On the complexity of cover-incomparability graphs of posets

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Abstract

In this paper we show that the recognition problem for C-I graphs of posets is NP-complete. On the other hand, we prove that induced subgraphs of C-I graphs are exactly complements of comparability graphs, and hence the recognition problem for induced subgraphs of C-I graphs of posets is polynomial.

1 Introduction

In this paper we deal with posets and graphs associated to them. There are several ways how to associate a graph G to a given poset P. The vertex set $V(G)$ is usually the set of points of P. Depending on the edge-set $E(G)$, we may obtain among others the *comparability graph* of P (x and y are adjacent iff $x < y$ or $y < x$), the *incomparability graph* of P (x and y are adjacent iff x and y are incomparable), the *cover graph* of P (x and y are adjacent iff x covers y or vice versa) or the *cover-incomparability graph* of P (x and y are adjacent iff x covers y, or y covers x, or x and y are incomparable). The incomparability graph of P is of course just the complement of the comparability graph, while the cover-incomparability graph of P is the union of the cover graph and the incomparability graph of G.

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Cover graphs, comparability graphs and incomparability graphs are standard ways how to associate a graph to a given poset, see e.g. [10, 11], while the notion of cover-incomparability graph is new. It was introduced in [2]. This notion was motivated by the theory of transit functions on posets. It turns out that the underlying graph G_{T_P} of the standard transit function T_P on the poset P is exactly the C-I graph of P (see [2] for details).

Complexity problems concerning various graphs associated to posets were extensively studied. While the recognition problem for cover graphs is NP-complete [7, 8] or [3], the recognition problem for comparability graphs (and hence also for incomparability graphs) is polynomial, see [1]. Moreover, due to Gallai [4] the minimum family of forbidden subgraphs for the class of comparability graphs is known.

Our study was focused on the following open problems posed in [2]:

Question 1: *Which graphs are C-I graphs?*

Question 2: *Can C-I graphs be recognized in polynomial time? In addition, do C-I graphs posses a forbidden (induced) subgraphs characterization?*

We define the following two classes. The class `C-I` of all C-I graphs of some poset and the class `SubC-I` of all induced subgraphs of C-I graphs of some poset. It is not hard to observe that the class `C-I` is not closed on induced subgraphs (see Section 3), hence the class `C-I` cannot be characterized by any list of forbidden induced subgraphs. On the other hand, the class `SubC-I` is closed on induced subgraphs and we show its minimal forbidden induced subgraphs. This is proved in Section 3 by showing that `SubC-I` are just complements of comparability graphs (i.e. graphs that admit a transitive orientation of edges). This implies that the recognition problem for the class `SubC-I` is polynomial.

In Section 4 we deal with the class `C-I`. The main theorem in this Section is Theorem 3 which says that the recognition problem for the class `C-I` is NP-complete.

2 Terminology and basic properties of C-I graphs

Let $P = (V, \leq)$ be a poset. We will use the following notation. We write:

- $u < v$ if $u \leq v$ and $u \neq v$.
- $u \lhd v$ if $u < v$ and there is no z such that $u < z < v$. We say that v *covers* u.
- $u \lhd\lhd v$ if $u < v$ and $\neg(u \lhd v)$.
- $u \parallel v$ if u and v are incomparable.

Definition 1 *For a given poset $P = (V, \leq)$, let $G(P) = (V, E)$ be a graph with $E = \{uv \mid u \lhd v$ or $u \parallel v\}$. Then we say that $G(P)$ is the* **cover-incomparability graph** *of P (or C-I graph of P for short).*

Note that for any $u, v \in V(G)$, $u \neq v$, we have

$$uv \notin E(G) \Leftrightarrow u \lhd\lhd v \text{ or } v \lhd\lhd u \ .$$

Here are two easy examples of C-I graphs:

- If P is a linear order on n points, then the C-I graph $G(P)$ is equal to the cover graph of P which is the path on n vertices.
- If P is an antichain of size n, then $G(P)$ is just the incomparability graph of G, that is the complete graph on n vertices.

Now let us collect a few easy observations about C-I graphs. They follow immediately from the definition.

Lemma 1 *Let $P = (V, \leq)$ be a poset and $G(P) = (V, E)$ its C-I graph. Then the following holds.*

(i) *$G(P)$ is connected.*

(ii) *If $U \subseteq V$ is an antichain in P then U induces a complete subgraph in $G(P)$.*

(iii) *If $I \subseteq V$ is an independent set in $G(P)$ then all points of I lie on a common chain in P.*

(iv) *There are at most* 2 *vertices of degree* 1 *in* $G(P)$.

(v) *If* $P^* = (V, \leq^*)$ *is the dual poset to* P *(i.e.* $u \leq v$ *in* $P \Leftrightarrow v \leq^* u$ *in* P^**), then* $G(P^*) = G(P)$.

(vi) *If the vertices* x, y, z *form a triangle in G(P) then at least two of them are incomparable.*

(vii) *Let* x, y, z *be vertices of* $G(P)$ *such that* $xy \in E$, $xz \notin E$, $yz \notin E$. *Then* $(x \lhd\lhd z \text{ and } y \lhd\lhd z)$ *or* $(z \lhd\lhd x \text{ and } z \lhd\lhd y)$.

The following Lemma 2 is not new. It was already proved in [2] by a tedious case analysis. Together with our Theorem 1 it also easily follows from Gallai's characterization of comparability graphs by forbidden subgraphs [4]. However, our proof of Lemma 2 is short and simple so we present it here, too.

Lemma 2 *Let* G *be a graph that contains* C_n, $n \geq 5$ *as an induced subgraph. Then* G *cannot be the C-I graph of any poset.*

Proof: Suppose for a contradiction that $C_n = G(P)$ for a poset P. Let $x_1, x_2, \ldots, x_n$ be the vertices of C_n (in this order). As $x_1x_3 \notin E(G)$ we may suppose that $x_1 \lhd\lhd x_3$ (the other case when $x_3 \lhd\lhd x_1$ is symmetric). It follows that for every $i \neq 2, 3, 4$ $x_i \lhd\lhd x_3$. To show this we repeatedly use Lemma 1 part (vii) on triples $\{x_3, x_1, x_n\}, \{x_3, x_n, x_{n-1}\}$ etc. Thus also $x_5 \lhd\lhd x_3$. Similarly applying Lemma1 part (vii) on triples $\{x_2, x_3, x_5\}$ and $\{x_1, x_3, x_4\}$ we get that $x_5 \lhd\lhd x_2$ and $x_1 \lhd\lhd x_4$ holds. This already gives a contradiction as for $x_2x_4 \notin E(G)$ neither $x_2 \lhd\lhd x_4$ nor $x_4 \lhd\lhd x_2$ is possible (again according to Lemma1 part (vii)). □

3 Subgraphs of C-I graphs

Recall that `C-I` denotes the class of all C-I graphs and `SubC-I` denotes the class of all C-I graphs and their induced subgraphs. First we observe that `C-I` ≠ `SubC-I`. In other words, the class `C-I` is not closed on induced subgraphs. This follows immediately from the fact that an induced subgraph of a C-I graph need not be connected. Another example is the following.

Example 1 *Consider the poset P and its C-I graph $G(P)$ depicted in Fig. 1. (Only the relation $\lhd$ is pictured.) $G(P)$ contains $K_{1,3}$ as an induced subgraph. According to Lemma 1 part (iv) $K_{1,3}$ is not the C-I graph of any poset.*

Figure 1: Poset P and its C-I graph $G(P)$

Hence, the class `C-I` cannot have a forbidden induced subgraphs characterization. In Section 4 we prove that the recognition problem for `C-I` is NP-complete.

Now we turn our attention to the class `SubC-I`. Theorem 1 gives a good characterization of this class.

For an undirected graph $G = (V, E)$ an orientation G is the oriented graph $D = (V, A)$ that arises from G by replacing each edge uv by **one** of the arcs $\overrightarrow{uv}$ or $\overrightarrow{vu}$. The orientation D is said to be *transitive* if for any two consecutive arcs $\overrightarrow{uv} \in A(D)$, $\overrightarrow{vw} \in A(D)$ also $\overrightarrow{uw} \in A(D)$.

Theorem 1 *For an arbitrary graph H, the following two conditions are equivalent.*

1. *There exist a poset P such that H is an induced subgraph of its C-I graph $G(P)$.*
2. *$\overline{H}$ (the complement of H) admits a transitive orientation.*

Proof:
(1. $\Rightarrow$ 2.)

Recall that for any $uv \notin E(H)$ (and thus also $uv \notin E(G)$), $u \lhd\lhd v$ or $v \lhd\lhd u$ in P. We define an orientation D on the non-edges of H (i.e. edges of $\overline{H}$) by

$$\overrightarrow{uv} \in A(D) \Leftrightarrow u \lhd\lhd v.$$

As the relation $\lhd\lhd$ is transitive, D is a transitive orientation of $\overline{H}$.

(2. $\Rightarrow$ 1.)

Let H be a graph such that $\overline{H}$ admits a transitive orientation. This orientation defines a partial order $P = (V(H), \leq)$ on $V(H)$ by

$$u \leq v \Leftrightarrow \overrightarrow{uv} \in A(D).$$

We extend the partial order P by adding a new point v_{xy} and new relations $x \leq v_{xy}$, $v_{xy} \leq y$ for every arc $\overrightarrow{xy} \in A(D)$.

More precisely, we define a new partial order $P' = (V', \leq')$ by

$$V' = V(H) \cup \{v_{xy} \mid \overrightarrow{xy} \in A(D)\}$$
$$\leq' = \leq \ \cup \{(u, v_{xy}), (v_{xy}, w) \mid \overrightarrow{xy} \in A(D),\ u \leq x,\ y \leq w\}.$$

Now let $G = G(P')$ be the C-I graph of P'. It remains to show that H is an induced subgraph of G.

- If $xy \in E(H)$, then $x \parallel y$ in P and by the construction of P' also $x \parallel y$ in P'. It follows that $xy \in E(G)$.
- If $xy \notin E(H)$, then $\overrightarrow{xy} \in A(D)$ or $\overrightarrow{yx} \in A(D)$. Thus $x \lhd\lhd y$ or $y \lhd\lhd x$ in P'. In both cases $xy \notin E(G)$.

$\square$

It is a well-known fact (see e.g. [6]) that graphs that admit a transitive orientation are just comparability graphs. This implies that Theorem 1 can be reformulated as follows.

Theorem 2 *Graphs in the class* `SubC-I` *are exactly the incomparability graphs (i.e. the complements of comparability graphs).*

This theorem and above mentioned results have several interesting consequences.

Corollary 1 *Graphs in the class* `SubC-I` *can be recognized in polynomial time.*

Corollary 2 *Minimal forbidden subgraphs for the class* `SubC-I` *are known and they are complements of minimal forbidden subgraphs for comparability graphs.*

As comparability graphs and their complements are perfect, see e.g. [6], we have

Corollary 3 *C-I graphs are perfect graphs.*

4 Complexity of C-I graphs

The problem we are concerned with is the following:

C-I testing
Instance: A graph $G = (V, E)$.
Question: Is there a poset $P = (V, \leq)$ such that G is the C-I graph of P, i.e. $G = G(P)$?

Now we state the main result of this section.

Theorem 3 *The C-I testing problem, i.e. the decision whether a given graph is a C-I graph, is the NP-complete problem.*

We shall prove this Theorem by giving a polynomial-time transformation from Covering by complete bipartite subgraphs, which is the NP-complete problem, see [9] or [5]. This problem can be stated as follows.

Covering by complete bipartite subgraphs
Instance: A bipartite graph $G = (V, E)$ and positive integer $k \leq |E|$.
Question: Are there k subsets $V_1, \ldots, V_k$ of V such that each V_i induces a complete bipartite subgraph of G and such that for each edge $uv \in E$ there is some V_i that contains both u and v?

Before we prove Theorem 3 we state a simple Lemma which we use in the proof of it.

Lemma 3 *Let $P = (W, \leq)$, $|W| \geq 3$, be a poset and $H = (W, F)$ its C-I graph. Let $ab \in F$ and $\deg_H(a) = 1$. Then the following holds.*

(i) *The point a is the greatest or the least element of P.*

(ii) *If a is the greatest element of P, then $b \geq x$ for every $x \in W \setminus \{a\}$.*

(iii) *If $xb, yb \in F$ are two edges of H such that $x \neq a$, $y \neq a$, then x and y are incomparable.*

Proof: As $\deg_H(a) = 1$ and $ab \in F$ there is some $x \in W$ such that $ax \notin F$ and we have $b \lhd a$ or $a \lhd b$. Suppose that $b \lhd a$. Then for every $x \in W$, $x \neq a, b$ we have $x < b \lhd a$, otherwise $\deg_H(a) > 1$. This proves (i) and (ii). To prove (iii) suppose for a contradiction that $x < y$. Then using (ii) (or its symmetric version) either $x < y < b$ or $b < x < y$ both contradict the assumption $xb, yb \in F$. $\square$

Proof of Theorem 3:
Given a bipartite graph $G = (V, E)$ and a positive integer k, $k \leq |E|$ we construct a graph $H = H_G^k = (W, F)$ such that H is a C-I graph iff G can be covered by k complete subgraphs. The construction is the following. Let V_1, V_2 be a partition of V such that each edge of E is incident to one vertex in V_1 and one vertex in V_2. Set

$$W = V \cup \{a_1, a_2, b_1, b_2, w_1, \ldots, w_k\}$$

(i.e. we add $k + 4$ new vertices to the vertices of G) and

$$\begin{aligned} F &= \overline{E} \cup \{a_1b_1, a_2b_2\} \cup \{b_1v \mid v \in V_1\} \cup \{b_2v \mid v \in V_2\} \\ &\quad \cup \{w_iv \mid i = 1, \ldots, k\ , v \in V\} \cup \{w_iw_j \mid i, j = 1, \ldots, k\ , i \neq j\}\ . \end{aligned}$$

(Here $\overline{G} = (V, \overline{E})$ is the complement of G.) The graph H_G^k is pictured in the Fig. 2. (The edges of complete subgraphs on sets V_1, V_2 and $\{w_1, \ldots, w_k\ \}$ are not depicted.)

The next Theorem will finish the proof.

Theorem 4 *A bipartite graph G can be covered by k complete subgraphs iff the graph $H = H_G^k$ is a C-I graph.*

Proof:
1. Let $G_i = (U_i, E_i)$, $i = 1, \ldots, k$ be complete bipartite subgraphs of G such that $E = \bigcup_{i=1}^k E_i$. Define the poset $P = (W, \leq)$ as follows:

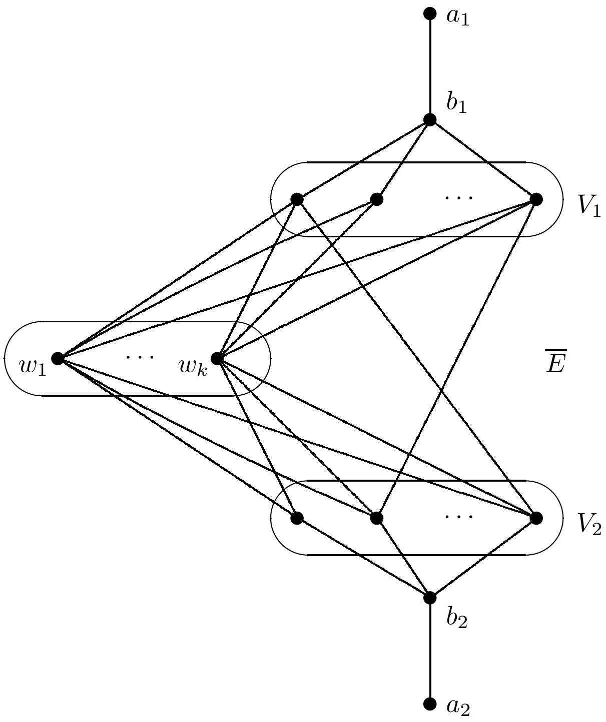

Figure 2: The graph H_G^k

- For every $x \in W$ set $a_2 \leq x \leq a_1$, i.e. a_1 is the greatest and a_2 the least element of P.
- For every $x \in W \setminus \{a_1, a_2\}$ set $b_2 \leq x \leq b_1$.
- For every $x \in V_1$ and for every $y \in V_2$ set $y \leq x$.
- For every $i = 1, \ldots, k$, for every $x \in U_i \cap V_1$, and for every $y \in U_i \cap V_2$, set $y \leq w_i \leq x$.

It is easy to check that $P = (W, \leq)$ is a poset and that $G(P) = H$.

2. Let H be a C-I graph, i.e. there is a poset $P = (W, \leq)$ such that $H = G(P)$. By Lemma 3 we may suppose that a_1 and a_2 are the greatest and the least element of P. Moreover for every $x \in W \setminus \{a_1, a_2\}$ we have $b_2 \leq x \leq b_1$ and all elements of V_1 and also all elements of V_2 are incomparable. Let $x \in V_1$ and $y \in V_2$ then $\neg(x \leq y)$ otherwise $x < y < b_1$ and hence $xb_1 \notin F$, a contradiction. Consider an edge $xy \in E$, $x \in V_1$, $y \in V_2$. By the definition of F we have $xy \notin F$ and there must be some $i = 1, \ldots, k$ such that $y < w_i < x$. To finish the proof set

$$X_i = \{x \in V_1 \mid x > w_i\} \quad \text{and} \quad Y_i = \{y \in V_2 \mid y < w_i\} ,$$

$i = 1, \ldots, k$. Then the induced subgraph of G on the set of verteces $X_i \cup Y_i$ is the complete bipartite subgraph of G, $i = 1, \ldots k$, and these subgraphs cover all edges of G. $\square$

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