$ the existence of polynomial sized homogeneous sets is open.

2 Graphs without induced C_5

For examining the structure of graphs without induced C_5 one might ask what a typical member of this class looks like. To this end Prömel and Steger [4] proved the following theorem.

Theorem 3 *Almost all graphs without induced C_5 are generalized split graphs.*

Here G is a *generalized split graph* if the vertex set of G or its complement can be partitioned into disjoint cliques $V_1, \ldots, V_\ell$ such that there are no edges between V_i and V_j for all $1 < i < j \leqslant \ell$. Generalized split graphs are perfect and thus it follows that almost all graphs without induced C_5 have polynomially sized homogeneous sets.

In order to take a closer look at the structure of graphs without induced C_5 we want to restrict our attention to graphs of a given density. More precisely we are interested in the following parameters. Let $0 < d < \frac{1}{2}$ be fixed. Then we define $C_d(n)$ to be the number of graphs on n vertices without induced C_5 and with dn^2 edges and we let $S_d(n)$ count those of them that are generalized split graphs. In joint work with Anusch Taraz and Andreas Würfl we were able to prove the following.

Theorem 4 *For all $0 < d < \frac{1}{2}$ we have*

$$\lim_{n\to\infty} \frac{\log C_d(n)}{n^2} = \lim_{n\to\infty} \frac{\log S_d(n)}{n^2} = \begin{cases} \frac{1}{4} & \frac{1}{8} \leqslant d \leqslant \frac{3}{8} \\ \frac{1}{4}H(4d) & otherwise, \end{cases}$$

where $H(x) := -x \log x - (1-x)\log(1-x)$ is the binary entropy function.

This result suggests (but unfortunately does not quite prove) that also almost all induced-C_5-free graphs with a fixed (positive) density are generalized split graphs. For proving this theorem we use a variant of Szemerédis regularity lemma tailored for handling induced subgraphs by using so-called strong regularity (cf. [1]).

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Alternating Paths on Bicolored Point Sets

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(joint work with Jan Kynčl, Viola Mészáros, Rudolf Stolař and Pavel Valtr)

1 Previous Results

One of the basic problems in geometric graph theory is to decide if a given graph can be drawn on a given planar point set using pairwise non-crossing straight-line edges. In a more demanding version, the points and the vertices of the graph are colored and each vertex has to be placed in a point of the same color (see the survey [4] for further references). Interesting and non-trivial questions arise already if we want to embed a 2-colored path on a 2-colored point set. The authors of several papers have focused on embeddings of so-called alternating paths, which are paths with no monochromatic edge. Since the colors on a 2-colored alternating path must alternate along the path, a 2-colored point set S may admit a Hamiltonian alternating path only if the coloring of S is equitable, i.e., the sizes of the color classes differ by at most one.

Let S be an equitably 2-colored set of points in general position in the plane. It is known that if the two color classes of S can be separated by a line then there is a non-crossing Hamiltonian alternating path on S [1]. The same result holds if one of the color classes is exactly the set of vertices of the convex hull [1]. Kaneko et al. [5] proved that any equitably 2-colored set S of at most 12 points or of 14 points admits a non-crossing Hamiltonian alternating path. On the other hand, Kaneko et al. [5] gave examples of equitably 2-colored sets S of n points admitting no non-crossing Hamiltonian alternating path for any $n > 12$, $n \neq 14$.

The above result on sets with color classes separated by a line easily implies that any equitably 2-colored set S of size n admits a non-crossing alternating path on at least $n/2$ points of S.

Problem 1 *Does any equitably 2-colored set S of size n admit a non-crossing alternating path on at least $n/2 + f(n)$ points, where $f(n)$ is unbounded?*

On the other hand, there are equitably 2-colored sets admitting no non-crossing alternating path of length more than $\approx 2n/3$ [2, 6]. This upper bound is proved for certain colorings of sets in convex position. The above general lower bound $n/2$ can be slightly improved to $n/2 + \Omega(\sqrt{n/\log n})$ for sets in convex position [6].

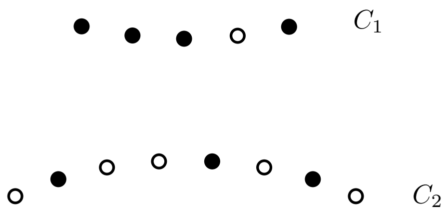

Figure 1: an equitably 2-colored double-chain (C_1, C_2)

Problem 2 *Determine the largest number c, such that in any equitably colored set of n points in convex position one can always find a non-crosing alternating path of length cn.*

In this paper we find arbitrarily large "universal" sets for which any equitable 2-coloring admits a non-crossing Hamiltonian alternating path. We prove the "universality" for so-called double-chains with each chain containing at least one fifth of all the points. Double-chains were first considered in [3].

2 Our Results

A *convex* or a *concave chain* is a finite set of points in the plane lying on the graph of a strictly convex or a strictly concave function, respectively. A *double-chain* (C_1, C_2) consists of a convex chain C_1 and a concave chain C_2 such that each point of C_2 lies strictly below every line determined by C_1 and similarly, each point of C_1 lies strictly above every line determined by C_2 (see Fig. 1). Note that we allow different sizes of the chains C_1 and C_2.

Let (C_1, C_2) be a double-chain, and let $p_1, p_2, \ldots, p_k \in C_1 \cup C_2$ be distinct points of $C_1 \cup C_2$. The polygonal line $p_1 p_2 \ldots p_k$ consisting of the $k-1$ straight-line segments $p_1p_2, p_2p_3, \ldots, p_{k-1}p_k$ is shortly called *the path* $p_1 p_2 \ldots p_k$. The path $p_1 p_2 \ldots p_k$ is *non-crossing* if any two non-consecutive segments in it are disjoint. The path $p_1 p_2 \ldots p_k$ is *Hamiltonian* (*for the double-chain* (C_1, C_2)) if it visits all the points of $C_1 \cup C_2$ (i.e., $k = |C_1| + |C_2|$).

Suppose that the points of a double-chain (C_1, C_2) are colored by two colors. Then a path $p_1 p_2 \ldots p_k$ is *alternating* if the endpoints of each segment are colored by different colors. A path on $C_1 \cup C_2$ is a *good path* if it is non-crossing, Hamiltonian and alternating.

An *equitable* 2*-coloring* of a double-chain (C_1, C_2) is a coloring of $C_1 \cup C_2$ by two colors such that the sizes of the color classes differ by at most one. We use *black* and *white* as the colors in the colorings. Here is our main result:

Theorem 3 *Let (C_1, C_2) be a double-chain whose points are colored by an equitable 2-coloring, and let $|C_i| \geqslant \frac{1}{5}(|C_1| + |C_2|)$ for $i = 1, 2$. Then (C_1, C_2) has a good path. Moreover, a good path on (C_1, C_2) can be found in linear time.*

On the other hand, we show that double-chains with highly unbalanced sizes of chains do not admit a good path for some equitable 2-colorings:

Theorem 4 *Let (C_1, C_2) be a double-chain whose points are colored by an equitable 2-coloring, and let C_1 be periodic with the following period of length 16: 2 black, 4 white, 6 black and 4 white points. If $|C_1| \geqslant 28(|C_2| + 1)$, then (C_1, C_2) has no good path.*

3 Unbalanced Double-Chains with no Good Path

This section sketches the proof of Theorem 4. Let (C_1, C_2) be a double-chain whose points are colored by an equitable 2-coloring, and let C_1 be periodic with the following period: 2 black, 4 white, 6 black and 4 white points. Let $|C_1| \geqslant 28(|C_2| + 1)$. We want to show that (C_1, C_2) has no good path.

Suppose on the contrary that (C_1, C_2) has a good path. Let $P_1, P_2, \ldots, P_t$ denote the maximal subpaths of the good path containing only points of C_1. Since between every two consecutive paths P_i, P_j in the good path there is at least one point of C_2, we have $t \leqslant |C_2| + 1$. In the following we think of C_1 as of a cyclic sequence of points on the circle. Note that we get more intervals in this way. Theorem 4 now directly follows from the following theorem.

Theorem 5 *Let C_1 be a set of points on a circle periodically 2-colored with the following period of length 16: 2 black, 4 white, 6 black and 4 white points. Suppose that all points of C_1 are covered by a set of t non-crossing alternating and pairwise disjoint paths $P_1, P_2, \ldots, P_t$. Then $t > |C_1|/28$.*

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Classes of 3-regular graphs that are $(7, 2)$-edge-choosable

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(joint work with Douglas B. West[1], University of Illinois, Urbana, IL)

Abstract

A graph is $(7, 2)$-edge-choosable if, for every assignment of lists of size 7 to the edges, it is possible to choose two colors for each edge from its list so that no color is chosen for two incident edges. We show that every 3-edge-colorable graph is $(7, 2)$-edge-choosable and also that many non-3-edge-colorable 3-regular graphs are $(7, 2)$-edge-choosable.

An s*-set edge-coloring* of a graph is an assignment of s-sets (of colors) to the edges such that incident edges receive disjoint sets. A graph G is (r, s)*-edge-colorable* if some s-set coloring uses at most r colors; the s*-set edge-chromatic number* is the least such r. In a more general model, each edge e is assigned a list $L(e)$ of available colors, and colors used on e must lie in $L(e)$. An assignment is r*-uniform* if each list has size k. A graph G is r*-edge-choosable* if a proper coloring can be chosen from any r-uniform list assignment. It is s*-set* r*-edge-choosable* if an s-set edge-coloring can be chosen from any r-uniform list assignment.

We seek the least r such that every 3-regular graph is 2-set r-edge-choosable. Tuza and Voigt [5] proved that if a connected graph G with maximum degree k is not complete and is not an odd cycle, then G is (km, m)-choosable whenever $m \geqslant 1$. Since the line graph of a 3-regular graph has maximum degree 4, every 3-regular graph is thus $(8, 2)$-edge-choosable. On his website, Bojan Mohar conjectured that every 3-regular graph is $(7, 2)$-edge-choosable.

Some 3-regular graphs are not even $(6, 2)$-edge-colorable). The 3-regular graph G formed from two copies of K_4 by subdividing one edge in each and making the two new vertices adjacent has 10 vertices and 15 edges. Hence in a $(6, 2)$-edge-coloring each color class is a perfect matching. Every perfect matching contains the central cut-edge, but only two color classes can. Thus Mohar's conjecture is sharp if true.

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We show that every 3-edge-colorable graph and many 3-regular non-3-edge-colorable graphs are $(7,2)$-edge-choosable. Some classes are $(6,2)$-edge-choosable. Since planar d-regular d-edge-colorable multigraphs are d-edge-choosable (see [2]), doubling the edges shows that planar 3-regular graphs are $(6,2)$-edge-choosable. Also, the Petersen graph is $(6,2)$-edge-choosable ([3]). These results use the Alon-Tarsi Theorem and thus provide only existence proofs. Our proofs of $(7,2)$-edge-choosability are constructive.

Our approach is to choose colors for edges in a careful order. In a 3-regular graph, each edge e is incident to four others. Colors chosen for them are forbidden at e. We want to choose the four colors on two edges incident to e using at most three colors from $L(e)$. Reducing the number of available colors at e by less than the number of colors chosen on edges incident to e is called *saving a color on* e. Doing this while choosing four colors on two edges incident to e retains at least four available colors at e. We choose colors on some edges so that the remaining graph is $(4,2)$-edge-choosable and retains lists of size 4.

We first generalize the Tuza–Voigt [4] result that even cycles are $(2m,m)$-edge-choosable.

Lemma 1 *Let A and B be sets of edges in G, with $A=\{a_1,\dots,a_k\}$ and $B=\{b_1,\dots,b_k\}$. Suppose that A is a matching and that b_i is incident to a_i and a_{i+1} but to no other edge in A (the indices are modulo k). From a uniform list assignment L, one can choose one color at each edge of A so that for each i, together a_i and a_{i+1} use at most one color from $L(b_i)$.*

Proof. When the lists are identical, use the same color on each edge of A. Index so that $L(a_1)\neq L(b_k)$. The lists have the same size, so choose $\varphi(a_1)\in L(a_1)-L(b_k)$.

If $\varphi(a_1)\notin L(b_1)$, choose $\varphi(a_2)\in L(a_2)$ arbitrarily. If $\varphi(a_1)\in L(b_1)\cap L(a_2)$, let $\varphi(a_2)=\varphi(a_1)$. If $\varphi(a_1)\in L(b_1)-L(a_2)$, choose $\varphi(a_2)\in L(a_2)-L(b_1)$. Repeat this for $a_3,\dots a_k$. At most one color from $L(b_k)$ appears in $\{\varphi(a_k),\varphi(a_1)\}$, since $\varphi(a_1)\notin L(b_k)$.

Corollary 2 (Tuza and Voigt [4]) *Even cycles are $(2m,m)$-edge-choosable.*

Proof. The cycle is the union of two matchings. Applying Lemma 1 m times to color the first matching leaves m colors for each edge of the second matching.

Theorem 3 *Every 3-edge-colorable graph is $(7,2)$-edge-choosable.*

Proof. Consider a proper 3-edge-coloring using red, green, and blue. Apply Lemma 1 twice; first pick one color for each red edge and save one color on each green edge, and next pick one color for each red edge and save one color on each blue edge. Now the green and blue edges each have 4 colors available, and they form disjoint even cycles. Applying Corollary 2 finishes the coloring.

Next we develop our main result. We use the term *double-star* for the 6-vertex tree with two vertices of degree 3. This is slightly non-standard; the term often indicates any tree with two non-leaves. A subgraph F in a graph G *consists of independent double-stars* if the components of F are double-stars and the leaves of all the components of F together form an independent set of vertices in G.

Definition 4 *A* MED decomposition *of a* 3*-regular graph is a decomposition of it into subgraphs G_1, G_2, G_3, where G_1 is a Matching, the components of G_2 are Even cycles, and G_3 consists of independent Double-stars.*

Note that a proper 3-edge-coloring is a MED decomposition in which G_3 is empty. Hence Theorem 3 is a special case of our main result Theorem 5, but the full proof of Theorem 5 is much more difficult. The details appear in [1].

Theorem 5 *Every 3-regular graph G having a MED decomposition is* $(7, 2)$*-edge-choosable.*

Proof. [Idea of Proof] We choose colors for each edge in G_1 and G_3 to save a color on each edge of G_2. Since even cycles are $(4, 2)$-edge-choosable, Corollary 2 then completes the proof.

The procedure for saving colors pays particular attention to the subgraph H obtained from G_3 by deleting the central edge of each double-star. The first phase considers components of $G_1 \cup G_2 \cup H$ independently. It results in saving one color on all edges of G_2 except a matching, while choosing two colors for each edge of G_1 and one color for each edge of H. The second phase uses a stronger form of Lemma 1 to complete the coloring of G_3 while saving a coloring on the remaining edges of G_2.

Conjecture 6 *Every 2-connected 3-regular graph has a MED decomposition.*

If Conjecture 6 is true, then by Theorem 5 all 2-connected 3-regular graphs are $(7, 2)$-edge-choosable. The argument does not extend to graphs with cut-vertices.

There is a small 3-regular graph having no MED decomposition; it is the 16-vertex graph with no perfect matching having one vertex adjacent to the vertex of degree 2 in three copies of the graph obtained from K_4 by subdividing one edge. Therefore, something stronger than Theorem 5 will be needed to prove Mohar's conjecture. Some generalization of MED decomposition may suffice, using more complicated structures than double-stars in addition to the matching and even cycles. In any case it would still be interesting to know whether every 2-connected 3-regular graph has a MED decomposition.

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New upper bound on the threshold for 3-SAT formulas

Josep Díaz

(joint work with L. Kirousis, D. Mitsche and X. Pérez-Giménez)

Satisfiability of Boolean formulas is a problem universally believed to be hard. Determining the source of this hardness will lead, as is often stressed, to applications in domains even outside the realm of mathematics or computer science; moreover, and perhaps more importantly, it will enhance our understanding of the foundations of computing.

In the beginning of the 90's several groups of experimentalists chose to examine the source of this hardness from the following viewpoint: consider a *random* 3-CNF formula with a given clauses-to-variables ratio, which is known as the *density* of the formula, here denoted γ. What is the probability of it being satisfied and how does this probability depend on the density? Their simulation results led to the conclusion that if the density is fixed and below a number approximately equal to 4.25, then for large n, a randomly chosen formula is almost always satisfiable, whereas if the density is fixed and above 4.25, a randomly chosen formula is, for large n, almost always unsatisfiable. More importantly, around 4.25 the complexity of several well known complete algorithms for checking satisfiability reaches a steep peak So, in a certain sense, 4.25 is the point where from an empirical, statistical viewpoint, the "hard" instances of SAT are to be found. Similar results were obtained for other combinatorial problems, and also for k-SAT for values of $k > 3$. These experimental results were followed by an intense activity to provide "rigorous results".

In the present, we show that 4.4898 is an upper bound. Our approach builds upon previous work. It makes use (i) of single-flip satisfying truth assignments, (ii) of formulas with a Poisson degree sequence and (iii) of positively unbalanced formulas.

First, we start by recursively eliminating one-by-one the occurrences of pure literals from the random formula, until we get its *impure core*, i.e. the largest subformula with no pure literals (a pure literal is one that has at least one occurrence in the formula but whose negation has none). Obviously this elimination has no effect on the satisfiability of the formula. Since we consider random formulas with a given degree sequence, we first have to determine what is the degree sequence of the impure core. For this, we use the differential equation method. The setting of the differential equations is more conveniently carried out in the so called *configuration* model, where the random formula is constructed by starting with as many labelled copies of each literal as its occurrences and then by considering random 3 dimensional matchings of these copies. The matchings define the clauses. We must take care of the fact that the configuration model allows formulas with (i) multiple clauses and (ii) multiple occurrences of the same variable in a clause, whereas we are interested in simple formulas, i.e. formulas where neither (i) nor (ii) holds. For our purposes, it is enough to bound from below the probability of getting a simple formula in the configuration model by $e^{-\Theta(n^{1/3}\log n)}$. The differential equations are then analytically solved, and we thus obtain the degree sequence of the core.

Second, we require that not only the degree sequence is Poisson, but also that the numbers of clauses with none, one, two and three positive literals, respectively, are close to the expected numbers. Notice that these expected numbers have to reflect the fact that we consider positively unbalanced formulas.

We compute the expectation of the number of satisfying assignments, in the framework determined by all the restrictions above, is computed. This expectation turns out to be given by a sum of polynomially many terms of functions that are exponential in n, as it is usual in such computations. We estimate this sum by its maximum term, using a standard technique. However in this case, finding the maximum term entails maximizing a function of many variables whose number depends on n. To avoid a maximization that depends on n, since then computer programs cannot be used, we prove a truncation result which allows us to consider formulas that have a Poisson degree sequence only for *light* variables, i.e. variables whose number of occurrences, either as positive or negated literals, is at most a constant independent of n.

After, we carry out the maximization. The technique we use is the standard one by Lagrange multipliers. We get a complex 3×3 system which can be solved numerically. We formally prove that the system does not maximize on the boundary of the system and we make a sweep over the domain which confirms the results of the numerical solution.

Our main result is the following:

Theorem 1 *Let* $\gamma = 4.4898$ *and* $m = \lfloor \gamma n \rfloor$. *A random* 3*-CNF formula in with* n *variables and* m *clauses, no repetition of clauses and no repetition of variables in a clause, is not satisfiable a.a.s.*

Duality in homomorphism complexes of graphs

Anton Dochtermann

(joint work with Carsten Schultz)

1 Introduction

In this extended abstract we discuss some connections between homomorphism duality and the equivariant topology of homomorphism (Hom) complexes of graphs. These observations arise from the more general results and constructions obtained in [DS], where missing proofs and further discussions are provided. The notion of duality in homomorphisms of graphs and other relational structures has been extensively studied in the work of Nešetřil and his coauthors (see for instance [NT00]); the basic idea is to identify a family $\mathcal{F}$ of obstructions to the existence of a homomorphism into a given graph G. For us the exemplary example is the collection of all odd cycles $\mathcal{C}_{odd}$, a family which provides obstructions to homomorphisms to the edge K_2. If we let $\hom(G, H)$ denote the set of (graph) homomorphisms between G and H, this duality can be expressed as:

$$\hom(\mathcal{C}_{odd}, G) \neq \emptyset \Leftrightarrow \hom(G, K_2) = \emptyset.$$

The primary focus in [NT00] is the study of *finite dualities* (where the family $\mathcal{F}$ is required to be a finite set). Although an infinite set, $\mathcal{C}_{odd}$ represents a particularly nice family in that there exist homomorphisms $C_{2r+3} \rightarrow C_{2r+1}$ which form a *linear* direct set:

$$\cdots \rightarrow C_{2r+3} \rightarrow C_{2r+1} \rightarrow \cdots \rightarrow C_5 \rightarrow C_3.$$

A naive hope would be to search for a similar directed set $\mathcal{F}_k$ which provided obstructions to homomorphisms into larger complete graphs:

$$\hom(\mathcal{F}_k, G) \neq \emptyset \Leftrightarrow \hom(G, K_{k+2}).$$

Although we do not see any obvious reason why such a family could not exist, it does seems like a lot to hope for as it would for instance imply the long-standing conjecture of Hedetniemi which states that the chromatic number of the (categorical) product $G \times H$ is equal to the minimum of the chromatic numbers of G and H.

In this paper we modify the 'classical' duality picture and consider $\mathrm{Hom}(K_2, G)$, a topological *space* that one can associate to a graph G which parameterizes all homomorphisms $f : K_2 \to G$ from an edge into G. As first shown by Lovász in [Lov78], the complex $\mathrm{Hom}(K_2, G)$ has the property that a certain topological 'complexity' (measured in terms of numerical invariants) provides a lower bound on the chromatic number of G. We then search for a family $\mathcal{T}_k$ of graphs which have the property that

$$\hom(\mathcal{T}_k, G) \neq \emptyset \Leftrightarrow \mathrm{Hom}(K_2, G) \text{ has 'complexity } k\text{'}.$$

For us the relevant notions of complexity are the *height* and *coindex* of the $\mathbb{Z}_2$-space $\mathrm{Hom}(T, G)$ (see below for definitions). In this context our main result regarding duality of homomorphism complexes is the following.

Theorem 1 *For any graph G we have*

$$\hom(\mathcal{T}_k, G) \neq \emptyset \Rightarrow \mathrm{ht}\big(\mathrm{Hom}(K_2, G)\big) \geqslant k,$$

$$\hom(\mathcal{T}_k, G) \neq \emptyset \Leftarrow \mathrm{coind}\big(\mathrm{Hom}(K_2, G)\big) \geqslant k.$$

The construction of the family of graphs $\mathcal{T}_k$ is discussed below. For our applications, we will also need the following result from [DS] which says that the graphs $\mathcal{T}_k$ are *test graphs* (as defined in [Koz07]) in the following sense.

Theorem 2 *For any graph $T \in \mathcal{T}_k$ and any graph G we have*

$$\chi(G) \geqslant \mathrm{ht}\big(\mathrm{Hom}(T, G)\big) + k + 2.$$

2 Graphs, homomorphism complexes, and $\mathbb{Z}_2$-spaces

For us a graph $G = \big(V(G), E(G)\big)$ consists of a vertex set $V(G)$ with a symmetric binary relation $E(G)$; hence our graphs are undirected, do not have multiple edges, but may have loops. If v and w are vertices of G such that $(v, w) \in E(G)$ we will often say that v and w are *adjacent* and will denote the $v \sim w$. A graph *homomorphism* $f : G \to H$ is a mapping of the vertex set $V(G) \to V(H)$ which preserves adjacency: if $v \sim w$ in G then we require $f(v) \sim f(w)$ in H.

The space $\mathrm{Hom}(T, G)$ is a polyhedral complex with vertices given by all graph homomorphisms $f : T \to G$ and with higher dimensional cells given by all 'multi-homomorphisms'. A *multi-homomorphism* is a mapping $\varphi : V(T) \to 2^{V(G)} \backslash \{\emptyset\}$ such that if $t \sim t'$ in T then $g \sim g'$ for all $g \in \varphi(t)$, $g' \in \varphi(t')$.

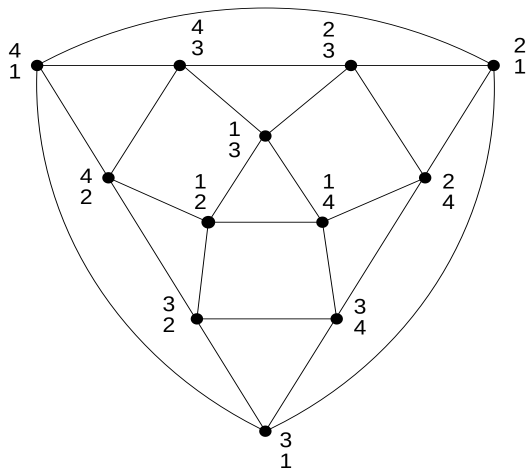

Figure 1: $\mathrm{Hom}(K_2, K_3)$

Example 3 *For $T = K_2$ and $G = K_4$, the complex* $\mathrm{Hom}(T, G)$ *is depicted in Figure 1 below. Here the vertices (which are by definition homomorphisms $K_2 \to K_4$) are labeled by the images of the vertices of $V(T)$.*

One can check that if G has no loops the space $\mathrm{Hom}(T, G)$ carries a *free* $\mathbb{Z}_2$-action induced by the nontrivial automorphisms of K_2. In general one can see that $\mathrm{Hom}(T, G)$ is a free $\mathbb{Z}_2$-space whenever T has an automorphism which flips an edge.

Given any (free) $\mathbb{Z}_2$-space X there are a number of numerical invariants used to measure the 'complexity' of the action. Perhaps the most traditional such measures are the index and coindex of X, which involve equivariant maps to and from spheres with the (free) antipodal action.

$$\mathrm{coind}(X) := \max\{n : \mathbb{S}^n \to_{\mathbb{Z}_2} X\}, \quad \mathrm{ind}(X) := \min\{n : X \to_{\mathbb{Z}_2} \mathbb{S}^n\}.$$

We also need the notion of the 'height' of a free $\mathbb{Z}_2$-space. If X is a free $\mathbb{Z}_2$-space, then basic bundle theory gives us a (unique up to homotopy) classifying map:

$$
\begin{array}{ccc}
X & \xrightarrow{\bar{f}} & \mathbb{S}^\infty \\
\downarrow & & \downarrow \\
X/\mathbb{Z}_2 & \xrightarrow[f]{} & \mathbb{R}P^\infty
\end{array}
$$

This then induces a map on cohomology $f^* : \mathbb{Z}_2[x] \cong H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \to H^*(X/\mathbb{Z}_2; \mathbb{Z}_2)$, and we define the *height* of X according to $\mathrm{ht}(X) := \max\{i : f^*(x^i) \neq 0\}$. The height is yet another measure of the complexity of the $\mathbb{Z}_2$-action on X and relates to the other invariants:

$$\mathrm{conn}(X) + 1 \leqslant \mathrm{coind}(X) \leqslant \mathrm{ht}(X) \leqslant \mathrm{ind}(X).$$

3 Constructing the graphs and a sketch of the proof

The construction of the graphs $\mathcal{T}_k$ mentioned in Theorems 1 and 2 involve taking quotients of products of graphs with a $\mathbb{Z}_2$-action. If G and H are both graphs with an involution α, then the product $G \times H$ has a $\mathbb{Z}_2$-action given by $\alpha \cdot (g, h) = (\alpha g, \alpha h)$. We define $G \times_{\mathbb{Z}_2} H$ to be the quotient graph under this action.

We define C'_m to be the cycle of length $2m$ with looped vertices labeled by $\{0, 1, \ldots, 2m-1\}$. The graph C'_m has a pair of $\mathbb{Z}_2$-actions: the antipodal left action given by $i \mapsto i+m \pmod{2m}$, and the reflection right action $i \mapsto 2m-1-i \pmod{2m}$. Now, given any graph G with a (right) $\mathbb{Z}_2$-action, we will want to consider the graph $G \times_{\mathbb{Z}_2} C'_m$ (see Figure 2).

The graph C'_m has the right (reflection) $\mathbb{Z}_2$-action which extends to $G \times_{\mathbb{Z}_2} C'_m$, and hence we can consider iterations of the $? \times_{\mathbb{Z}_2} C'_m$ construction.

Definition 4 *For integers $k, m \geqslant 1$ we define the graph*

$$T_{k,m} := K_2 \underbrace{\times_{\mathbb{Z}_2} C'_m \times_{\mathbb{Z}_2} \cdots \times_{\mathbb{Z}_2} C'_m}_{k-times}.$$

The example in Figure 2 is $T_{1,3}$, and we point out that each $T_{k,m}$ is a graph without loops. We note that there exist graph homomorphisms $f_i : C'_{3^i} \to C'_{3^{i-1}}$ given by $j \mapsto \lfloor \frac{j}{3} \rfloor$, which is equivariant with respect to both $\mathbb{Z}_2$-actions on C'_m described above. We then define $\mathcal{T}_k$ to be the collection of all graphs $\{T_{k,m} : m = 3^i, i \geqslant 1\}$ with the linear directed system induced by these homomorphisms.

Sketch proof of Theorems 1 and 2. Both theorems follow from the following more general fact proved in [DS]. If T is a graph with a $\mathbb{Z}_2$-action, there exists a

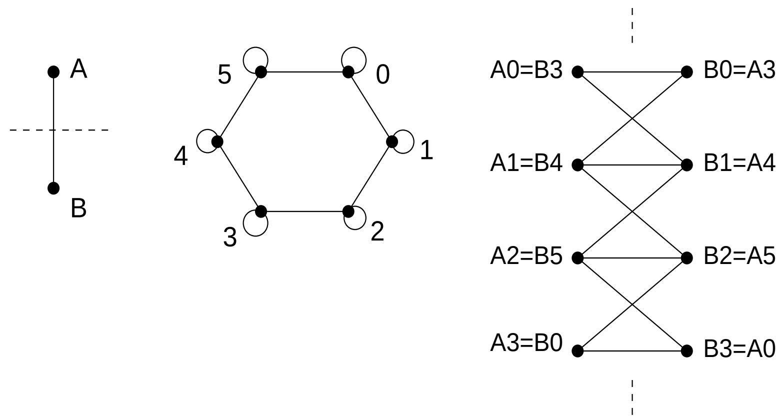

Figure 2: The graphs K_2, C'_3, and $K_2 \times_{\mathbb{Z}_2} C'_3$.

$\mathbb{Z}_2$-homotopy equivalence

$$\bigcup_{m=3^i} \mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m, G) \simeq_{\mathbb{Z}_2} \mathrm{Map}_{\mathbb{Z}_2}\big(\mathbb{S}^1, \mathrm{Hom}(T,G)\big),$$

where $\mathrm{Map}_{\mathbb{Z}_2}(X,Y)$ is the space of $\mathbb{Z}_2$-equivariant maps (with the appropriate topology) between $\mathbb{Z}_2$-spaces X and Y; here $\mathbb{S}^1$ carries a pair of $\mathbb{Z}_2$-actions given by the antipodal action and reflection. This implies the existence of a topological map $\mathbb{S}^1 \times_{\mathbb{Z}_2} |\mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m, G)| \to |\mathrm{Hom}(T,G)|$. In [Sch] it is shown that $\mathrm{ht}(\mathbb{S}^1 \times_{\mathbb{Z}_2} X) \geqslant \mathrm{ht}(X) + 1$ for any $\mathbb{Z}_2$-space X. Hence if $\mathrm{ht}\big(\mathrm{Hom}(T,G)\big) \geqslant s$ for some $T \in \mathcal{T}_k$ then we have $\mathrm{ht}\big(\mathrm{Hom}(K_2,G)\big) \geqslant k+s$. From this we conclude Theorem 2 and the first half of Theorem 1.

Similarly, if $\mathrm{coind}\big(\mathrm{Hom}(K_2,G)\big) \geqslant k$ then we obtain a map $\mathbb{S}^{k-1} \to \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathrm{Hom}(K_2,G))$ by adjunction. Iterating this procedure implies that for sufficiently large m there exists a graph homomorphism $T_{k,m} \to G$, which gives us the second half of Theorem 1 For the complete proofs we refer to [DS].

4 Applications

Our duality results allow us to prove a weakened version of the following conjecture.

Conjecture 5 (Lovàsz) *Let G be a graph with no loops. If* $\mathrm{Hom}(T, G)$ *is empty or k-connected for all graphs T of maximum degree $\leqslant d$, then $\chi(G) \geqslant k + d + 2$.*

In [BW04] Brightwell and Winkler have managed to prove a weaker version of this conjecture for the case $j = 0$. The $d = 2$ case of Conjecture 5 follows from a simple application of the Babson-Kozlov result regarding topological bounds obtained from odd cycles (see [BK07]). We can apply a similar argument with the graphs $T_{k,m}$ to obtain the following result.

Proposition 6 *Suppose G is a graph with at least one edge. If* $\mathrm{Hom}(T, G)$ *is empty or n-connected for every graph T with maximum degree $\leqslant d$, then*

$$\chi(G) \geqslant \min\{n + 1, \log_3 d\} + n + 3.$$

Proof. Since G has an edge, $\mathrm{Hom}(K_2, G)$ is nonempty and hence by assumption is n-connected. So then we have $\mathrm{coind}\big(\mathrm{Hom}(K_2, G)\big)$, and hence by Theorem 1 we have a graph map $T_{k,m} \to G$ for some m for all $k \leqslant n + 1$. We see that $T_{k,m}$ has maximum degree 3^k. We pick the maximum $k \leqslant n + 1$ such that $3^k \leqslant d$, and by assumption we get that $\mathrm{Hom}(T_{k,m}, G)$ is n-connected. By Theorem 2 this implies that

$$\chi(G) \geqslant (n + 1) + k + 2 \geqslant \min\{n + 1, \log_3 d\} + n + 3.$$

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Small Graph Classes and Bounded Expansion

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We work with simple undirected graphs, without loops or parallel edges. A class of graphs is *small* if it contains at most $n!\alpha^n$ different (but not necessarily non-isomorphic) graphs on n vertices, for some constant α. For example, the class of all trees is small, as there are exactly $n^{n-2} < n!e^n$ trees on n vertices. Norine et al. [8] showed that all proper minor-closed classes of graphs are small, answering the question of Welsh [9]. This question was motivated by the results of McDiarmid et al. [2] regarding random planar graphs. These results in fact hold for any class of graphs that is small and addable. A class $\mathcal{G}$ is *addable* if

- $G \in \mathcal{G}$ if and only if every component of G belongs to $\mathcal{G}$, and
- if $G_1, G_2 \in \mathcal{G}$, $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, then the graph obtained from the disjoint union of G_1 and G_2 by adding the edge $\{v_1, v_2\}$ belongs to $\mathcal{G}$.

Many naturally defined graph classes are addable (for example, proper minor-closed classes excluding a 2-connected minor), and this condition is usually easy to verify. The more substantial assumption thus is that the class is small. The aim of this paper is to prove that classes of graphs with expansion bounded by a slowly growing function ($f(r) = O(r^{0.315})$) are small. This generalizes the result of Norine et al. [8], as proper minor-closed classes have expansion bounded by a constant.

Let us now recall the notion of classes of graphs with bounded expansion, as defined by Nešetřil and Ossona de Mendez [6, 3, 4, 5]. The *grad (Greatest Reduced Average Density) with rank* r of a graph G is equal to the largest average density of a graph G' that can be obtained from G by removing some of the vertices (and possibly edges) and then contracting vertex-disjoint subgraphs of radius at most r to single vertices (arising parallel edges are suppressed). The grad with rank r of G is denoted by $\nabla_r(G)$. In particular, $2\nabla_0(G)$ is the maximum average degree of a subgraph of G. Given a function $f : N \to R^+$, a graph has *expansion bounded by* f if $\nabla_r(G) \leqslant f(r)$ for every integer r. A class $\mathcal{G}$ of graphs has *expansion bounded by* f if the expansion of every $G \in \mathcal{G}$ is bounded by f. Finally, we say that a class of graphs $\mathcal{G}$ has *bounded expansion* if there exists a function f such that the expansion of $\mathcal{G}$ is bounded by f.

The concept of classes of graphs with bounded expansion proves surprisingly powerful. Many classes of graphs have bounded expansion (proper minor-closed classes, classes of graphs with bounded maximum degree, classes of graphs excluding subdivision of a fixed graph, …), and many results for proper minor-closed classes (existence of colorings, small separators, light subgraphs, …) generalize to classes of graphs with bounded expansion (possibly with further natural assumptions). The classes of graphs with bounded expansion are also interesting from the algorithmic point of view, as the proofs of the mentioned results usually give simple and efficient algorithms. Furthermore, fast algorithms and data structures for problems like deciding whether a graph contains a fixed subgraph, or for determining the distance between a pair vertices (assuming that the distance is bounded by a fixed constant), have been derived. The reader is referred to [7] for a survey of the results regarding the bounded expansion.

Our main result is the following:

Theorem 1 *For any $A, B > 0$ and $0 \leqslant c < \log_9 2$, the class of graphs whose expansion is bounded by $f(r) = A + B \cdot r^c$ is small.*

Proof. [Outline of the proof] We show that any graph G with $\nabla_0(G) \leqslant f(0)$ and $\nabla_1(G) \leqslant f(1)$ contains a set $S \subseteq V(G)$ and star forests $F_1 \subseteq G[S]$ and $F_2 \subseteq G - S$ such that

- S has size $(1-\varepsilon)|V(G)|$ and the degree of vertices of S is bounded by a constant (depending on $f(0)$ and $f(1)$), and
- F_1 does not contain isolated vertices and spans all vertices that are not isolated in $G[S]$, and
- the neighborhood of an isolated vertex of $G[S]$ in G induces a clique in the graph G' obtained from $G - S$ by contracting the edges of F_2.

Furthermore, using the observation of Norine et al. [8], the number of cliques in G' is at most $1 + 4^{f(1)}|V(G')|$, and we can ensure that F_2 has many edges, so that $|V(G')|$ is small. Let G'' be the graph obtained from G by removing the isolated vertices of $G[S]$ and contracting the edges of F_1. Note that the expansion of G'' is bounded by the function $f''(r) = f(3r+1)$, and $|V(G'')| \approx |V(G)|/2$. By induction, we can bound the number of choices of G''. Furthermore, using the properties of S, we bound the number of ways how to extend G'' to a graph with expansion bounded by f with $|V(G)|$ vertices, and conclude that the number of such graphs is at most $n!k^n$, for some constant k.

For any fixed $d > 2$, the results of Bender and Canfield [1] imply that the number of simple d-regular graphs on n vertices (with dn even) is $\Omega\left(\frac{(nd/2)!}{(d!)^n}\right)$. It follows that the class of 3-regular graphs (whose expansion is bounded by $f(n) = 3 \cdot 2^{n-1}$) is not small. This bound can be slightly improved:

Theorem 2 *There exists a constant $c > 0$ such that the class of graphs with expansion bounded by the function $f(r) = c \cdot e^{\sqrt{r}\log(r+e)}$ is not small.*

However, this bound is still exponential. Significantly improving the constant $\log_9 2 \approx 0.315$ would require a new idea; ideally, we would like to decrease the size of the graph about $d = \nabla_0(G)$ times, instead of roughly twice, in each iteration of the recurrence. However, this appears hard to achieve if no or only a few vertices have degree approximately d. Nevertheless, we propose the following conjecture. A class of graphs has *subexponential expansion* if its expansion is bounded by a function f with $\frac{\log\log f(r)}{\log r} = o(1)$.

Conjecture 3 *Any class of graphs with subexponential expansion is small.*

This is motivated by the fact that classes of graphs with subexponential expansion have separators of sublinear size (Nešetřil and Ossona de Mendez [4]), indicating that some kind of structure appears in such classes.

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Linear Complementarity and Unique-Sink Orientations

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(joint work with Komei Fukuda, Bernd Gärtner and Hans-Jakob Lüthi)

Let M be a real $n \times n$ matrix and let q be a real n-vector. The *linear complementarity problem* $\mathrm{LCP}(M, q)$ is to find two non-negative real n-vectors w, z such that

$$\begin{aligned} w - Mz &= q, \\ w^T z &= 0. \end{aligned}$$

Linear complementarity problems encompass many famous optimisation problems such as linear and quadratic programming and the bimatrix game problem. Applications include computing market equilibrium as well as the optimal stopping problem of Markov chains. LCP has thus received much attention from a large number of researchers. For details see the book [4].

Determining whether $\mathrm{LCP}(M, q)$ has a solution is NP-hard [3]. We concentrate on a particular case *P-LCP*, where the matrix M is a *P-matrix*, that is, the determinant of each of its principal submatrices is positive. This is an important case because for a P-matrix M there exists a unique solution to $\mathrm{LCP}(M, q)$ for any vector q. The goal is to find the solution quickly.

Hence the main open problem is the following.

Problem 1 *Is there a polynomial-time algorithm for finding the solution to* $\mathrm{LCP}(M, q)$ *where* M *is a P-matrix?*

Our motivation stems partly from the work of Kathie Cameron and Jack Edmonds, namely the existence of an EP-theorem (see [1, 6]). Since deciding whether a matrix M is a P-matrix is in coNP, and deciding whether there exists a solution to $\mathrm{LCP}(M, q)$ is in NP, there ought to be a polynomial-time algorithm that, given M and q, finds either a certificate for M not being a P-matrix, or a solution to $\mathrm{LCP}(M, q)$. This belief is strengthened by Nimrod Megiddo's [7] observation that if P-LCP is NP-hard, then $\mathsf{NP} = \mathrm{coNP}$.

On the other hand, the same results apply to computing the Nash equilibrium of a bimatrix game. However, computing the Nash equilibrium was found to be PPAD-complete [2], which some researchers reckon to suggest there is no poly-time algorithm. (At present, the evidence for such a conjecture is rather weak, though.) While P-LCP also belongs to the class PPAD, no hardness result is known.

Problem 2 *Is P-LCP PPAD-complete?*

After primality testing has been resolved, P-LCP is now probably the most prominent problem for which neither polynomial-time algorithms nor hardness results are available.

A well-known algorithm for solving P-LCP is the *simple principal pivoting method* (see [4, Sec. 4.2]), which is in fact a class of algorithms. A particular algorithm in this class is then determined by a *pivot rule*, just as in the case of the simplex method for linear programming. This algorithm naturally opens another question: the existence or non-existence of a combinatorial (or strongly) poly-time algorithm. In particular, we have the following.

Problem 3 *Is there a pivot rule under which the simple principal pivoting method solves the P-LCP within a polynomial number of pivot steps?*

A combinatorist will like principal pivoting because it is equivalent to finding the sink of an orientation of the n-cube (as was shown by Alan Stickney and Layne Watson [10]). More precisely, a (combinatorial) n-*cube* is a graph whose vertices are n-dimensional $(0, 1)$-vectors, and two vertices are adjacent if their Hamming distance is exactly 1. A *unique-sink orientation* (USO) is an orientation of the n-cube in which every subcube (face) has a unique sink.

The reader is now kindly requested to consult [9, Sec. 3.2] or [5] for the precise description of how a P-LCP determines a unique-sink orientation of a cube. Here we just note that the run of a simple principal pivoting method may be described as following a directed path in a unique-sink orientation, and a pivot rule corresponds to a rule for choosing the outgoing edge if there is more than one option.

In fact, finding the sink of the corresponding USO is all we need to solve the P-LCP. Therefore we examine the problem of finding the sink in an abstract setting (see [11]): We assume that the USO is given by an oracle that,

given a vertex of the USO, returns the orientation of incident edges, and we want an algorithm that finds the sink of an USO. In this situation, the number of questions to the oracle should be bounded by a polynomial in the dimension n of the cube. Such an algorithm would in turn provide a strongly polynomial algorithm for P-LCP.

There is little hope that a deterministic polynomial algorithm would exist that finds the sink of any USO. It is much more likely that we have to impose some conditions met by LCP-induced orientations. One such condition is the *Holt-Klee condition*: in each face of dimension d there exist d vertex disjoint paths from the (unique) source to the (unique) sink. The importance of this class of USOs remains unclear but it seems that this condition is still insufficient (the class is too large).

The simple principle pivoting method can be described as follows.

Algorithm 1 Simple principal pivoting

start in any vertex of the n-cube USO
while not in the sink **do**
    choose an outgoing edge according to the *pivot rule*
    move to the neighbouring vertex along this edge
end while
return the sink of the USO

The lack of success with deterministic pivot rules motivated research into *randomised* pivot rules. The simplest rule, which chooses the outgoing edge uniformly at random in each step, has been found to take substantially more steps on some examples than the exhaustive enumeration of all 2^n vertices [8].

More hope rests on the following rule: In the beginning, choose a random permutation π from the symmetric group S_n. In each step, select the outgoing edge in the coordinate with the least index according to π. Let us refer to the simple principal pivoting method with this randomised pivot rule as the *Random-Permutation* algorithm. Currently no slow examples for Random-Permutation are known that satisfy the Holt-Klee condition.

Problem 4 *Does the Random-Permutation algorithm find the sink of any P-LCP-induced unique-sink orientation in expected polynomial time?*

The analysis seems to be so hard that even positive results for some special classes of USOs would be a fair success. Two special cases have recently been resolved in [5].

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Paths of low weight and connectivity in planar graphs

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In this paper we consider only simple graphs. We use a standard graph theory terminology except of terms defined below. Let a k-vertex be a vertex of degree k. Let an (a, b)-edge be an edge h if an a-vertex and a b-vertex are endvertices of h. Let a k-path and a k-cycle be a path and a cycle on k vertices respectively. For a subgraph H of a planar graph G the *weight* $w_G(H)$ of H is defined to be the sum of vertices of H in G, namely

$$w_G(H) = \sum_{n \in V(H)} \deg_G(n)\,.$$

It is well known that every planar graph contains a vertex of degree at most 5. In 1955 Kotzig [K1] proved that every 3-connected planar graph contains an edge of the weight at most 13 in general and at most 11 in the absence of 3-vertices, respectively. These bounds are best possible. Erdös conjectured that the requirement on 3-connectivity can be replaced with the condition on minimum degree at least 3. This was proved in 1971 by Barnette, see [G], and later independently by Borodin [B]. In 1993 Ando, Iwasaki and Kaneko [AIK] proved that every 3-connected planar graph G contains a 3-path P such that $w_G(P) \leqslant 21$. The bound 21 is tight. There are two natural directions in generalizing the above theorems.

(A) Find the smallest integer $f(\varkappa, k)$ such that whenever a $\varkappa$-connected planar graph G contains at least k vertices, $k \geqslant 1$, then it also contains a connected subgraph H on k vertices (of order k) for which there is

$$w_G(H) = \sum_{v \in V(H)} \deg_H(v) \leqslant f(\varkappa, k)\,.$$

(B) Find the smallest integer $w = w(\varkappa, k)$ such that whenever a $\varkappa$-connected planar graph G contains a k-path, then it also contains a k-path P with weight

$$w_G(P) = \sum_{v \in V(P)} \deg_P(v) \leqslant w(\varkappa, k)\,.$$

The possibility **(A)** was investigated by Enomoto and Ota [EO]. They proved that for every $k \geqslant 4$

$$8k - 5 \leqslant f(3,k) \leqslant 8k - 1$$

and conjectured the precise value of $f(3,k)$ to be $8k-5$.

The problem **(B)** was formulated in [FJ1]. The precise values of $w(\varkappa,k)$ are known only for small values of k, namely $w(1,1)=5$, $w(3,2)=13$, $w(4,2)=11=w(5,2)$. From the papers of Fabrici and Jendrol' [FJ1], [FJ2] it follows

Theorem 1 ([FJ1], [FJ2]). *For every $k \geqslant 1$ there is*

$$k \log k \leqslant w(3,k) \leqslant 5k^2\,.$$

Madaras [Ma] *improved the upper bound showing that $w(3,k) \leqslant \frac{5}{2}(k+1)$.*

Recently Fabrici, Harant and Jendrol' [FHJ] proved

Theorem 2 ([FHJ]). *Let k be an integer $k \geqslant 4$. Then*

(i) *Every plane triangulation T, that contains a k-path, also contains a k-path P such that*

$$w_T(P) \leqslant k^2 + 13k\,.$$

(ii)

$$w(3,k) \leqslant \frac{3}{2}\,k^2 + \mathcal{O}(k)\,.$$

The following beautiful result was proved by Mohar

Theorem 3 ([Mo]). *Let $k \geqslant 1$ be an integer. Then*

$$w(4,k) = 6k - 1\,.$$

For 2-connected planar graph we have the following [J]

Theorem 4 ([J]). *Let H be a connected planar graph of order at least 3 and let m be an inter. Then there exists a 2-connected planar graph G of minimum degree $\delta(G) \geqslant 3$ such that each copy K of H in G has weight*

$$w_G(K) \geqslant m\,.$$

Corollary.

$$w(2,k) = +\infty\,.$$

For any $k \geqslant 3$.

Notice that if we replace the path in the problem **(B)** we obtain

Theorem 5 ([FJ1]). *Let H be a connected planar graph different from any path and let m be an integer. Then there exists a 3-connected planar graph G such that each copy K of H in G has weight*

$$w_G(K) \geqslant m\,.$$

On the other hand using an idea of Mohar [Mo], Fabrici, Harant and Jendrol' proved

Theorem 6 ([FHJ]). *Let G be a 3-connected n-vertex planar graph and let $c(G)$ be the circumference of G. Let $\sigma = \frac{c(G)}{n}$ and $3 \leqslant k \leqslant c(G)$. Then G contains a k-path P with*

$$w_G(P) = \sum_{u \in V(P)} \deg_G(u) < \Big(\frac{3}{\sigma} + 3\Big)k\,.$$

So I believe that the following is true

Conjecture.

$$w(3,k) = \mathcal{O}(k \log\ k)\,.$$

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Cover-incomparability graphs - new graphs associated to posets

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1 The concept and its motivation

There are three standard ways in which one can associate a graph to a given poset P. In the *cover graph* of P points x and y are adjacent if either x covers y or y covers x, and in the *compararability graph* they are adjacent if they are comparable in P. The *incomparability graph* is the complement of the comparability graph.

A *transit function* on a non empty set V is a function $T : V \times V \to 2^V$ satisfying the following transit axioms:

(t1) $u \in T(u, v)$ for any u and $v \in V$.
(t2) $T(u, v) = T(v, u)$ for all u and $v \in V$.
(t3) $T(u, u) = \{u\}$ for all $u \in V$.

This notion was introduced by Mulder about ten years ago and written up in [3], see also [1].

The *underlying graph* G_T of a transit function T on a set V is the graph with vertex set V, where distinct u and v in V are joined by an edge if $|T(u, v)| = 2$.

For a poset $P = (V, \leqslant)$, the *standard poset transit function* $T_P : V \times V \to 2^V$ is defined in the following way [2]:

(i) If x and y are incomparable, then $T_P(x, y) = \{x, y\}$.
(ii) If $x \leqslant y$, then $T_P(x, y) = \{z \mid x \leqslant z \leqslant y\}$.
(iii) If $y \leqslant x$, then $T_P(x, y) = \{z \mid y \leqslant z \leqslant x\}$.

Clearly, T_P is a transit function. The underlying graph G_{T_P} of T_P is obtained from the cover graph of P by adding an edge between any pair of incomparable elements of P. Thus the edges of G_{T_P} are the union of the edges of the cover graph of P and the incomparability graph of P. Hence we call G_{T_P} the *cover-incomparability graph* (*C-I graph*) of P.

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2 Results and problems

Theorem 1 *Let $\mathcal{G}$ be a class of graphs with a forbidden induced subgraphs characterization. Let*

$$\mathcal{P} = \{P \mid P \text{ is a poset with } G_{T_P} \in \mathcal{G}\}.$$

Then $\mathcal{P}$ has a forbidden subposets characterization.

This theorem was applied in particular to chordal graphs and distance-hereditary graphs, in both cases there are three forbidden posets. (Recall that a connected graph is *distance-hereditary* if every induced path is a shortest path.) For instance:

Theorem 2 *Let P be a poset. Then G_{T_P} is chordal if and only if P is P_1-, P_2- and P_3-free; see Fig. 1.*

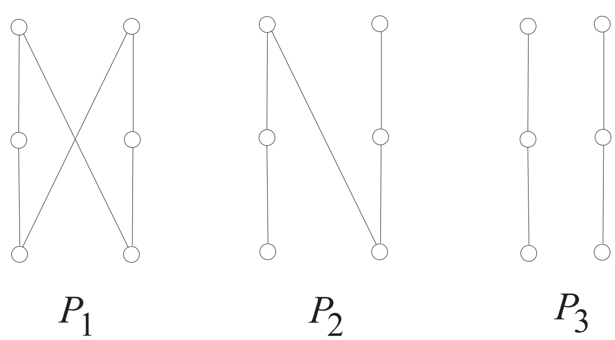

Figure 1: Forbidden subposets for C_4

There are many questions and problems one can ask about C-I graphs.

Problem 3 *Find forbidden subposets for other classes of graphs that allow forbidden subgraphs characterization. For instance, line graphs.*

Theorem 2 has the following reverse question.

Problem 4 *Which chordal graphs are C-I graphs?*

More generally:

Problem 5 *Which graphs are C-I graphs?*

And finally:

Problem 6 *Is the recognition problem for C-I graphs hard?*

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Markov bases of graph models of K_4-free graphs

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Sturmfels and Sullivant [Michigan Math. J. 57, 2008] introduced cut ideals and ideals associated with binary graph models. We study the latter notion. If G is an n-vertex graph, let $\Psi_G : K[p_{i_1 \dots i_n}] \to K[q^e_{k\ell}]$ be a homomorphism between polynomial rings over a field K with variables $p_{i_1 \dots, i_n}$, $i_k \in \{0,1\}$, and $q^e_{k\ell}$, $e \in E(G)$ and $k, \ell \in \{0,1\}$, that is defined as the linear extension of the mapping $p_{i_1 \dots i_n} \to \prod_{e=jj' \in E(G)} q^e_{i_j i_{j'}}$. The ideal J_G which we study is the kernel of Ψ_G. We prove a conjectrue of Develin and Sullivant that the degree $\mu(J_G)$ of a largest minimal generator of J_G is at most four if and only if G is a graph with no K_4 minor, i.e., the tree-width of G is at most two.

Let us also mention some interesting open problems in this area, mostly due to Develin and Sullivant:

- Is $\mu(J_G)$ a function of the tree-width of G only?
- Is $\mu(J_G)$ bounded by six for planar graphs G? Or at least, bounded by a constant?
- What is $\mu(J_G)$ if G is a complete graph?

With respect to the first two questions, it would be interesting to determine $\mu(J_G)$ for the graph G obtained from a complete graph of order six by removing a perfect matching. This graph is planar and has tree-width four. Since $\mu(J_{K_5}) = 10$, computing the value $\mu(J_G)$ will answer at least one of the first two questions.

Some classes of finite homomorphism-homogeneous point-line geometries

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A structure is *homogeneous* if every isomorphism between finite substructures of the structure extends to an automorphism of the structure. The theory of (countable) homogeneous structures gained its momentum in 1953 with the famous theorem of Fraïssé [6] which states that countable homogeneous structures can be recognized by the fact that their collections of finitely induced substructures have the amalgamation property. Nowdays it is a well-established theory with deep consequences in many areas of mathematics.

Homogeneous objects have been determined for many important classes of structures. For example, countably infinite homogeneous posets were characterized in [12]; countably infinite homogeneous graphs were described in [9], while the finite ones were determined in [7]; countably infinite homogeneous digraphs were described in [3] while finite and countably infinite homogeneous tournaments were described in [8]. As in this paper we are particularly interested in finite geometries, let us finally mention that homogeneous linear spaces were characterized in [5], and homogeneous semilinear spaces in [4].

In their recent paper [2] the authors discuss a generalization of homogeneity to various types of morphisms of structures, and in particular introduce the notion of homomorphism-homogeneous structures:

Definition 1 (Cameron, Nešetřil [2]) *A structure is called* homomorphism-homogeneous *if every homomorphism between finite substructures of the structure extends to an endomorphism of the structure.*

Not much is known about homomorphism-homogeneous structures. Homomorphism-homogeneous posets were characterized in [10] and the characterization of countable posets with respect to various types of morphisms can be found in [1]. Moreover, finite homomorphism-homogeneous tournaments (with loops) were characterized in [11].

In this paper we characterize several classes of finite homomorphism-homogeneous point-line geometries. The full characterization of finite homomorphism-homogeneous point-line geometries is hard because it includes the characterization of finite homomorphism-homogeneous graphs with loops, which is hard. Therefore, in this paper we restrict our attention to finite point-line geometries with "enough geometry". We consider three types of point-line geometries: disconnected point-line geometries, linear spaces, and connected proper point-line geometries.

A *point-line geometry* is an ordered pair $(X, \mathcal{L})$ where X is a set of *points*, $\mathcal{L} \subseteq \mathcal{P}(X)$ is a set of *lines* and the following is satisfied:

- every line contains at least two points, and
- every pair of distinct points is contained in *at most one* line.

A *linear space* is a point-line geometry where every pair of distinct points is contained in *exactly one* line.

An *isolated point* is a point which belongs to no line. A line with precisely two points will be referred to as an *edge*, whereas a line with three or more points will be referred to as a *proper line*. Thus, a point-line geometry where every line is an edge is nothing but a graph. A point-line geometry is *projective* if every pair of distinct lines in the geometry has a point in common, and it is *proper* if it contains a pair of distinct intersecting proper lines. A point-line geometry which is not proper will be referred to as *improper*.

Example 2 *Let I denote the linear space with exactly one point; let L_n denote the linear space with n points lying on a single line; let L_n^* denote the linear space with $n+1$ points which is obtined from L_n by adding a new point and joining the new point by an edge to each of the n points lying on the line; and let K_n denote the complete graph on n points (see Fig. 1).*

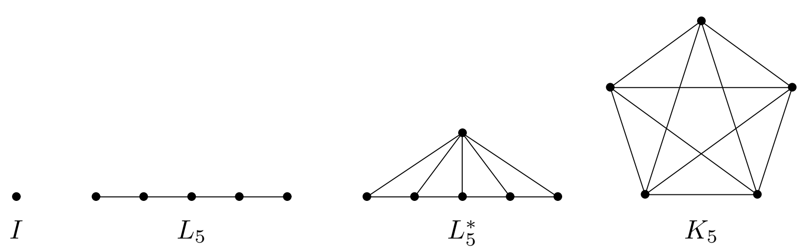

Figure 1: Four trivial linear spaces

Example 3 *A* pencil of lines *is a point-line geometry where all the lines pass through a common point, Fig. 2. The point which belongs to every line of the geometry is called the* center *of the pencil.*

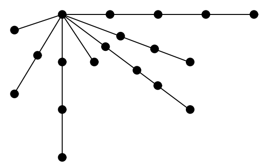

Figure 2: A pencil of lines

Example 4 *For nonnegative integers α, β and γ, let $\Delta(\alpha, \beta, \gamma)$ denote the point-line geometry $(X, \mathcal{L})$ with $2+\alpha+\beta+\gamma$ points x, y, a_1, …, a_α, b_1, …, b_β, c_1, …, c_γ and $2+\gamma$ lines k, l, m_1, …, m_γ, where (see Fig. 3)*

$$\begin{aligned} k &= \{x, y, a_1, \ldots, a_\alpha\} \\ l &= \{x, b_1, \ldots, b_\beta, c_1, \ldots, c_\gamma\} \\ m_i &= \{y, c_i\}, \qquad i \in \{1, \ldots, \gamma\}. \end{aligned}$$

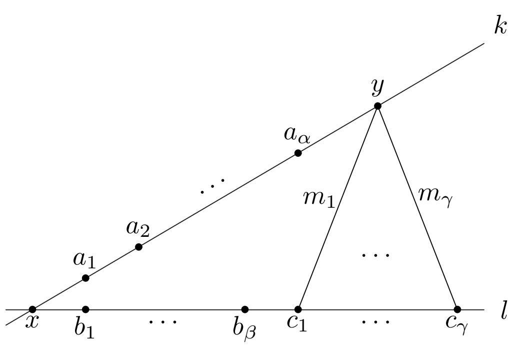

Figure 3: The point-line geometry $\Delta(\alpha, \beta, \gamma)$

Example 5 *The* Fano plane *is the projective linear space with 7 points and 7 lines depicted in Fig. 4.*

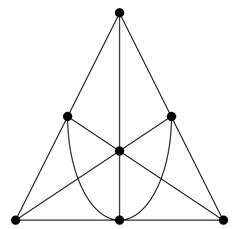

Figure 4: The Fano plane

Example 6 *A* triangular space *is a finite projective point-line geometry where point lies on exactly two lines. Let us recall a few facts about triangular spaces: all the lines in a triangular space have the same number of points* $q+1$*, where* $q \geqslant 1$ *is the* order *of the semilinear space. Let* $T(q)$ *denote the triangular space of order* q*. Fig. 5 depicts* $T(2)$*,* $T(3)$*,* $T(4)$ *and* $T(5)$*.*

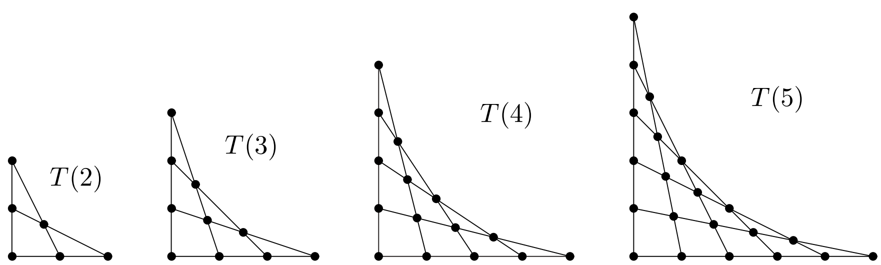

Figure 5: Triangular spaces $T(2)$, $T(3)$, $T(4)$ and $T(5)$

A point-line geometry $(Y, \mathcal{M})$ is a *subgeometry* of a point-line geometry $(X, \mathcal{L})$ if $Y \subseteq X$ and

$$\forall m \in \mathcal{M}\ \exists l \in \mathcal{L}\ :\ m \subseteq l.$$

The *subgeometry of* X *induced by* Y is the point-line geometry $(Y, \mathcal{L}|_Y)$ where

$$\mathcal{L}|_Y = \{l \cap Y : |l \cap Y| \geqslant 2\}.$$

A point-line geometry $(Y, \mathcal{M})$ is a *subdivision* of a point-line geometry $(X, \mathcal{L})$ if $Y \supseteq X$ and there is a mapping $f : Y \setminus X \to \mathcal{L}$ such that

$$\mathcal{M} = \{l \cup f^{-1}(l) : l \in \mathcal{L}\}$$

(Fig. 6). In other words, $(Y, \mathcal{M})$ is a subdivision of $(X, \mathcal{L})$ if $(Y, \mathcal{M})$ can be obtained from $(X, \mathcal{L})$ by adding new points in such a way that every new point belongs to precisely one line from $\mathcal{L}$. Note that in case $X = Y$ the mapping f is empty and $\mathcal{M} = \mathcal{L}$.

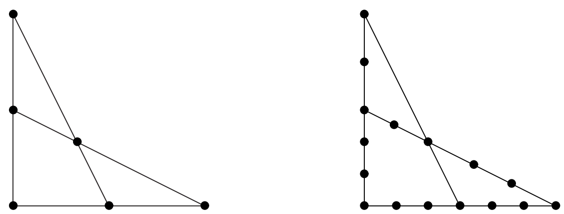

Figure 6: A point-line geometry and one of its subdivisions

A mapping $f : X \to Y$ is a *homomorphism* from a point-line geometry $(X, \mathcal{L})$ to a point-line geometry $(Y, \mathcal{M})$ if f maps collinear points onto collinear points, i.e.

$$\forall l \in \mathcal{L} \quad \exists m \in \mathcal{M} : \quad f(l) \subseteq m.$$

As usual, an *endomorphism* of a point-line geometry is a homomorphism from the point-line geometry into itself. A homomorphism f between two arbitrary finitely induced subgeometries $(S, \mathcal{L}|_S)$ and $(T, \mathcal{L}|_T)$ of a point-line geometry $(X, \mathcal{L})$ will be referred to as a *local homomorphism of* $(X, \mathcal{L})$.

Definition 7 *A point-line geometry* $(X, \mathcal{L})$ *is* homomorphism-homogeneous *if every local homomorphism extends to an endomorphism of* $(X, \mathcal{L})$.

A *chain* from x_0 to x_k in a point-line geometry $(X, \mathcal{L})$ is a sequence

$$x_0 \; l_1 \; x_1 \; \ldots \; x_{k-1} \; l_k \; x_k$$

of points and lines such that $x_0, \ldots, x_k \in X$, $l_1, \ldots, l_k \in \mathcal{L}$ and $x_{i-1}, x_i \in l_i$ for all $i \in \{1, \ldots, k\}$. A point-line geometry $(X, \mathcal{L})$ is *connected* if there is a chain from any point to any other point of the geometry. We say that $(X, \mathcal{L})$ is *disconnected* if it is not connected. A *connected component* of a point-line geometry $(X, \mathcal{L})$ is a maximal subset S of X such that the induced subgeometry $(S, \mathcal{L}|_S)$ is connected.

Our main results are the following three theorems, characterizing homomorphism-homogeneous linear spaces, disconnected point-line geometries, and proper connected point-line geometries, respectively:

Theorem 8 *A finite linear space is homomorphism-homogeneous if and only if it belongs to one of the following classes:*

(1) improper point-line geometries (this class includes some trivial geometries such as a single point, a single line and the complete graph K_n), or

(2) the Fano plane.

Theorem 9 *A finite disconnected point-line geometry is homomorphism-homogeneous if and only if*

(1) every connected component is an isolated point,

(2) every connected component induces an improper linear space with at least two points, or

(3) every connected component is isomorphic to the Fano plane, L_n or L_n^*.

Theorem 10 *A finite connected proper point-line geometry is homomorphism-homogeneous if and only if it is one of the following:*

(1) a pencil of lines,

(2) the Fano plane,

(3) a subdivision of $T(q)$, $q \geqslant 1$, or

(4) the point-line geometry $\Delta(\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \geqslant 1$ are arbitrary.

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Hardness of embedding simplicial complexes in $\mathbb{R}^d$

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(joint work with Martin Tancer and Uli Wagner)

Let $\text{EMBED}_{k\to d}$ be the following algorithmic problem: Given a finite simplicial complex K of dimension at most k, does there exist a (piecewise linear) embedding of K into $\mathbb{R}^d$? Known results easily imply polynomiality of $\text{EMBED}_{k\to 2}$ ($k = 1, 2$; the case $k = 1$, $d = 2$ is graph planarity) and of $\text{EMBED}_{k\to 2k}$ for all $k \geqslant 3$ (even if k is not considered fixed).

We show that the celebrated result of Novikov on the algorithmic unsolvability of recognizing the 5-sphere implies that $\text{EMBED}_{d\to d}$ and $\text{EMBED}_{(d-1)\to d}$ are undecidable for each $d \geqslant 5$. Our main result is NP-hardness of $\text{EMBED}_{2\to 4}$ and, more generally, of $\text{EMBED}_{k\to d}$ for all k, d with $d \geqslant 4$ and $d \geqslant k \geqslant (2d-2)/3$.

These dimensions fall outside the so-called metastable range of a theorem of Haefliger and Weber, which characterizes embeddability using the deleted product obstruction. Our reductions are based on examples, due to Segal, Spież, Freedman, Krushkal, Teichner, and Skopenkov, showing that outside the metastable range the deleted product obstruction is not sufficient to characterize embeddability.

Six-decomposition of snarks and induced colourings on 6-cuts

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(joint work with Ján Karabáš[1] and Edita Máčajová[2])

1 Introduction

A graph $G = (V, E)$ consists of the *vertex set* V and the *edge set* E. Each edge $e \in E$ has two ends and each end can, but need not be incident with a vertex. An end of an edge that is not incident with a vertex is called a *free end*. If an edge has exactly one free end, it is called a *free-edge*. A k*-pole* is a cubic graph with k free ends. If the free ends of a k-pole are endowed with a linear ordering $s_1, s_2, \dots, s_k$, $M(s_1, s_2, \dots, s_k)$ is called an *ordered* k*-pole*.

Let M be a k-pole and let x and y be free ends of M. We say that M' is formed by a *junction* of x and y if M' arises from M by identifying (gluing) x and y. Junction of two free ends of a free edge e is equivalent to the removal of e.

Let $M(e_1, e_2, \dots, e_k)$ and $N(g_1, g_2, \dots, g_m)$ be poles, f be a partial bijection matching a subset of $\{e_1, e_2, \dots, e_k\}$ to a subset of $\{g_1, g_2, \dots, g_m\}$. A *junction* $M *_f N$ of poles M and N with respect to f is a pole formed from a disjoint union of M and N gluing consecutively free ends e_i and $f(e_i)$ for every free end e_i for which the image $f(e_i)$ is defined. In particular, $M *_f N$ is a cubic graph without free ends if and only if f is a bijection matching the set of free ends of M to the set of free ends of N. A notion of a (regular) 3-edge-colouring naturally extends to cubic graphs with semiedges and free edges

The following statement is basic but fundamental.

Lemma 1 (Parity lemma) *Let φ be a 3-edge-colouring of a k-pole P. Then the number of free ends of P that are in φ coloured by any given colour has the same parity as k.*

A non 3-edge-colourable cubic graph without free ends is called a *snark*. A snark is non-trivial or *solid* if it is cyclically 4-connected and its girth is at least 5. Given k-edge cut in a snark P determines two ordered k-poles $M(e_1, e_2, \dots, e_k)$ and $N(f_1, f_2, \dots, f_k)$, where $P = M *_g N$ and $g(e_i) = f_i$ for $i = 1, 2, \dots, k$. In

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other words, the edges of the cut arise by junctions e_i and f_i, $i = 1, 2, \ldots, k$. In case one of M and N is not colourable, then one of the k-poles in the junction can be replaced by a set of free edges if k is even, or by a disjoint union of free edges and a single vertex with 3 free ends attached if k is odd. This way P is reduced to a smaller snark and the respective operation is called a *k-reduction*. Snarks which do not admit a k-reduction are called *k-irreducible*. In 1996 Nedela and Škoviera [5] have characterised k-irreducible snarks. It follows from their characterisation that a snark is k-irreducible for some $k \geqslant 6$ if and only if it is 6-irreducible. It is proved that 6-irreducibility of a simple snark X is characterized by the property that removal of any 2-vertex pole (connected or not) yields a 3-edge-colourable graph.

Assume a snark $P = M *_f N$ where both M and N are 3-edge-colourable k-poles. Adding small 3-edge-colourable k-poles M' and N' to both M and N we may form two new snarks $M *_f M'$ and $N *_g N'$ with $|M * M'| \leqslant |P|$ and $|N * N'| \leqslant |P|$. Such an operation is called a *k-decomposition*, where $|P|$ denotes the number of vertices of respective pole.

Theorem 2 (General k-Decomposition Theorem [5]) *There is a function $\Phi(k)$ such that for every snark $G = M * N$ which is a junction of k-poles M, N*

(a) *either one of M, N is not colourable,*

(b) *or both M, N are colourable and by adding at most $\Phi(k)$ vertices both M, N can be completed to snarks.*

A k-decomposition (k-reduction) is *proper* if the resulting snarks have smaller order than the original one. A snark is called *k-irreducible* and *k-indecomposable* if it admits no proper k-reduction and no proper k-decomposition, respectively. In [5] k-irreducible snarks for all k are characterised. The problem of determining $\Phi(k)$ is discussed in [5] as well. Known values are $\Phi(2) = 0$, $\Phi(3) = 1$, $\Phi(4) = 2$ [3] and $\Phi(5) = 5$ [2]. In particular, it follows from the k-decomposition theorems for $k \leqslant 5$ that a k-irreducible and k-indecomposable snark ($k \leqslant 5$) is either cyclically 6-connected, or it is cyclically 5-connected and every cycle-separating 5-cut separates a 5-cycle.

Our aim is to investigate the problem of determining an upper-bound on $\Phi(6)$ formulated explicitly in [5] and more generally, to study colourings of free ends of 6-poles induced by their 3-edge colourings. The importance of determination of $\Phi(6)$ which would result in a 6-decomposition theorem is stressed by the following long time open conjecture of Jaeger [4].

Conjecture 3 *[4] There does not exist a cyclically 7-connected snark.*

2 Kempe-closed Colour Sets

Given k-pole $P = P(x_1, x_2, \ldots, x_k)$ we define the *colour set* of P to be

$$
\begin{aligned}
Col(P) = \{&\text{3-partition } \cup_{i=1}^{3} X_i \text{ of } \{1, 2, \ldots, k\} | \\
&\exists \text{ a 3-edge-colouring } \gamma \text{ of } P \text{ such that} \\
&\text{for every free end } x_j, \text{ if } j \in X_i \text{ then } \gamma(x_j) = i\}
\end{aligned}
$$

A 3-partition of the set $\{1, 2, \ldots, k\}$ will said to be of *degree* k. In the context of this paper we allow one, or even two sets X_i of a 3-partition to be empty. In general, we define a *colour set* to be any subset of the set of all 3-partitions of the same degree. A colour set $\mathcal{P}$ is called *realisable* if $\mathcal{P} = Col(P)$ for some k-pole P. A 3-partition $\cup_{i=1}^{3} X_i$ of $\{1, 2, \ldots k\}$ is called *parity-admissible* if $|X_i| \equiv k \pmod 2$ for $i = 1, 2, 3$. A colour set $\mathcal{P}$ is called *parity-admissible* if every 3-partition of $\mathcal{P}$ is parity-admissible. By the Parity lemma, $Col(P)$ of a k-pole P is parity admissible. The set of all parity-admissible 3-partitions of degree k will be denoted $\mathcal{A}_k$. A 3-partition $a = \{X_1, X_2, X_3\}$ will be coded by a *colour vector* $a = a_1 a_2 \ldots a_k$, where $a_i \in \{1, 2, 3\}$ and $a_i = j$ means $i \in X_j$. Permuting indexes of the partition sets we get a set of at most 6 colour vectors coding the same partition. To have a uniquely determined representation we choose the lexically minimal colour vector $lexmin(a)$ to represent a. In what follows we shall identify a 3-partition a with its representation by a colour vector $lexmin(a)$. In particular, there is one parity-admissible 3-partition of degree 2 and 3, there are four parity-admissible 3-partitions of degree 4, there are 10 parity-admissible 3-partitions of degree 5 and 31 parity-admissible 3-partitions of degree 6.

Inspection of the proofs of 4- and 5-decomposition theorems shows importance of Kempe alternating chains starting and terminating at free ends of a pole. Recall that a *Kempe switch* is an operation on a 3-edge-coloured pole P interchanging two colours on a non-extendable alternating path joining two free ends. Let P be a k-pole and let γ be a colouring of P with corresponding 3-partition $a = \cup_{i=1}^{3} X_i$. Any Kempe switch on P results into a colouring γ with corresponding 3-partition $a' = \cup_{i=1}^{3} X'_i$ such that $X_j = X'_j$ for some $j \in \{1, 2, 3\}$ and for any $i \neq j$ we have $|X_i \div X'_i| = 2$.

In what follows we formalise an idea of the operation of Kempe switch. Two parity-admissible 3-partitions $\cup X_i$ and $\cup X'_i$ are *adjacent* if there is a permutation ψ of $\{1, 2, 3\}$ such that $X_r = {X'}_{\psi(r)}$ for some $r \in \{1, 2, 3\}$ and the symmetric difference $|X_i \div X'_{\psi(i)}| = 2$, if $i \neq r$. The above adjacency relation defines a graph $\mathcal{G}_k$ with the vertex set $\mathcal{A}_k$. Given a subset $\mathcal{S} \subseteq \mathcal{A}_k$, for $a = \cup X_i \in \mathcal{S}$ of $\{1, 2, \ldots, k\}$ we define an $\mathcal{R}_j^{\mathcal{S}}(a)$ to be the set of 3-partitions adjacent to a in $\mathcal{S}$ and containing the set X_j, for $j \in \{1, 2, 3\}$. For $r \in \{1, 2, 3\}$ we define a *transition graph* $T_r^{\mathcal{S}}(a)$ to be a simple graph, whose vertex set is $V = \{1, 2, \ldots, k\} \setminus X_r$ and $\{x, y\}$ is an edge if there is $a' = \cup X_i' \in \mathcal{R}_r^{\mathcal{S}}(a)$ such that $\{x, y\} = X_i \div X'_{\psi(i)}$ for every $i \neq r$.

We say that a subset $\mathcal{S}$ of parity-admissible 3-partitions of $\{1, 2, \ldots, k\}$ is *closed on Kempe switches*, or simply *Kempe-closed* if for each partition a in $\mathcal{S}$ and for each $i \in \{1, 2, 3\}$ the i-th transition graph $T_i^{\mathcal{S}}(a)$ satisfies the following conditions:

(1) if $|V(T_i^{\mathcal{S}}(a))| \geqslant 2$ then minimal degree of $T_i^{\mathcal{S}}(a)$ is at least 1

(2) every edge in $T_i^{\mathcal{S}}(a)$ extends into a perfect matching.

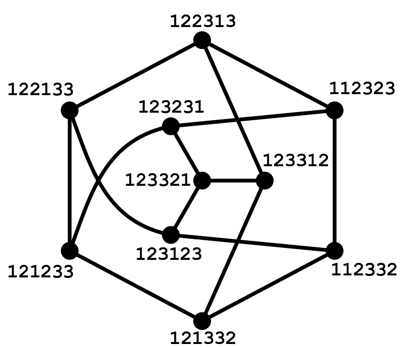

Figure 1: Subgraph of $\mathcal{G}_6$ induced by a Kempe-closed colour set

The following proposition highlights the importance of Kempe-closed colour sets.

Proposition 4 *Any realisable colour set is Kempe-closed.*

We say that a Kempe-closed colour set $\mathcal{S} \subseteq \mathcal{A}_k$ is *complementable* if there exist a non-trivial Kempe-closed colour set $\mathcal{T}$ such that $\mathcal{S} \cap \mathcal{T} = \emptyset$.

Proposition 5 *Let a snark G be a junction of 3-colourable k-poles P and Q, $G = P *_f Q$. Then $Col(P) \cap Col(Q) = \emptyset$, in particular both $Col(P)$ and $Col(Q)$ are complementable.*

The action of the symmetric group $Sym(k)$ on $\{1, 2, 3, \dots, k\}$ extends to 3-partitions and the action on 3-partitions induces an action of $Sym(k)$ on subsets of $\mathcal{A}_k$. Given subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{A}_k$ we say that $\mathcal{A} \preceq \mathcal{B}$ if there is $\psi \in S_k$ such that $\psi(\mathcal{A}) \subseteq \mathcal{B}$. Obviously $\preceq$ is a quasi-order on the set of all colour sets of degree k. Moreover, it induces a partial order on the orbits of $Sym(k)$ in the action on the set of all subsets of $\mathcal{A}_k$.

A family Ω of non-empty Kempe-closed colour sets of degree k is *set-complete* if for every Kempe-closed, complementable colour set $\mathcal{A}$ there exists $\mathcal{B} \in \Omega$ and $\psi \in Sym(k)$ such that $\mathcal{A} \cap \psi(\mathcal{B}) = \emptyset$.

A family Ω of non-empty realisable colour sets of degree k is *complete* if for every realisable, complementable colour set $\mathcal{A}$ there exists $\mathcal{B} \in \Omega$ and $\psi \in Sym(k)$ such that $\mathcal{A} \cap \psi(\mathcal{B}) = \emptyset$.

A complete (set-complete) family of colour sets is *minimal* if it is an anti-chain with respect to $\preceq$. Clearly, given complete family of colour sets of 3-partitions of degree k determines an upper bound on $\Phi(k)$ in the General k-Decomposition Theorem. In particular, the investigation of complete families, for $k = 6$, is a central problem considered in this paper. The following proposition can be viewed as a reformulation of k-Decomposition Theorems for k at most 5.

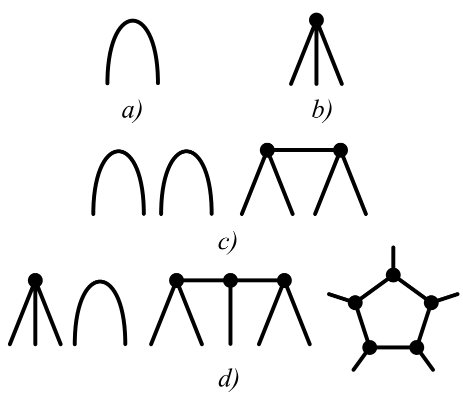

Figure 2: Poles representing Kempe-closed sets in cases $k = 2, 3, 4, 5$.

Proposition 6 *The following families of colour sets of degree* $2 \leqslant k \leqslant 5$ *are minimal, complete, and set-complete:*

(1) if $k = 2$*:* $\{\{\mathtt{11}\}\}$,

(2) if $k = 3$*:* $\{\{\texttt{123}\}\}$,

(3) if $k = 4$*:* $\{\{\texttt{1111}, \texttt{1122}\}, \{\texttt{1212}, \texttt{1221}\}\}$,

(4) if $k = 5$*:*

$$\{\{\texttt{11231}, \texttt{11213}, \texttt{11123}\}, \{\texttt{12113}, \texttt{12131}, \texttt{12223}, \texttt{12231}\},$$

$$\{\texttt{12333}, \texttt{12311}, \texttt{11231}, \texttt{11123}, \texttt{12223}\}\}.$$

The above sets are unique minimal complete (set-complete) sets of degree $k = 2, 3, 4, 5$ up to the action of $Sym(k)$.

The associated poles whose colour sets respectively coincide with the minimal Kempe-closed sets of 3-partitions are depicted on Figure 2.

For $k = 6$ the investigation of Kempe-closed sets is much harder than for $k \leqslant 5$. One reason is that the size of the power set jumps from 2^{10} in case $k = 5$ to 2^{31} for $k = 6$. A solution at a set-theoretical level is given in the following theorem whose proof is computer aided.

Theorem 7 *A minimal set-complete family* Ω *of degree* 6 *consists of the* 38 *colour sets. Moreover, every minimal set complete family of degree* 6 *arises by replacing every element of* Ω *by an equivalent colour set.*

By Proposition 4, every realisable colour set is Kempe-closed. We have checked that for $k \leqslant 5$ that the reverse implication holds. In particular, for $k = 5$ there are 10 Kempe-closed colour sets and we have found a realisation for each one of them. The following problem is central.

Problem 8 *Is every Kempe-closed colour set of degree* 6 *realisable?*

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K_t minors in large t-connected graphs

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A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An H-minor is a minor isomorphic to the graph H. A rich theory of graph minors has been developed by Robertson and Seymour in a series of 23 papers [5, 6]. One of the central results in this theory is an approximate structural description of graphs that do not contain a fixed graph as a minor.

We have been working on obtaining an exact description of large graphs that do not contain a complete graph on t vertices K_t as a minor. A characterization of such t-connected graphs has been conjectured by Thomas.

Conjecture 1 *For every positive integer t there exists a positive integer $N = N(t)$ such that for every t-connected graph G with $|V(G)| \geqslant N$ and no K_t minor contains a set X of $t - 5$ vertices such that $G - X$ is planar.*

If true the conjecture gives an exact structural description of graphs with no K_t minor that is best possible from several points of view. First, several infinite classes of 5-connected graphs with no K_6 minor are known, and the general structure of such graphs is believed to be extremely complex. Therefore, in general, one can not hope for a concise description of (large) $(t-1)$-connected graphs with no K_t-minor for $t \geqslant 6$, as such graphs can be obtained from 5-connected graphs with no K_6 minor by adding $t - 5$ universal vertices. Second, Thomason [7] has shown that there exist $\Omega(t\sqrt{\log t})$-connected graphs with no K_t-minor, and therefore the restriction on the size of $|V(G)|$ is also necessary. In fact, already for $t = 8$, the graph obtained from K_{10} by deleting a perfect matching is 8-connected, yet can not be made planar by deletion of any three vertices.

In earlier joint work with Kawarabayashi and Wollan [3, 4], we have established Conjecture 1 for $t \leqslant 6$. For $t = 6$, it was conjectured by Jörgensen [2] that every 6-connected graph that can not be made planar by deleting a vertex, regardless of its size, contains a K_t minor. Jörgensen's conjecture is of interest, in particular, because if true it would provide another proof of the following famous Hadwiger's conjecture [1] for $t = 5$.

Conjecture 2 *For every integer $t \geqslant 1$, if a loopless graph has no K_{t+1} minor then it is t-colorable.*

More recently we have established the conjecture for $t \leqslant 8$. A significant difference between $t = 8$ and $t \leqslant 6$ cases is that the bound on the number of vertices is necessary for $t = 8$, as mentioned above. Additionally, in our proof for $t = 8$ we have to account for vortices, an important structural element of Robertson and Seymour's description of graphs with a fixed forbidden minor. Vortices can not appear when the excluded graph is K_6.

Our proof of Conjecture 1 for $t = 8$ establishes the existence of certain "dense linear substructures" in any sufficiently large 8-connected graph with no fixed minor. For purposes of Conjecture 1 we use these substructures to find a K_8 minor, or to identify a set $X \subseteq V(G)$ as required. We hope that our technique could be sufficiently robust not only to extend to a proof of Conjecture 1 for general t, but also to establish optimal bounds on connectivity of a graph G that guarantee that G has certain (rooted) minors, provided that G is large.

One example of such a conjectured bound concerns linkages. A graph G is said to be *k-linked* if it has at least $2k$ vertices and for any choice of $2k$ distinct vertices $s_1, \ldots, s_k$, $t_1, \ldots, t_k$ of G, there exist pairwise vertex disjoint paths $P_1, \ldots, P_k$ such that the path P_i has ends s_i and t_i for $i = 1, \ldots, k$. Thomassen conjectured the following.

Conjecture 3 *For every positive integer t there exists a positive integer $N = N(t)$ such that every $2t + 2$ connected graph G with $|V(G)| \geqslant N$ is t-linked.*

If true, the conjecture is again best possible, as the connectivity can not be lowered and the assumption that $|V(G)|$ is large is necessary. It is open for $t \geqslant 3$.

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Heuristics for locally constrained homomorphisms

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We consider simple, undirected, possibly infinite but connected graphs. A *(graph) homomorphism* $f : G \to H$ from a graph $G = (V_G, E_G)$ to a graph $H = (V_H, E_H)$ is a mapping $V_G \to V_H$ such that $(f(u), f(v)) \in E_H$ whenever $(u, v) \in E_G$. A graph homomorphism f from a graph G to a graph H can be required to satisfy some local constraint. If, for every $u \in V_G$ the restriction of f, i.e. the mapping $f_u : N(u) \to N(f(u))$, is bijective, we say that f is *locally bijective*, and we write $G \xrightarrow{B} H$. If, for every $u \in V_G$, f_u is injective, we say that f is *locally injective*, and we write $G \xrightarrow{I} H$. If, for every $u \in V_G$, f_u is surjective, we say that f is *locally surjective*, and we write $G \xrightarrow{S} H$.

The main computational question is whether for every graph H the problem of deciding if an input graph G has a homomorphism of given type $* = B, I$ or S to the fixed graph H can be classified as either NP-complete or polynomially solvable. For the locally surjective homomorphisms this classification is known [3], with the problem for every connected H on at least three vertices being NP-complete. For the locally bijective and injective cases there are many partial results, see e.g. [2, 5], but even conjecturing a classification for these two cases is problematic. We continue the study started in [4] in order to get more insight in these computational issues.

The existence of a locally constrained homomorphism imposes a partial order on the class of connected graphs $\mathcal{C}$ for each of the three local constraints B, I, and S [4]. We can relax these three orders in two different ways. This leads to two different heuristics for testing if $G \xrightarrow{*} H$ for two given graphs G and H under each type $* = B, I, S$.

Firstly, we can transform the partial orders from the domain of finite graphs to the domain of matrices. An *equitable partition* of a connected graph G is a partition of its vertex set in blocks $B_1, \dots, B_k$ such that each vertex in each B_i has the same number $m_{i,j}$ of neighbors in B_j, and we call the $k \times k$ matrix $M = (m_{i,j})_{1 \leqslant i,j \leqslant k}$ a *degree matrix* of G. We say that a vertex u is of the *i-th sort* if $u \in B_i$. Note that the degree refinement matrix of G is the degree matrix corresponding to the equitable partition of G with the smallest number of blocks (which are ordered in a unique way), and an adjacency matrix of G is a degree matrix with maximum number of rows.

Let $\mathcal{M}$ be the set of all degree matrices. We define three relations $(\mathcal{M}, \xrightarrow{\exists B})$, $(\mathcal{M}, \xrightarrow{\exists I})$ and $(\mathcal{M}, \xrightarrow{\exists S})$ imposed on the set of degree matrices by the existence of graph homomorphisms of the corresponding local constraint, i.e., $M \xrightarrow{\exists *} N$ if and only if there exist two graphs $G, H \in \mathcal{C}$ with degree matrix M, N, respectively, such that $G \xrightarrow{*} H$. All three relations are quasi-orders (and partial orders if we restrict to degree refinement matrices) [4] and a successful matrix comparison of each type is a necessary condition for the corresponding graph comparison.

Secondly, we can transform the partial orders from the domain of finite graphs to the domain of possibly infinite trees. The *universal cover* T_G of a connected graph G is the only tree that allows a locally bijective homomorphism $T_G \xrightarrow{B} G$ [1]. A generic construction of the universal cover takes as vertices of T_G all finite walks in G that start from an arbitrary fixed vertex in G and that do not traverse the same edge in two consecutive steps. Two such vertices are adjacent in T_G if the associated walks differ only in the presence of the last edge. The required homomorphism $T_G \xrightarrow{B} G$ can be taken as the mapping that assigns every walk its last vertex.

Also universal covers can be equipped with a structure that impose a necessary condition for the existence of a locally constrained homomorphism. There are two options: either the existence of a locally constrained homomorphism or a simple inclusion (as a subtree). In the latter case, $T_G = T_H$, $T_G \subseteq T_H$, and $T_G \supseteq T_H$ are necessary conditions for $G \xrightarrow{B} H$, $G \xrightarrow{I} H$ and $G \xrightarrow{S} H$, respectively [4]. Moreover, a result in [4] states that the universal cover T_G is equal to the *universal cover* T_M of any degree matrix M of G which is constructed in the following way. Take as root a vertex corresponding to row 1 of M, thus of the 1st sort, and inductively add a new level of vertices while maintaining the property that each vertex of the i-th sort has exactly $m_{i,j}$ neighbors of the j-th sort. Hence, a successful universal cover comparison is a necessary condition for the corresponding graph comparison as well.

The following theorem is known. It can be obtained by combining known results from the literature with a few extra arguments, see [4] for more details.

Theorem 1 *Let G and H be connected graphs with degree matrices M and N,* *resp. Then the following holds:*

$$\begin{array}{ccccccc} G \xrightarrow{B} H & \Longrightarrow & M \xrightarrow{\exists B} N & \Longleftrightarrow & T_G \xrightarrow{B} T_H & \Longleftrightarrow & T_G = T_H \\ G \xrightarrow{I} H & \Longrightarrow & M \xrightarrow{\exists I} N & \Longrightarrow & T_G \xrightarrow{I} T_H & \Longleftrightarrow & T_G \subseteq T_H \\ G \xrightarrow{S} H & \Longrightarrow & M \xrightarrow{\exists S} N & \Longrightarrow & T_G \xrightarrow{S} T_H & \Longrightarrow & T_G \supseteq T_H \end{array}$$

We present a counter example for the backward implication $M \xrightarrow{\exists S} N \Longleftarrow T_G \xrightarrow{S} T_H$. Also the other missing backward implications in Theorem 1 can be excluded [4].

The problem of deciding $G \xrightarrow{*} H$ is NP-complete for all three local constraints. $M \xrightarrow{\exists B} N$ can be verified in polynomial time, but so far only membership to the class NP could be shown for the matrix comparison problem $M \xrightarrow{\exists *} N$ for $* = I, S$ [4]. Testing if $T_G = T_H$ can be done in polynomial time by checking if G and H share the same the degree refinement matrix [1]. We show the following result.

Theorem 2 *The problems of deciding if $T_G \xrightarrow{I} T_H$ (or equivalently $T_G \subseteq T_H$) or $T_G \xrightarrow{S} T_H$ for two given connected graphs G and H are polynomially solvable.*

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An efficient heuristic for finding dense subgraphs in a directed graph

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Finding complete directed bipartite sub-graphs in a directed graph is an important problem in the analysis of social/technological/natural networks such as web-graphs, protein-protein interactions networks, phone-calls networks, etc. Since finding complete bipartite sub-graphs is NP-hard in general, efficient methods provably correct when certain conditions are met can be useful. Here we report on a result of this nature especially useful for very large graphs like the web-graph.

A *directed graph* $G = (V, E)$ consists of a set V of *vertices* and a set E of *arcs*, where an arc is an ordered pair of vertices. Let u, v be any vertices of a directed graph G, if there exists an arc $a = (u, v)$, then a is an *outlink* of u, and an *inlink* of v. Moreover, v is called a *successor* of u, and u a *predecessor* of v. For every vertex u, $\mathrm{N}^+(u)$ denotes the set of its successors, and $\mathrm{N}^-(u)$ the set of its predecessors. Then, the *outdegree* and the *indegree* of u are respectively $\mathrm{d}^+(u) = |\mathrm{N}^+(u)|$ and $\mathrm{d}^-(u) = |\mathrm{N}^-(u)|$. Let X be any subset of V, the successors and the predecessors of X are respectively defined by: $\mathrm{N}^+(X) = \bigcup_{u \in X} \mathrm{N}^+(u)$ and $\mathrm{N}^-(X) = \bigcup_{u \in X} \mathrm{N}^-(u)$. Observe that $X \cap \mathrm{N}^+(X) \neq \emptyset$ is possible. A graph $G = (V, E)$ is called a *complete directed bipartite graph*, if V can be partitioned into two disjoint subsets X and Y, such that, for every vertex u of X, the set of successors of u is exactly Y, i.e., $\forall u \in X,\ \mathrm{N}^+(u) = Y$. Consequently for every node $v \in Y$ its predecessor set is X. Finally, let $\widetilde{\mathrm{N}}(u)$ be the co-citation set of u, the set of vertices that share at least one successor with u: $\widetilde{\mathrm{N}}(u) = \{w \in V \mid \mathrm{N}^+(u) \cap \mathrm{N}^+(w) \neq \emptyset\}$. As before for a set of nodes X its co-citation set is: $\widetilde{\mathrm{N}}(X) = \bigcup_{u \in X} \widetilde{\mathrm{N}}(u)$.

In Y. Dourisboure, F. Geraci and M. Pellegrini "Extraction and classification of dense communities in the Web" In Proceedings of the 16th International World Wide Web Conference (WWW2007), Banff, Alberta, Canada. May 2007, we have described a function $F(u)$ that is computed using only the indegree and out degree of nodes in $\widetilde{\mathrm{N}}(u) \cup \mathrm{N}^+(\widetilde{\mathrm{N}}(u))$ and for which the following holds. Let $X, Y \subset V$ be two subsets of nodes so that the induced subgraph in G is a complete directed bipartite graph then there exists a node $u \in X$ such that:

$$\lim_{|X| \to \infty} F(u) = 0$$

when it holds that that
1) $|\widetilde{\mathrm{N}}(u) \cup \mathrm{N}^{+}(\widetilde{\mathrm{N}}(u))|$ is $O(|X|+|Y|)$;
2) the number of edges in the subgraph induced by $\widetilde{\mathrm{N}}(u) \cup \mathrm{N}^{+}(\widetilde{\mathrm{N}}(u))$ minus the number of edges induced by $X \cup Y$ is $o(|X||Y|)$.
An open problem is to determine other natural sufficient conditions for which $F(u)$ (or a similar formula) converges when evaluated at a node of a dense subgraph of a given graph.

Loebl-Komlós-Sós Conjecture: dense case

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Abstract

Loebl Komlós and Sós conjectured that if a graph has at least half of its vertices of degree at least k, then it contains a copy of any tree with k edges. We expose some known partial results and sketch the proof of the conjecture for large dense graphs.

1 Introduction

Loebl (see [3]) conjectured that any graph G with at least $n/2$ vertices with degrees at least k contains any tree with at most $n/2$ edges as a subgraph. Ajtai Komlós and Szemerédi proved an approximate version of the conjecture [1].

Theorem 1 (Ajtai, Komlós, Szemerédi) *For any $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$, any graph on n vertices that has at least $(1+\varepsilon)\frac{n}{2}$ vertices of degrees at least $(1+\varepsilon)\frac{n}{2}$ contains a copy of any tree with at most $n/2$ edges.*

Zhao proved the Loebl conjecture for large graphs [7].

Theorem 2 (Zhao) *There is a n_0 such that for any for any $n \geqslant n_0$, every graph on n vertices that has at least $\frac{n}{2}$ vertices of degrees at least $\frac{n}{2}$ contains a copy of any tree with at most $n/2$ edges.*

The Loebl conjecture was then generalized by Komlós and Sós to the following.

Conjecture 3 (Loebl, Komós, Sós) *If a graph G of order n has at least $n/2$ vertices of degrees at least k, then G contains any tree with at most k edges as a subgraph.*

This conjecture was solved approximately by D. P., and M. Stein [6].

Theorem 4 (D. P., M. Stein) *For any $\varepsilon, q > 0$, there exists an $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$ and any $k \geqslant qn$, every graph on n vertices that has at least $(1+\varepsilon)\frac{n}{2}$ vertices of degrees at least $(1+\varepsilon)k$ contains a copy of any tree with at most k edges.*

We strengthened Theorem 4 and proved the following in [4].

Theorem 5 (J. H., D. P.) *For any $q > 0$, there exists an $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$ and any $k \geqslant qn$, every graph on n vertices with at least half of its vertices of degrees at least k contains a copy of any tree with at most k edges.*

Conjecture 3 is closely related to a conjecture of Erős and Sós, where they consider the average degree instead of the median degree of the host graph.

Conjecture 6 (Erdős, Sós) *If a graph G of order n has more than $n(k-1)/2$ edges, then it contains the copy of any tree with at most k edges.*

A solution of Conjecture 6 for large values of k has been announced by Ajtai, Komlós, Simonovits and Szemerédi [2].

2 Sketch of the Proof

Let G be a graph on n vertices with at least $\frac{n}{2}$ vertices of degree at least k. Let T be any tree of order $k+1$. The proof is divided into two main parts: the *iterative part* and the *extremal part*.

The **iterative part** consists of iterative steps, where at each step one considers a subgraph H of the host graph G and either finds a copy of T in H, or extracts from H a subgraph H' of H with a particular configuration. The first subgraph to be considered is G itself. In the iterative step, we first exclude two special cases of configuration of H or T. Then we apply the regularity lemma on the subgraph H to obtain a cluster graph $\mathbf{H}$. On this cluster graph $\mathbf{H}$, we apply the Gallai-Edmonds matching theorem and deduce two possible configurations of $\mathbf{H}$:

(1) there are two connected clusters A and B such that the average degree of the vertices in cluster A is at least $k - o(k)$, the average degree of the vertices in cluster B is at least $\frac{k}{2} + o(k)$, and there is a matching $\mathbf{M}$ in $\mathbf{H}$ such that for each matching edge $CD \in \mathbf{M}$, at most one of the clusters C and D is connected to the cluster A, or

(2) there are two connected clusters A and B such that the average degree of the vertices in cluster A, as well as the average degree of the vertices in cluster B is at least $k - o(k)$.

If the first configuration occurs, one can reduce it to a setting analogue to the approximate version of the conjecture and is thus solved similarly as in [6]. If the second configuration occurs, one uses some saving arguments to try to embed the tree T in H. If this attempt is not successful, one can extract from H a subgraph H' of size approximately k, that is only sparsely connected to $G - H'$ and has roughly half of its vertices of degree k. In the next iteration step, we then consider the subgraph $H - H'$.

The **extremal part** is considered when none of the iterative steps lead to the embedding of the tree T. We obtain a configuration of G, consisting of n/k sets of size approximately k, nearly completely disconnected one from the others, where in each such set, about half of the vertices have degree k. The existence of a copy of T is proved by case-analysis, without the use of the Regularity lemma.

3 Open Problem

An interesting open problem is whether Theorems 1, 2, 4, and 5 can be proved without the use of the regularity lemma. This would significantly improve the upper bound on the minimal number n_0 of vertices of the host graph. Similar results, as replacing the regularity lemma with other arguments, were already found (see e. g. [5]).

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Turán Problem for Book Hypergraphs

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The *Turán function* $\mathrm{ex}(n, F)$ of a k-graph F is the maximum size of an F-free k-graph H on n vertices. The *Turán density* of F is

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{k}}.$$

Determining $\mathrm{ex}(n, F)$ for hypergraphs F is in general a very difficult problem. For example, the \$1000 prize of Erdős for computing the Turán density for an *at least one* complete k-graph of order m with $m > k \geqslant 3$ is still unclaimed. Last decade saw a surge of activity in this field, with discovery of a variety of new results and methods. Still, we know $\pi(F)$ exactly for very few non-trivial hypergraphs F.

Here we survey known results for what we call "book hypergraphs": given k and m with $k \geqslant m$, the *book k-graph* $B_{k,m}$ is

$$B_{k,m} = \Big\{\{k, \ldots, 2k-1\}\Big\} \cup \Big\{\{1, \ldots, k-1, i\} \mid k \leqslant i \leqslant k+m-1\}\Big\}.$$

It is easy to see that $\pi(B_{k,0}) = \pi(B_{k,1}) = 0$, so we further assume that $m \geqslant 2$.

Bollobás [2] proved that $\pi(B_{3,2}) = 2/9$. This was extended by Frankl and Füredi [4] who proved that for $n > 3000$,

$$\mathrm{ex}(n, B_{3,2}) = \left\lfloor \frac{n+2}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor.$$

Keevash and Mubayi [10] reduced the threshold 3000 to 33 here.

Mubayi and Rödl [11] showed that $\pi(B_{3,3}) \leqslant 1/2$ and conjectured that the Turán density is $4/9$. This was proved by Füredi, Pikhurko, and Simonovits [7]. Later, Füredi, Pikhurko, and Simonovits [8] computed $\mathrm{ex}(n, B_{3,3})$ exactly for all large n.

Sidorenko [13] showed that $\pi(B_{4,2}) = 3/32$; Pikhurko [12] computed $\mathrm{ex}(n, B_{4,2})$ exactly for all large n (see later for more details).

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The complete solution for $B_{4,3}$ and $B_{4,4}$ (exact computation of the Turán function as well as the corresponding stability result) was obtained respectively by Füredi, Pikhurko, and Simonovits [9] and Füredi, Mubayi, and Pikhurko [6].

Frankl and Furedi [5] showed that $\pi(B_{5,2}) = 720/14641$ and $\pi(B_{6,2}) = 55/1728$, where the lower bound is given by blowing up perfect quadruple and quintuple Steiner systems on 11 and 12 vertices respectively.

The property of being $B_{k,k}$-free is also called the *empty neighborhoods* property. The solved cases $k = 2, 3$ suggest that $\pi(B_{k,k})$ is attained by a bipartite construction: partition the vertex set into two parts and take for edges all k-tuples that intersect the first part in an odd number of vertices. Füredi, Mubayi, and Pikhurko [6] proves this for $k = 4$ and conjectured that it is the case for all $k \geqslant 5$. However, this conjecture is false: very recently, Bohman, Frieze, Mubayi, and Pikhurko [1] showed that it fails for all $k \geqslant 7$. Also, tight bounds as $k \to \infty$ are obtained in [1]:

$$\pi(B_{k,k}) = 1 - \frac{2\log k}{k} + \Theta(1) \times \frac{\log\log k}{k}.$$

Still, the conjecture may be true for $k = 5, 6$ in which case the following would holds.

Conjecture 1 (Füredi, Mubayi, and Pikhurko [6])

$$\begin{aligned}\pi(B_{5,5}) &= \frac{40}{81}\\ \pi(B_{6,6}) &= \frac{1}{2}\end{aligned}$$

Let us call a k-graph F *stable* if for any $\varepsilon > 0$ there is $\delta > 0$ such that any F-free k-graph of order n and size at least $\mathrm{ex}(n, F) - \delta n^k$ can be made into a maximum F-free k-graph by changing (removing and/or adding) at most εn^k edges. For $k = 2$ any graph is stable (Erdős [3], Simonovits [14]). Stability is useful for the hypergraph Turán problem as it often helps in proving exact results. For example, the stability approach was used by Keevash and Mubayi [10] to compute the exact value of $\mathrm{ex}(n, B_{3,2})$.

The above papers by Sidorenko [13] and Füredi [5] use the so-call *Lagrange polynomial* of a hypergraph H,

$$\lambda_H(y_1, \ldots, y_m) = \sum_{D \in H} \prod_{i \in D} y_i,$$

and the the *Lagrangian* of H

$$\Lambda_H = \max\{\lambda_H(y_1, \ldots, y_m) \mid y_i \in \mathbb{R},\ y_i \geqslant 0,\ y_1 + \cdots + y_m = 1\}$$

Sidorenko [13] proved that any $B_{4,2}$-free 4-graph H satisfies

$$\Lambda_H \leqslant 1/4^4 \tag{1}$$

which implies that $\pi(B_{4,2}) \leqslant 4!/4^4$ in view of inequality $|H| \leqslant n^4 \lambda_H$ where n is the number of vertices of H. Since it is easy to show that $\pi(B_{4,2}) \geqslant 4!/4^4$ (just take the maximum complete 4-partite 4-graph on n vertices), we have $\pi(B_{4,2}) = 4!/4^4$.

Here is the exact result for $B_{4,2}$.

Theorem 2 ([12]) *There is an n_0 such that for all $n \geqslant n_0$ we have*

$$\mathrm{ex}(n, B_{4,2}) = \left\lfloor \frac{n}{4} \right\rfloor \times \left\lfloor \frac{n+1}{4} \right\rfloor \times \left\lfloor \frac{n+2}{4} \right\rfloor \times \left\lfloor \frac{n+3}{4} \right\rfloor,$$

and, moreover, the complete balanced 4*-partite* 4*-graph is the unique extremal configuration.*

The proof goes by showing that $B_{4,2}$ is stable. This in turn requires to establish the appropriately defined stability property for the problem of maximizing λ_H given that H is $B_{4,2}$-free.

We still do not have the exact result for $\mathrm{ex}(n, B_{i,2})$ for $i \geqslant 5$. The method in [12] seems promising in attacking the cases $k = 5, 6$, given the results of Frankl and Füredi [5]. One of the difficult steps here is to prove the following conjecture.

Conjecture 3 ([12]) *Both $B_{5,2}$ and $B_{6,2}$ are stable.*

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The 3-colorability of planar graphs : Steinberg's conjecture, Havel's problem

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1 Introduction

In 1976, Appel and Haken proved that every planar graph is 4-colorable [2, 3], and as early as 1959, Grötzsch [7] proved that every planar graph without 3-cycles is 3-colorable. As proved by Garey, Johnson and Stockmeyer [6], the problem of deciding whether a planar graph is 3-colorable is NP-complete. Therefore, some sufficient conditions for planar graphs to be 3-colorable were stated. In 1976, Steinberg [12] raised the following:

Steinberg's Conjecture '76 *Every planar graph without 4- and 5-cycles is 3-colorable.*

In 1969, Havel [8, 9] posed the following problem:

Havel's Problem '69 *Does there exist a constant C such that every planar graph with the minimum distance between triangles at least C is 3-colorable?*

Many other sufficient conditions for the 3-colorability of planar graphs were proposed in which cycles with lengths from specific sets are forbidden (for example, see [15]). In this talk we present two results. One is a new approach based on the adjacencies of cycles, the second one introduced the notion of distance-(k, l)-property.

2 Non adjacencies Theorem

GA - Graph of Non-Adjacencies A *graph of non-adjacencies* is one whose vertices are labelled by integers greater than two and each integer appears at most once. Given a graph $G_{\mathcal{A}}$ of non-adjacencies, we say that a graph G *respects* $G_{\mathcal{A}}$

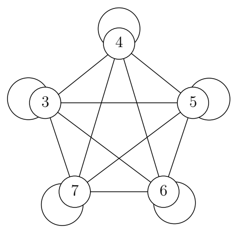

Figure 1: A graph of non-adjacencies.

if no two cycles of lengths i and j are adjacent in G if the vertices labelled with i and j are adjacent in $G_{\mathcal{A}}$.

Example. Let $G_{\mathcal{A}}$ be the graph depicted by Figure 1. A graph G respecting $G_{\mathcal{A}}$ is a graph in which there is no i-cycle adjacent to a j-cycle for $3 \leqslant i \leqslant j \leqslant 7$.

We propose the following natural general question:

Problem 1 *Under which conditions of adjacencies is a planar graph 3-colorable?*

In this talk the first presented result is that each planar graph respecting the graph $G_{\mathcal{A}}$ depicted by Figure 1 is 3-colorable.

Theorem 2 (Borodin, Montassier, Raspaud '08) *[5] Every planar graph in which no i-cycle is adjacent to a j-cycle whenever $3 \leqslant i \leqslant j \leqslant 7$ is 3-colorable.*

3 The Havel's problem

By Grötzsch's theorem, every triangle-free planar graph is 3-colorable. In [8, 9] Havel asked if there exists d such that every planar graph with the minimum distance between triangles at least d is 3-colorable. In 1976, Aksionov and Mel'nikov [1] and, independently, Steinberg (see [1]) proved that $d \geqslant 4$.

In this talk we present the following theorem:

Theorem 3 (Montassier, Raspaud, Wang, Wang '08) *[10] Every planar graph in which the cycles of length 3, 4, and 5 are at distance at least 4 from each other is 3-choosable.*

This is a partial answer to a relaxation of Havel's problem.

The *distance between two vertices x and y in G*, denoted by $d_G(x, y)$, is the length (number of edges) of a shortest path between x and y in G. The *distance between two cycles C and C'* of G, denoted by $d(C, C')$, is defined as follows:

$$d(C, C') = min\{d_G(x, y) : x \in V(C), y \in V(C')\}$$

The distance-(k, l)-property A planar graph G satisfies the *distance-(k, l)-property* if every pair of cycles $\{C, C'\}$ of G of length at most k verifies $d(C, C') \geqslant l$.

A relaxation of Havel's problem Given an integer $i \geqslant 3$, does there exist d_i (resp. d_i^l) such that any planar graph G with the distance-(i, d_i)-property is 3-colorable (resp. 3-choosable)? Moreover, for a given i, if d_i (resp. d_i^l) exists, what is the smallest value of d_i?

In addition, if d_i does not exist, we say that $d_i = \infty$.

The case "$i = 3$" corresponds to the Havel's problem. We recall that Aksionov and Mel'nikov [1] proved that if d_3 exists, then $d_3 \geqslant 4$, and conjectured that $d_3 = 5$. Since Voigt proved that there exist non 3-choosable planar graphs without triangle [14], it implies that $d_3^l = \infty$.

Using ideas developped in [4], it is not hard to prove that a planar graph with the distance-(9,1)-property is 3-choosable. Hence, $d_9^l = 1$.

Using the fact that every planar graph with the distance-$(7, 2)$-property (resp. the distance-$(6, 3)$-property) is 2-degenerate, we obtain that $d_7^l \leqslant 2$ and $d_6^l \leqslant 3$. The proof of 2-degeneracy can be easily done using a discharging procedure in which each face of length at least 8 (resp. 7) gives some charge to each adjacent face of length at most 5.

Finally, Theorem 3 implies that $d_5^l \leqslant 4$.

The following table summerizes the bounds obtained for the relaxation of Havel's problem - Choosability version (which give bounds for the relaxation of Havel's problem - Colorability version):

i	d_i^l
3	∞
4	?
5	4
6	3
7	2
9	1

Table 1: Every planar graph with the distance-(i, d_i^l)-property is 3-choosable.

We propose the following natural questions:

Problem 4 *For a given i, decrease the upper bound given by Table 1.*

Observe that proving the existence of d_4^l will strengthen Thomassen's theorem on the 3-choosability of planar graphs with girth at least 5 [13].

In [11], it is proved that there exist non 3-choosable planar graphs without cycles of lengths 4 and 5 and without intersecting triangles. It follows that $d_5^l \geqslant 2$ and if d_4^l exists, then $d_4^l \geqslant 2$.

Problem 5 *For a given i, exhibit a graph which gives lower bounds on d_i^l.*

Problem 6 *For a given i, improve bounds in the "colorability case".*

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Complexity of Pure Partition Constraint Satisfaction Problems

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1 Introduction

Constraint Satisfaction Problems (CSP) constitute a powerful framework for modelling many natural combinatorial problems. One of the most important open questions about CSP problems is the classification of the complexity of CSP(S) where S, named the template, is a set of relations on which we can build CSP instances. Such classifications are known for many special cases, for example when S is a set of relations over a Boolean [6] or three element domain [3]. In a recent paper, Feder, Madelaine and Stewart [4] have proved that the CSP($F^\bullet$) problem, with F being a set of two unary functions and $F^\bullet$ the set of its graphs, is as powerful as the general CSP(S) problem. Determining the complexity of *Pure Partition Constraint Satisfaction Problem* is a step toward the study of the CSP($F^\bullet$) problem.

2 Definitions

All along this abstract, a *partition function* f is an unary function $f\colon D \to \{a, b\}$ with D being a finite domain and a, b being elements from D. We denote by F a finite set of pure partition functions, i.e., every partition function in F share the same range $\{a, b\}$. The *graph* of an unary function f is the binary relation $f^\bullet = \{(x, y) \in D^2 \mid f(x) = y\}$. Extending this notion to F, we obtain the set of graphs $F^\bullet = \{f^\bullet \mid f \in F\}$.

A *polymorphism* of a relation R is an k-ary function f such that R is closed under f applied component-wise. By extension, f is a polymorphism of a set of relations S (and equivalently S is closed under f) if f is a polymorphism of every relation R from S.

A set of relations S is a *core* if all unary polymorphisms of S are injective. Let $F_c^\bullet$ denote the core of $F^\bullet$. Let Con_D denote the set of all (unary) constant functions over the domain D.

3 Dichotomy for Pure Partition CSP

To classify the complexity of $\mathrm{CSP}(F^{\bullet})$ for any set of functions F, it is sufficient to classify the complexity of sets of functions F containing all unary constant functions.

Lemma 1 *For every set of functions F on a domain D' there exists a set of functions G on domain $D \subseteq D'$ such that $\mathrm{CSP}(F^{\bullet}) \equiv_m^p \mathrm{CSP}((G \cup \mathrm{Con}_D)^{\bullet})$. More specifically, G can be chosen such that $G^{\bullet}$ is the core of $F^{\bullet}$ and D the domain elements present in the core of $F^{\bullet}$.*

>From now on, we assume that we work only with cores and that we have every constant function in our template.

When studying the complexity of $\mathrm{CSP}(F^{\bullet})$ for a set of unary functions F, it is convenient to represent $F^{\bullet}$ in the H-normal form. The *H-normal form* of the graphs of a set of unary functions $F = \{f_1, \ldots, f_k\}$ is the $(k+1)$-ary relation F^H expressed by the following formula $\bigwedge_{i \in \{1,\ldots,k\}} f_i(x) = y_i$.

When speaking about F^H we call the first column for the left hand side and the other columns the right hand side, and we denote it by F^H_{rhs}. Hence, we see that each column in F^H_{rhs} corresponds to the graph of a function f in F. The following Lemma 2 allow us to work and focus only on the H-normal form F^H of a set F of pure partition functions.

Lemma 2 $\mathrm{CSP}(F^{\bullet}) \equiv_m^p \mathrm{CSP}(F^H)$.

Without loss of generality, we identify the range $\{a, b\}$ of functions in F with the Boolean domain $\{0, 1\}$. This allows us to introduce the following Dichotomy Theorem for the Pure Partition CSP Problem.

Theorem 3 (Dichotomy Theorem) *Let F be a set of pure partition functions. If F^H_{rhs} is closed under a majority operation or a minority operation, or if it is closed under* $\max$ *or* $\min$ *and the two first tuples $(0, \vec{x})$ and $(1, \vec{y})$ of F^H satisfy $\vec{x} \leqslant \vec{y}$ component-wise, then $\mathrm{CSP}(F^{\bullet})$ is polynomial-time decidable. Otherwise it is* NP-*complete.*

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Two Ramsey Problems

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Abstract

The author who is a regular visitor of the Midsummer Prague Combinatorial Meetings is very thankful to the people in the department for their kind hospitality which makes always fun to visit Prague.

1 Problem 1

If $G_1, G_2, \ldots, G_r$ are graphs, then the Ramsey number $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then at least one of the subgraphs H_i $(1 \leqslant i \leqslant r)$ has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's original paper [8]. The number $R(G_1, G_2, \ldots, G_r)$ is called the Ramsey number for the graphs $G_1, G_2, \ldots, G_r$. There is very little known about $R(G_1, G_2, \ldots, G_r)$ for $r \geqslant 3$ even for very special graphs. In the first problem we consider the case when each G_i is a path P_n on n vertices. For $r = 2$ a well-known theorem of Gerencsér and Gyárfás [5] states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

Let

$$f(n) = \begin{cases} 2n-1 & \text{for odd } n, \\ 2n-2 & \text{for even } n. \end{cases}$$

For $r = 3$ in [3] Faudree and Schelp wrote that they felt that $R(P_n, P_n, P_n) = f(n)$. In fact, the lower bound of this conjecture holds for the three-color Ramsey number of any connected graph G_n. Figaj and Łuczak [2] in 2007 proved the asymptotic version of this conjecture, i.e., $R(P_n, P_n, P_n) = (2+o(1))n$. Gyárfás, Ruszinkó, Sárközy and Szemerédi [7] proved the conjecture in its full strength, i.e.,

Theorem 1 *For n sufficiently large* $R(P_n, P_n, P_n) = f(n)$.

In both papers the Regularity Lemma was used and the object desired to be found in the reduced graph is a large connected monochromatic matching. This is a matching where all of the edges are in the same color component. If MC_n is such a matching covering n vertices, then it follows from the Regularity Lemma that

$$R(P_n, \dots, P_n) \sim R(MC_n, \dots, MC_n)$$

for arbitrary number of colors r.

For $r \geqslant 4$ the Ramsey numbers for P_n are not known. On the other hand, in case of r colors the following inequality holds

$$\frac{n}{r}\left(1 - o(1)\right) \leqslant R(P_n, \dots, P_n) \leqslant \frac{n}{r-1}\left(1 + o(1)\right).$$

The upper bound follows by using the most frequent color of K_n and apply the Erdős-Gallai [1] extremal theorem for cycles. The lower bound follows from the well known fact (see e.g., Füredi, Gyárfás [4]) that in arbitrary r coloring of K_n there is a monochromatic component of size $\sim \frac{n}{r-1}$.

By the approach of connected monochromatic matchings in order to show that in case of r colors $R(P_n, \dots, P_n) \sim \frac{n}{r-1}$ it would be enough to prove that $R(MC_n, \dots, MC_n) \sim \frac{n}{r-1}$ which we pose as an open problem, i.e.,

Problem 2 *Is it true that in arbitrary r-coloring of the edges of K_n, if n is large enough then one can always find a connected monochromatic matching of size $\sim \frac{n}{r-1}$?*

2 Problem 2

Since the problem mentioned about was studied quite extensively, it can be difficult. But maybe the following one has not been considered before.

It is well known that in arbitrary 2-coloring of the edges of K_n in one of the colors it is connected. Moreover, it is also known, that the diameter of the connected component is bounded by some constant. Can we say something similar in the case of $r \geqslant 3$ colors?

Problem 3 *Is it true that in arbitrary r-coloring of the edges of K_n there is a monochromatic component of size $\sim \frac{n}{r-1}$ of diameter bounded by some constant d?*

We can show that there is such a constant $d = d(r)$, moreover the following also holds.

Theorem 4 *For arbitrary $\varepsilon > 0$ $\exists d = d(\varepsilon)$ such that in arbitrary r-coloring of the edges of K_n there is a monochromatic component of size $\geqslant \frac{n}{r-1}(1-\varepsilon)$ which has diameter $\leqslant d$.*

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Bounds for χ' depending on Δ and μ

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(joint work with Diego Scheide)

In his diploma thesis [3] the first author investigated the function f defined by

$$f(\Delta, \mu) = \max\{\chi'(G) \,|\, \Delta(G) = \Delta,\, \mu(G) = \mu\},$$

i.e., $f(\Delta, \mu)$ is the maximum chromatic index χ' possible for a graph with maximum degree Δ and maximum multiplicity μ. Observe that $f(0, 0) = 0$ and that beside this value the function f is defined for all pairs (Δ, μ) with $1 \leqslant \mu \leqslant \Delta$. To present the results from [3], we introduce the following sets:

$$\begin{aligned}
\mathcal{P}_1^p &= \{(\Delta, \mu) \,|\, 2p\mu + 1 \leqslant \Delta \leqslant 2(p+1)\mu - (2p+1)\} \\
\mathcal{P}_1 &= \bigcup_{p=1}^{\mu-1} \mathcal{P}_1^p \\
\mathcal{P}_2 &= \{(\Delta, \mu) \,|\, \mu \leqslant \Delta < 2\mu\}.
\end{aligned}$$

Observe that $\mathcal{P}_1^p = \emptyset$ if $p \geqslant \mu$.

Vizing's bound says that $f(\Delta, \mu) \leqslant \Delta + \mu$, but already Vizing himself [8] noticed that $f(\Delta, \mu) \leqslant \Delta + \mu - 1$ if $\Delta = 2\mu + 1 \geqslant 5$. In [9] Vizing proposed a further investigation of his estimate. For a proof of the next theorem the reader is referred to [3, 5].

Theorem 1 *For a pair (Δ, μ) of positive integers with $\Delta \geqslant \mu$ the following statements hold.*

(a) *If $(\Delta, \mu) \notin \mathcal{P}_1 \cup \mathcal{P}_2$, then $f(\Delta, \mu) = \Delta + \mu$.*

(b) *If $(\Delta, \mu) \in \mathcal{P}_2$, then $f(\Delta, \mu) \leqslant \Delta + \mu - 1$.*

(c) *If $(\Delta, \mu) \in \mathcal{P}_1$, then $f(\Delta, \mu) \leqslant \Delta + \mu - 1$ provided that Goldberg's Conjecture is true.*

(d) *If $1 \leqslant p \leqslant 5$ and $(\Delta, \mu) \in \mathcal{P}_1^p$, then $f(\Delta, \mu) \leqslant \Delta + \mu - 1$.*

The proof of statement (a) is based on a construction of graphs attaining Vizing's bound. This construction, however, works only for pairs (Δ, μ) such that $2p(\mu - 1) + 2 \leqslant \Delta \leqslant 2p\mu$ holds for some integer $p \geqslant 1$. Statement (b) is a direct consequence of Shannon's bound [7] saying that $f(\Delta, \mu) \leqslant \lfloor 3\Delta/2 \rfloor$.

Apart from the maximum degree there is another trivial lower bound for the chromatic index of a graph G, sometimes called the *density* $\mathcal{W}(G)$ of G defined by

$$\mathcal{W}(G) = \max_{H \subseteq G, |V(H)| \geqslant 2} \left\lceil \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} \right\rceil.$$

A conjecture made by Goldberg [1, 2] around 1970 says that every graph G satisfies $\chi'(G) \leqslant \max\{\Delta(G) + 1, \mathcal{W}(G)\}$. Statement (c) then follows from a result of [3] saying that every χ'-critical graph G with $\chi'(G) = \mathcal{W}(G)$ and $(\Delta(G), \mu(G)) \in \mathcal{P}_1$ satisfies $\chi'(G) \leqslant \Delta(G) + \mu(G) - 1$. Statement (d) is a consequence of statement (c) and a result of Scheide [4] supporting Goldberg's conjecture.

Our knowledge about the function f is far from being complete. If $\Delta < 2\mu$, then Shannon's bound implies that the gap between $f(\Delta, \mu)$ and Vizing's bound $\Delta + \mu$ can be arbitrarily large. The first author proved in his diploma thesis [3, 6] that this is also the case for $\Delta \geqslant 2\mu$. More precisely, he proved that if Δ, μ are positive integers satisfying $p\mu + r \leqslant \Delta \leqslant (p + 1)\mu - S$, where p, r are positive integers and $S = \lfloor \frac{rp^2+10(r-1)p+9r-3}{4} \rfloor$, then $f(\Delta, \mu) \leqslant \Delta + \mu - r$.

Based on the function f, we define the graph parameter θ by

$$\theta(G) = f(\Delta(G), \mu(G)).$$

Obviously, θ is the best upper bound for χ' in terms of Δ and μ. An upper bound ρ of χ' is said to be *efficiently realizable* if there exists an algorithm that finds, for every graph $G = (V, E)$, an edge colouring of G using at most $p(G)$ colors and has time complexity bounded from above by a polynomial in $|V|$ and $|E|$.

Problem 2 *Is θ an efficiently realizable upper bound for χ'?*

It might be also an interesting task to investigate the function

$$g(\Delta, \mu) = \max\{\mathcal{W}(G) \mid \Delta(G) = \Delta,\ \mu(G) = \mu\}.$$

Obviously, $g(\Delta, \mu) \leqslant f(\Delta, \mu)$ for all feasible pairs (Δ, μ) and Goldberg's Conjecture implies that equality holds provided that $f(\Delta, \mu) \geqslant \Delta + 2$. From the proof of Theorem 1 it follows that $f(\Delta, 1) = g(\Delta, 1) = \Delta + 1$ for every $\Delta \geqslant 2$. This supports the conjecture that $g = f$.

Problem 3 *Is $g = f$?*

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Unexpected behaviour of crossing sequences

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Abstract

The n^{th} crossing number of a graph G, denoted $cr_n(G)$, is the minimum number of crossings in a drawing of G on an orientable surface of genus n. We prove that for every $a > b > 0$, there exists a graph G for which $cr_0(G) = a$, $cr_1(G) = b$, and $cr_2(G) = 0$. This provides support for a conjecture of Archdeacon et al. and resolves a problem of Salazar.

Planarity is ubiquitous in the world of structural graph theory, and perhaps the two most obvious generalizations of this concept—crossing number, and embeddings in more complicated surfaces—are topics which have been thoroughly researched. Despite this, relatively little work has been done on the common generalization of these two: crossing numbers of graphs drawn on surfaces. This subject seems to have been introduced in [3], and studied further in [1]. Following these authors, we define for every nonnegative integer i and every graph G, the i^{th} *crossing number*, $cr_i(G)$, (and also the i^{th} *nonorientable crossing number*, $\tilde{cr}_i(G)$) to be the minimum number of crossings in a drawing of G in the orientable (nonorientable, respectively) surface of genus i. Observe that $cr_i(G) = 0$ (respectively, $\tilde{cr}_i(G) = 0$) if and only if i is greater or equal to the genus (resp., nonorientable genus) of G. This gives, for every graph G, two finite sequences of integers, $(cr_0(G), cr_1(G), \ldots, 0)$ and $(\tilde{cr}_0(G), \tilde{cr}_1(G), \ldots, 0)$, both of which end up with a zero. The first of these is the *orientable crossing sequence* of G, the second the *nonorientable crossing sequence* of G.

The natural question is to characterize crossing sequences of graphs. This is the focus of both [3] and [1]. If we are given a drawing of a graph in a surface $\mathcal{S}$ with at least one crossing, then modifying our surface in the neighborhood of this crossing by either adding a crosscap or a handle gives rise to a drawing of G in a higher genus surface with one crossing less. It follows from this that every orientable and nonorientable crossing sequence is strictly decreasing until it hits 0. This necessary condition was conjectured to be sufficient in [1].

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Conjecture 1 (Archdeacon, Bonnington, and Širáň)
If $(a_1, a_2, \ldots, 0)$ *is a sequence of nonnegative integers which strictly decreases until* 0*, then there is a graph whose crossing sequence (nonorientable crossing sequence) is*
$(a_1, a_2, \ldots, 0)$.

To date, there has been very little progress on this appealing conjecture. For the special case of sequences of the form $(a, b, 0)$, Archdeacon, Bonnington, and Širáň [1] constructed some interesting examples for both the orientable and nonorientable cases. We shall postpone discussion of their examples for the oriented case until later, but let us highlight their result for the nonorientable case here.

Theorem 2 (Archdeacon, Bonnington, and Širáň) *If* a *and* b *are integers with* $a > b > 0$*, then there exists a graph* G *with nonorientable crossing sequence* $(a, b, 0)$.

It has been believed by many that such a result cannot hold for the orientable case. For the most extreme special case $(N, N-1, 0)$, where N is a large integer, Salazar asked [2] if this sequence could really be the crossing sequence of a graph. The following almost contradictory quote of Dan Archdeacon illustrates why such crossing sequences are counterintuitive:

> If G has crossing sequence $(N, N-1, 0)$, then adding one handle enables us to get rid of no more than a single crossing, but by adding the second handle, we get rid of many. So, why would we not rather add the second handle first?

Our main theorem is an analogue of Theorem 2 for the orientable case, and its special case $a = N$, $b = N - 1$ resolves a question of Salazar [2].

Theorem 3 *If* a *and* b *are integers with* $a > b > 0$*, then there exists a graph* G *whose orientable crossing sequence is* $(a, b, 0)$.

Quite little is known about constructions of graphs for more general crossing sequences. Next we shall discuss the only such construction we know of. Consider a sequence $\mathbf{a} = (a_0, a_1, \ldots, a_g)$ and define the sequence $(d_1, \ldots, d_g)$ by the rule $d_i = a_{i-1} - a_i$. Then, roughly speaking, d_i is the number of crossings which can be saved by adding the i^{th} handle. It seems intuitively clear that sequences for which $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_g$ should be crossing sequences, since here we receive diminishing returns for each extra handle we use. Indeed, Širáň [3] constructed a graph with crossing sequence $\mathbf{a}$ whenever $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_g$.

Constructing graphs for sequences which violate the above condition is rather more difficult. For instance, it was previously open whether there exist graphs with crossing sequence $(a, b, 0)$ where a/b is arbitrarily close to 1. The most extreme examples are due to Archdeacon, Bonnington and Širáň [1] and have a/b approximately equal to $6/5$. Although our main theorem gives us a graph with every possible crossing sequence of the form $(a, b, 0)$, we don't know what happens for longer sequences.

In the rest of this extended abstract we will describe the graphs that we use to prove Theorem 3. We need a couple of gadgets which are common in the study of crossing numbers. A *special graph* is a graph G together with a distinguished subset $T \subseteq E(G)$ of *thick* edges, a subset $U \subseteq V(G)$ of *rigid* vertices and a family $\{\pi_u\}_{u \in U}$ of prescribed *local rotations* for the rigid vertices. Here, π_u describes the cyclic ordering of the ends of edges incident with u. A *drawing* of a special graph G in a surface Σ is a drawing of the underlying graph G with the added property that for every $u \in U$, the local rotation of the edges incident with u given by this drawing either in the local clockwise or counterclockwise order matches π_u. The *crossing number* of a drawing of the special graph G is ∞ if there is an edge in T which contains a crossing, and otherwise it is the same as the crossing number of the drawing of the underlying graph. We define the *crossing number* of a special graph G in a surface Σ to be the minimum crossing number of a drawing of G in Σ, and $cr_i(G)$ to be the crossing number of G in a surface of genus i.

Lemma 4 *If G is a special graph with crossing sequence* $\mathbf{a}$*, then there exists an (ordinary) simple graph with crossing sequence* $\mathbf{a}$*.*

(Proof is omitted in the extended abstract.) This result permits us to use special graphs in our constructions. When defining a (special) graph with a diagram, we shall use the convention that thick edges are drawn thicker, and vertices which are marked with a box instead of a circle have the distinguished rotation scheme as given by the figure. With this terminology, we can now introduce our principal family of graphs.

The n^{th} *hamburger graph* H_n is a special graph with $3n + 8$ vertices. Its thick edges form a cycle $C = qv_1 \dots v_n rr's'su_n \dots u_1tt'q'q$ of length $2n + 8$ together with two additional thick edges $\tau_0 = qr$ and $\tau_1 = st$. See Figure 1. In addition to these, H_n has n special vertices u'_i (for odd values of i) and v'_i (for even values of i) with rotation as shown in the figure. These vertices are of degree 4 and they lie on paths $r_1 = q'v'_2v'_4 \dots v'_m r'$ (where $m = n$ if n is even and $m = n - 1$ otherwise) and $r_2 = t'u'_1u'_3 \dots u'_l s'$ (where $l = n$ if n is odd and $m = n - 1$ otherwise). These two paths will be referred to as the *rows* of H_n. Each u'_i and each v'_i is

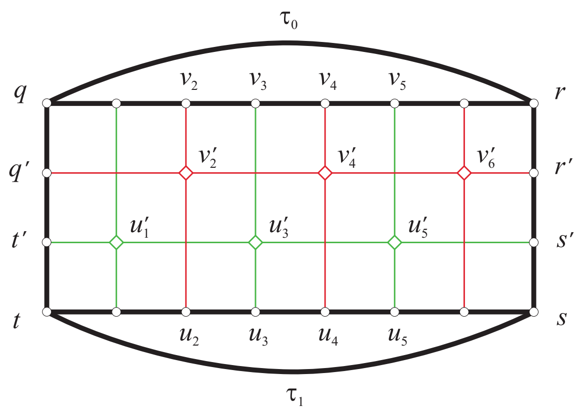

Figure 1: The graph H_n (for $n = 6$)

adjacent to u_i and v_i, and the 2-path $c_i = u_i u_i' v_i$ (or $c_i = u_i v_i' v_i$, depending on the parity of i) is called a *column* of H_n, $i = 1, \ldots, n$.

We claim that the hamburger graph H_n has crossing sequence $(n, n-1, 0)$ whenever $n \geqslant 5$ (or $n = 3$). Although this does not handle all possible sequences of the form $(a, b, 0)$, as discussed above, these are in some sense the most difficult and counterintuitive cases. Indeed, a rather trivial modification of these will be used to get all possible sequences.

Since it is quite easy to sketch proofs of $cr_0(H_n) = n$ and $cr_2(H_n) = 0$, let us do so here (rigorous arguments will be given in the full version). The first of these equalities follows from the observation that every row must meet every column in any planar drawing in which thick edges are crossing-free. The second equality follows from the observation that H_n minus the thick edges τ_0, τ_1 is a graph which can be embedded in the sphere. Using an extra handle for each of τ_0, τ_1 gives an embedding of the whole graph in a surface of genus 2. Of course, it is possible to draw H_n in the torus with only $n-1$ crossings by starting with the drawing in the figure and then adding a handle to remove one crossing. It remains to prove the inequality $cr_1(H_n) \geqslant n-1$ (which is true for $n = 3$ and $n \geqslant 5$). Not surprisingly, this is the difficult part. To prove this inequality, we analyze the influence of homotopy types of columns of H_n on the number of crossings.

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Hypergraph coloring: New kinds of problems[1]

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We discuss a new direction in hypergraph coloring theory, which emerged in the last one and half decades and has led to new kinds of phenomena with lots of challenging problems.

Hypergraphs. A *hypergraph* is a pair $\mathcal{H} = (X, \mathcal{E})$, where X is the *vertex set* and $\mathcal{E}$ is a set system over X, called the *edge set* of $\mathcal{H}$. In the present context, $|E| \geqslant 2$ is assumed for all edges $E \in \mathcal{E}$. The classical part of hypergraph coloring theory deals with colorings $\varphi : X \to \mathbb{N}$ such that no $E \in \mathcal{E}$ is monochromatic.

Substantial extension: Mixed hypergraphs. Around the mid-1990's, the fourth author [13, 14] introduced a second condition on vertex colorings of hypergraphs, which resulted in a remarkable twist in the behavior of colorability properties. A *mixed hypergraph* can be written as a triple $(X, \mathcal{C}, \mathcal{D})$, where both $\mathcal{C}$ and $\mathcal{D}$ are set systems over X, termed the sets of *C-edges* and *D-edges*, respectively. A coloring $\varphi : X \to \mathbb{N}$ has to satisfy the requirements that every C-edge has two vertices with a common color and every D-edge has two vertices with different colors. The extension of the traditional hypergraph model with C-edges changes the situation in a dramatic way in several aspects. A major source of information concerning the first decade of mixed hypergraph coloring is the monograph [15]; recent developments and open problems are surveyed in [11] and [1].

General model: Stably bounded hypergraphs. Our general approach is to put restrictions on each edge for the number of colors and for the largest multiplicity of a color. Given a hypergraph $\mathcal{H} = (X, \mathcal{E})$ together with four functions $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b} : \mathcal{E} \to \mathbb{N}$ on its edge set, a mapping $\varphi : X \to \mathbb{N}$ is a *proper coloring* if all the following conditions are met:

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- every $E \in \mathcal{E}$ contains at least $\boldsymbol{s}(E)$ colors,
- every $E \in \mathcal{E}$ contains at most $\boldsymbol{t}(E)$ colors,
- in every $E \in \mathcal{E}$ at least one color occurs at least $\boldsymbol{a}(E)$ times,
- in every $E \in \mathcal{E}$ the multiplicity of each color is at most $\boldsymbol{b}(E)$.

Then $(X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ is termed *stably bounded hypergraph* [4], whereas $(X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ without monochromatic bounds is called *color-bounded hypergraph* [3].

Subclasses. Beside the pair $(\boldsymbol{s}, \boldsymbol{t})$, other combinations of the four functions are of interest, too. Introducing new edges in the hypergraph if necessary, all restrictions can be expressed by using just the pair $(\boldsymbol{s}, \boldsymbol{a})$, but this may yield a loss in structural properties. It depends on the actual problems, which other combinations have equal strength to this universal pair $(\boldsymbol{s}, \boldsymbol{a})$.

Colorability. Some hypergraphs are uncolorable, e.g. the mixed hypergraph on three vertices, with three 2-element D-edges and one 3-element C-edge. Although decidable in polynomial time on some well-structured hypergraph classes, colorability is $\mathcal{NP}$-complete on general mixed hypergraphs and it is also hard for any function pair in $\{\boldsymbol{s}, \boldsymbol{b}\} \times \{\boldsymbol{t}, \boldsymbol{a}\}$ already on some classes for which the mixed hypergraph model admits an efficient solution. The *uniquely colorable hypergraphs* of various types (mixed or defined with some of $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$), which have just one coloring apart from renumbering of colors, also lead to interesting questions, see e.g. [2, 4, 12].

Feasible sets. A surprising fact concerning mixed hypergraphs is that if $\mathcal{H} = (X, \mathcal{E})$ has a coloring with exactly k' and k'' colors ($k' < k''$), there is no guarantee that $\mathcal{H}$ is colorable with k colors for any $k' < k < k''$. Actually, in the class of non-1-colorable mixed hypergraphs any finite set of integers greater than 1 occurs as the *feasible set* of $\mathcal{H}$, which means the set of possible numbers of colors in a proper coloring [8]. What is more, the numbers of colorings with the given numbers of colors can also be prescribed arbitrarily [9].

C-perfectness. An interesting part of the theory, which still offers a lot to be explored, deals with questions inspired by the notion of perfect graph. One can observe that the *C-stability number* of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, defined as the largest number of vertices not containing any C-edge, is an upper bound on the largest number of colors in a proper coloring of $\mathcal{H}$. A mixed hypergraph $\mathcal{H}$ is called *C-perfect* if those two maxima are equal in every induced subhypergraph of $\mathcal{H}$. Beside constructions and structural characterizations, algorithmic aspects are also very interesting and sometimes unexpected [5, 6, 7, 10].

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Precoloring extension

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Let $G = (V, E)$ be a graph where every vertex $v \in V$ is assigned a list of available colors $L(v)$. We say that G is *L-list colorable* for a given list assignment L if we can color every vertex with a color from its list such that adjacent vertices get different colors. If $L(v) = \{1, \ldots, k\}$ for all $v \in V$ then a corresponding list coloring is nothing else than an ordinary *k-coloring* of G. For $W \subseteq V$ denote the minimum distance between the components of $G[W]$ by $d(W)$.

Problem: *Let G be a graph and k be an integer. We consider subsets W of $V(G)$ where each component of $G[W]$ is precolored by at most $s = \chi(G[W]) \leqslant k$ colors. Determine bounds for $d(W)$ such that any precoloring of W extends to a k-coloring or an L-list coloring of V for every list assignment L with $|L(v)| = k$ for all $v \in V$.*

Some of the known results concerning ordinary colorings are summarized in the following table. The expression $(k{+}1) \rightarrow (k{+}1)$ means that a $(k{+}1)$-coloring of W extends to a $(k+1)$-coloring of V.

description of G	description of $G[W]$	$d(W)$	coloring	
$\chi(G) = k$	$\chi(G[W]) = 1$	$\geqslant 4$	$(k+1) \rightarrow (k+1)$	[1]
$\chi(G) = k$	$\chi(G[W]) = 1$	$\geqslant 3$	$(k+1) \rightarrow \lceil \frac{3k+1}{2} \rceil$	[4]
$\chi(G) = k$	$G[W]$ is a union of K_s's	$\geqslant 4s$	$(k+1) \rightarrow (k+1)$	[4]
$K_{\Delta+1} \not\subseteq G, \Delta(G) \geqslant 4$	$G[W]$ is a union of K_i's	$\geqslant 8$	$\Delta(G) \rightarrow \Delta(G)$	[3]
$K_4 \not\subseteq G, \Delta(G) = 3$	$G[W]$ is a union of K_i's	$\geqslant 10$	$\Delta(G) \rightarrow \Delta(G)$	[3]
$\chi(G) = k$	$\chi(G[W]) = s$	$\geqslant 4$	$(k+s) \rightarrow (k+s)$	[2]

There are some results for the list coloring version of this problem. Axenovich [6] and Albertson, Kostochka and West [3] proved that for independent W, $k = \Delta(G) \geqslant 3$ and $d(W) \geqslant 8$ a precoloring of W is always extendable to an L-list coloring if G does not contain a K_{k+1} as subgraph and $|L(v)| = k$ for all $v \in V$. Remarkably, the examples showing the sharpness of the result are 1-connected graphs. In fact, for t-connected graphs G with $t \geqslant 2$ and $\Delta(G) \geqslant 4$ the requirement $d(W) \geqslant 4$ and for $\Delta(G) = 3$ the requirement $d(W) \geqslant 6$ is sufficient to guarantee an extension of a precoloring to such a list coloring (see [9, 10])

Note that W is independent in the mentioned list coloring results. More general we may ask what happens if $G[W]$ is the union of complete graphs.

Problem 1: *Let G be a graph not containing K_{k+1} as subgraph where $k = \Delta(G)$ and let W be a subset of $V(G)$ such that $G[W]$ is the union of complete graphs. Is there an integer d such that a precoloring of W is always extendable to an L-list coloring of G whenever $d(W) \geqslant d$ and $|L(v)| = k$ for all $v \in V$?*

Considering the last row of the above table it is natural to ask which kind of requirements are necessary if we have only $k + s - 1$ colors instead of $k + s$.

Hutchinson and Moore [7] showed that without further constraints no distance can insure a color extension with $k + s - 1$ colors. They found several results for special graph classes. Let G be a simple graph with $\chi(G) = k$ and without a minor K_{k+1} and let $G[W]$ be s-colorable with every component of $G[W]$ precolored with at most s colors chosen from a set of $k + s - 1$.

$k\backslash s$	2	3	4	5
2	3	–	–	–
3	$7 \leqslant d \leqslant 8$	5	–	–
4	$7 \leqslant d \leqslant 8$	7	6	–
5	$7 \leqslant d \leqslant 8$	$7 \leqslant d \leqslant 8$	7	6

The above table shows bounds and exact values, respectively, for a smallest d such that $d(W) \geqslant d$ ensures the required coloring extension ([7]). Albertson and Moore [5] found several results for $s = 1$.

Recently, the case $k = 3, s = 2$ was solved in [8]. Let G be a K_4-minor-free graph and $W \subseteq V(G)$ where $G[W]$ is bipartite and $d(W) \geqslant 7$. Then any 4-coloring of $G[W]$ using at most two colors for every component of $G[W]$ can be extended to a 4-coloring of all of G. Moreover, an analogous list coloring result is shown if G is outerplanar instead of K_4-minor free (see [8]).

Thus the first open problem indicated by the above table is $k = 4, s = 2$.

Problem 2: *Let G be a K_5 minor-free graph, $W \subseteq V(G)$, and $G[W]$ bipartite. Furthermore, assume that the shortest distance $d(W)$ between components of $G[W]$ is at least 7 and each component of $G[W]$ is colored by at most 2 colors. Is such a coloring always extendable to 5-coloring of $V(G)$?*

In [7] it is proved that $d(W) \geqslant 8$ is sufficient. On the other hand, there are planar examples showing that an extension is not always possible if $d(W) = 6$.

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On Circles and Spheres Containing Many Points

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The Circle (or Disk) Containment Problem. The starting point of our discussion is the following result of Neumann-Lara and Urrutia [8].

Theorem 1 *Let S be a set of n points in the plane $\mathbf{R}^2$. Then there is a pair of points $a, b \in S$ such that any closed disk D that contains both a and b contains at least $c_2 \cdot n$ points of S, where $c_2 > 0$ is an absolute constant.*

The original proof of Neumann-Lara and Urrutia yields a constant of $1/60$, and naturally, a hunt for the best value of c_2 ensued [5, 2, 4]. The currently best lower bound of $c_2 \geqslant 1/2 - 1/\sqrt{12} \approx 1/4.73$ was obtained[2] by Edelsbrunner, Hasan, Seidel, and Shen [3]. On the other hand Hayward et al. [5] gave a construction of point sets that shows $c_2 \leqslant 1/4$, which is conjectured to be tight.

A few remarks are in order. First of all, it is easy to see that it suffices to consider point sets in *general position*, i.e., such that no three points are collinear and no four lie on a common circle. Moreover, it is not hard to convince oneself of the fact that it suffices to consider disks whose boundary circle passes through both points a and b. In this case, the proofs usually show that there is a pair for which any circle through a and b contains at least $c_2 n$ points inside and outside.

Depth, Centerpoints, and the Paraboloidal Lifting Map The circle containment problem is closely related to the well-known notions of *Tukey depth* and emphcenterpoints. Recall that for a set S of n points in $\mathbf{R}^d$, the *Tukey* or *halfspace depth* of a point $o \in \mathbf{R}^d$ (which may or may not belong to S) is defined as the minimum number of points of S in any halfspace containing o. Moreover, it is a basic fact of life (see, for instance, [7]) that for any set $S \subseteq \mathbf{R}^d$, there exists a *centerpoint*, i.e., a point of halfspace depth at least $n/(d+1)$. However, the centerpoint need not belong to the original point set.

We see that Theorem 2 is of a similar flavor: the first difference is simply that we consider depth with respect to circles or disks instead of halfplanes. The second

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[2]The proof is rather involved, and the proof of one of the key lemmas contains an essential gap; correct proofs of that lemma were furnished later [6] and [1].

difference is that we need two points (any one point is contained in a small disk containing nothing else), but on the other hand, we can guarantee a "center pair" that actually belongs to the original point set. In fact, the similarity between the results extends beyond the merely formal level, in fact circle (or sphere) depth and halfspace depth are closely related by the *paraboloidal lifting map* that is a standar tool in computational geometry. This map assigns to every point $p = (x, y) \in \mathbf{R}^2$ the point $\hat{p} := (x, y, x^2 + y^2)$ on the standard paraboloid $U = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 = z\}$ in $\mathbf{R}^3$. It is easy to see that this lifting map gives a bijection between circles C in $\mathbf{R}^2$ and non-empty intersections of $\hat{C} = U \cap H$ of with non-vertical planes H, and that p lies inside/on/outside of C iff $\hat{p}$ lies below/on/above the plane H.

The Sphere Containment Problem. It is also very natural to consider the generalization of the circle containment problem to higher dimensions. In general dimension d, two points are not enough: for instance, one can construct point sets in $\mathbf{R}^3$ such that through any pair of points there passes a sphere containing no other points in its interior. Bárány et al. [2] showed that Theorem 2 admits a generalization if $\lfloor (d+3)/3 \rfloor$ points are used:

Theorem 2 *Let $S \subseteq \mathbf{R}^d$ be a set of n points. Then there is a subset $A = \{a_1, \ldots, a_{\lfloor \frac{d+3}{2} \rfloor}\}$ such that any ball containing A contains at least $c_d \cdot n$ points of S, where $c_d > 0$ is a constant depending only on d.*

The original proof of Bárány et al. yields a lower bound for c_d that is exponentially small in d, roughly $c_d \geqslant \frac{\sqrt{\pi}(\frac{d+3}{2})^{3/2}}{2^{d+3}}$, while the best upper bound is $c_d \leqslant d + 2$. Again, it is easy to show that it suffices to consider balls that contain A in their boundary. (One can also use a larger number of points in A (up to d of them), to guarantee more points inside the ball.)

Recently, the lower bound [10] was improved to one which is polynomial in d, albeit still quite far from the upper bound, $c_d \geqslant \frac{4}{5ed^3}$. The proof combines the paraboloidal lifting map with results on k-sets of point sets in *higher dimensions* and is quite complicated. Instead of describing it, we finish our discussion by outlining a beautiful argument, due to Ramos and Viaña [9], that uses very similar ideas (discoved independently) and yields a very short and elegant new proof of the currently best lower bound for c_2.

Depth of Segments. Using the paraboloidal lifting map, it suffices to show the following (note that the paraboloid is a convex surface):

Theorem 3 *Given a set* $S \subseteq \mathbf{R}^3$ *of* n *points* in convex position*, there is a pair* $a, b \in S$ *such that any plane containing the segment* ab *contains at least* $(1/2 - 1/\sqrt{12})n$ *points of* S *on either side.*

This follows in a very slick way by a simple averaging argument from known results about k-facets. Recall that a k*-facet* of a point set $S \subseteq \mathbf{R}^3$, $k < \frac{n-3}{2}$, is a triangle abc, spanned by points of S, such that the plane spanned by abc contains exactly k points on one side. In general, the number of k-facets of a point set is difficult to analyze. For point sets in *convex position*, however, it is known exactly, see Welzl [11].

Proposition 4 *Let* S *be a set of* n *points in convex position in* $\mathbf{R}^3$*. Then the number* e_k *of* k*-facets of* S *equals* $e_k = 2(k+1)n - 2(k+1)(k+2)$*, for* $0 \leqslant k \leqslant \frac{n-4}{2}$.

The proof of the theorem is completed by the following argument: let s_k denote the number of segments ab of *depth* at most k, i.e., for which there exists a plane through ab that contains at most k points. We want to show that $s_k < \binom{n}{2}$ if $k < (1/2 - 1/\sqrt{12})n$. This follows easily from the above proposition and the following lemma.

Lemma 5 *Let* S *be a set of* n *points in general position* $\mathbf{R}^3$ *and let* $k < \frac{n-3}{2}$*. Then* $s_k \leqslant \frac{3}{2}e_k$*.*

Proof. We want to show that $s_k < \binom{n}{2}$ if $k < (1/2 - 1/\sqrt{12})n$. If abc is a k-facet of S then clearly each of its edges ab, ac, and bc contributes to s_k. On the other hand, given a segment ab of depth at most k, we claim that there exist at least two k-facets abc and abc' that contain ab on their boundary. To see this, consider any plane H through ab that contains $l \leqslant k$ points on one side, say its positive side. Rotating H continuously about ab in one direction, the count on the positive side changes by ± 1 whenever we pass over a point of S, and after a rotation of 180 degrees, we obtain a plane that contains $n - 3 - \ell > k$ points on its positive side. Therefore, we must have passed through a point c such that the plane through abc contains exactly k points on its positive side. Rotating in the other direction, we obtain the second k-facet abc'. It would be interesting to know if Theorem 3 also holds for point sets in non-convex position.

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