Generalized linear fractional programming under interval uncertainty

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Abstract

Data in many real-life problems suffer from inexactness. Herein we assume that we are given some intervals in which the data can simultaneously perturb. We consider a generalized linear fractional programming problem with interval data and present an efficient method for computing the range of optimal values. We consider also the inverse problem: How much can data of a real generalized linear fractional program vary such that the optimal values do not exceed some prescribed bounds. We illustrate the approach on a simple von Neumann economic growth model.

1 Introduction

Uncertainties in data measurement and observation is a common phenomenon in practice. Considering their interval envelopes is one way to tackle these uncertainties. Computing with interval values has many useful properties, e.g., it ensures that all possible instances of interval data are taken into account.

Mathematical programming problems with interval data have been investigated for several decades. Many papers studied the problem of computing the range of optimal values of linear programming problem with data varying inside intervals, see [5, 7, 10, 21] among others. Less people were involved nonlinear programming with data perturbing inside intervals. For instance, interval convex quadratic programming was studied in [9, 20], posynomial geometric programming in [9, 17, 18, 19], and a specific nonlinear programming problem with linear constraints in [29].

In this paper, we focus on a generalized linear fractional programming problem the data of which vary inside some given intervals. To the best of our knowledge, this problem itself has never been investigated. In the essence, it can be solved by the general method from [9], where a unified method for dealing with interval nonlinear programming problems was proposed. That approach was based on duality theory in nonlinear programming. For generalized linear fractional programming we have a developed duality [6, 15] to use. Nevertheless, stronger results are obtained by direct inspection, which is exactly what we do in Section 2.

We show that the exact range of optimal values can be calculated by solving up to four real-valued mathematical programs. Moreover, the method is easily adapted for solving the inverse problem: We are given real-valued a generalized linear fractional programming problem and some bounds on the optimal value function, and we calculate tolerances (intervals) for all required parameters such that the optimal values do not exceed the bounds while the parameters are perturbing inside thier intervals.

Many applications of the interval generalized linear fractional programming arise in the field of economics and optimization. For instance, von Neumann growth model of expanding economy [25], goal programming with rational criteria [3, 4], or Chebyshev discrete rational approximation. For another possible application, see e.g. [23, 24].

2 Range of optimal values

Consider a generalized linear fractional programming problem

$$f(A,B,C,c) := \inf\ \lambda \ \text{ subject to } \ Ax \le \lambda Bx,\ Cx \le c,\ x \ge 0, \qquad (1)$$

where $A, B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{l\times n}$ and $c \in \mathbb{R}^{l}$. Moreover, we assume that $Bx \ge 0$ holds for all x satisfying $Cx \le c$, $x \ge 0$. Such problems are solvable in polynomial time using an interior point method [8, 22].

Now suppose that the input data are not known exactly, and we are given only lower and upper bounds on their values. Formally, the matrix A varies in some interval matrix $\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}\}$, where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$ are given matrices. In a similar way we consider interval matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ and interval vector $\boldsymbol{c}$ in which B, C and c may perturb, respectively. Thus we have a family of problems (1) with $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$ and $c \in \boldsymbol{c}$. Any problem belonging to this family is referred as *an instance*.

To ensure that each instance is a proper generalized linear fractional programming problem we have to assume:

(A1) For every $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$ and $c \in \boldsymbol{c}$ any solution to $Cx \geq c$, $x \geq 0$ solves also $Bx \geq 0$.

Proposition 1 shows that to verify this assumption; it suffices to verify only one instance with $B = \underline{B}$, $C = \underline{C}$ and $c = \overline{c}$.

Proposition 1. *Assumption (A1) is true if and only if $\underline{B}x \geq 0$ holds for all x satisfying $\underline{C}x \leq \overline{c}$, $x \geq 0$.*

Proof. One implication is easily seen as $B = \underline{B}$, $C = \underline{C}$ and $c = \overline{c}$ is an instance of our family of problems.

Conversely, let $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$ and $c \in \boldsymbol{c}$ and suppose that any x satisfying $\underline{C}x \leq \overline{c}$, $x \geq 0$ is also a solution of $\underline{B}x \geq 0$. Now, let x^* be any solution to $Cx \leq c$, $x \geq 0$. Then

$$\underline{C}x^* \leq Cx^* \leq c \leq \overline{c}.$$

Thus x^* is a solution to $\underline{C}x \leq \overline{c}$, and by our supposition x^* solves also $\underline{B}x \geq 0$. Hence

$$Bx^* \geq \underline{B}x^* \geq 0.$$

Therefore x^* is a solution to $Bx \geq 0$. □

As data are perturbing in their intervals, the optimal value $f(A, B, C, c)$ ranges in some interval as well. Our aim is to determine the exact lower and upper bound on the optimal value. They are respectively defined as

$$\underline{f} := \inf\ f(A,B,C,c) \ \text{ subject to } \ A \in \boldsymbol{A},\ B \in \boldsymbol{B},\ C \in \boldsymbol{C},\ c \in \boldsymbol{c},$$
$$\overline{f} := \sup\ f(A,B,C,c) \ \text{ subject to } \ A \in \boldsymbol{A},\ B \in \boldsymbol{B},\ C \in \boldsymbol{C},\ c \in \boldsymbol{c}.$$

In the following theorem we say that each the bound can be calculated by solving 1 to 2 real-valued generalized linear fractional programming problems.

Theorem 1.

1. *(Lower bound) Let*

$$f_1 := \inf\ \lambda \ \ \textit{subject to}\ \ \underline{A}x \le \lambda \underline{B}x,\ \lambda \le 0,\ \underline{C}x \le \overline{c},\ x \ge 0. \tag{2}$$

   If $f_1 < 0$ *then* $\underline{f} = f_1$*, otherwise* $\underline{f} = f_2$ *with*

$$f_2 := \inf\ \lambda \ \ \textit{subject to}\ \ \underline{A}x \le \lambda \overline{B}x,\ \lambda \ge 0,\ \underline{C}x \le \overline{c},\ x \ge 0.$$

2. *(Upper bound) Let*

$$f_3 := \inf\ \lambda \ \ \textit{subject to}\ \ \overline{A}x \le \lambda \underline{B}x,\ \lambda \ge 0,\ \overline{C}x \le \underline{c},\ x \ge 0. \tag{3}$$

   If $f_3 > 0$ *then* $\overline{f} = f_3$*, otherwise* $\overline{f} = f_4$ *with*

$$f_4 := \inf\ \lambda \ \ \textit{subject to}\ \ \overline{A}x \le \lambda \overline{B}x,\ \lambda \le 0,\ \overline{C}x \le \underline{c},\ x \ge 0. \tag{4}$$

Proof. 1. (Lower bound) First we consider the case when $\underline{f} < 0$. There is at least one instance of (1) with negative optimal value, so we can restrict our considerations to the family

$$\inf\ \lambda \ \ \text{subject to}\ \ Ax \le \lambda Bx,\ \lambda \le 0,\ Cx \le c,\ x \ge 0 \tag{5}$$

with $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$, $c \in \boldsymbol{c}$. For any instance and any feasible point λ, x we have

$$\underline{A}x \le Ax \le \lambda Bx \le \lambda \underline{B}x,$$

and

$$\underline{C}x \le Cx \le c \le \overline{c}.$$

It means that λ, x is also a feasible solution to the problem

$$\inf\ \lambda \ \ \text{subject to}\ \ \underline{A}x \le \lambda \underline{B}x,\ \lambda \le 0,\ \underline{C}x \le \overline{c},\ x \ge 0. \tag{6}$$

This problem belongs to the family of problems (5), since $\underline{A} \in \boldsymbol{A}$, $\underline{B} \in \boldsymbol{B}$, $\underline{C} \in \boldsymbol{C}$, $\overline{c} \in \boldsymbol{c}$. That is, the feasible set to (6) covers feasible sets of all instances of problems (5). Therefore the lower bound $\underline{f}$ will be achieved for this instance.

Suppose now that $\underline{f} \geq 0$. In this case, all instances of (1) have non-negative optimal values, and all their feasible solutions λ, x have $\lambda \geq 0$. That is why (2) is either infeasible or its optimal value is zero. So it remains to show that $\underline{f} \geq 0$ implies $\underline{f} = f_2$. Herein, (1) takes the equivalent form

$$\inf\ \lambda \ \text{ subject to } \ Ax \leq \lambda Bx,\ \lambda \geq 0,\ Cx \leq c,\ x \geq 0.$$

For any instance and every feasible solution λ, x we have

$$\underline{A}x \leq Ax \leq \lambda Bx \leq \lambda \overline{B}x,$$

and

$$\underline{C}x \leq Cx \leq c \leq \overline{c}.$$

It means, λ, x is also a feasible solution to the problem

$$\inf\ \lambda \ \text{ subject to } \ \underline{A}x \leq \lambda \overline{B}x,\ \lambda \geq 0,\ \underline{C}x \leq \overline{c},\ x \geq 0. \tag{7}$$

This problem belongs to the family of problems (1), since $\underline{A} \in \boldsymbol{A}$, $\overline{B} \in \boldsymbol{B}$, $\underline{C} \in \boldsymbol{C}$, $\overline{c} \in \boldsymbol{c}$. It means that the feasible set to (7) covers feasible sets of all instances of problems (1). Therefore the lower bound $\underline{f}$ will be achieved for this instance.

2. (Upper bound) We proceed in the similar manner as in the previous. First we assume that $\overline{f} > 0$. Then there is at least one instance of (1) with positive optimal value, so we can restrict our considerations to the family

$$\inf\ \lambda \ \text{ subject to } \ Ax \leq \lambda Bx,\ \lambda \geq 0,\ Cx \leq c,\ x \geq 0 \tag{8}$$

with $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$, $c \in \boldsymbol{c}$.

Let λ, x be any feasible solution of (3); if (3) is infeasible then $\overline{f} = f_3 = \infty$ and we are finished. Then for any instance of (8) we have

$$Ax \leq \overline{A}x \leq \lambda \underline{B}x \leq \lambda Bx,$$

and

$$Cx \leq \overline{C}x \leq \underline{c} \leq c.$$

Thus λ, x is a feasible solution to any instance of (8). In other words, the feasible set to (3) is included in a feasible set of any instance of (8).

Therefore the highest optimal value will be achieved for $A = \overline{A}$, $B = \underline{B}$, $C = \overline{C}$, $c = \underline{c}$.

Suppose now that $\overline{f} \le 0$. In this case, all instances of (1) have non-positive optimal values, and all their feasible solutions λ, x have $\lambda \le 0$. That is why (3) is either infeasible or its optimal value is zero. It remains to show that $\overline{f} = f_4$. We rewrite (1) in the equivalent form

$$\inf\ \lambda \ \text{ subject to } \ Ax \le \lambda Bx,\ \lambda \le 0,\ Cx \le c,\ x \ge 0. \tag{9}$$

Let λ, x be any feasible solution of (4); if (4) is infeasible then $\overline{f} = f_4 = \infty$ contradicting our assumption. For any instance of (9) we have

$$Ax \le \overline{A}x \le \lambda \overline{B}x \le \lambda Bx,$$

and

$$Cx \le \overline{C}x \le \underline{c} \le c.$$

Thus the feasible set to (4) is included in a feasible set of any instance of (9). Therefore the highest optimal value will be achieved in the setting $A = \overline{A}$, $B = \overline{B}$, $C = \overline{C}$, $c = \underline{c}$. □

3 Tolerances of variations

In this section we consider the inverse problem to the previous one. We start with some real-valued problem and want to extend the reals to intervals such that the optimal value of all instances ranges in some prescribed bounds. Analogous problems were studied in linear programming [14], but—to the best of our knowledge—no one discussed any nonlinear case.

Similar kinds of problems are called tolerance analysis, and we usually study how much may certain parameters perturb while preserving some characteristics, e.g. optimality of some point or basis. They were dealt with mainly in linear programming [1, 2, 13, 26, 27, 28] concerning only selected parameters (in the objective function or in the right-hand side of constraints). Tolerance analysis for all objective function coefficients in multiobjective linear programming was done in [11, 12].

Consider a real-valued generalized linear fractional programming problem (1) with $A := A^0$, $B := B^0$, $C := C^0$, $c := c^0$, and let f^* be its optimal value. Next, $\underline{f}$ and $\overline{f}$, $\underline{f} \le f^* \le \overline{f}$, is a given lower and upper bound on the optimal value function, respectively. Our aim is to extend the real data of (1) to tolerance intervals such that optimal values of all instances ranges in $[\underline{f}, \overline{f}]$. For that purpose we introduce so called tolerance rates: Let $A^{\Delta}, B^{\Delta} \in \mathbb{R}^{m\times n}$ and $C^{\Delta} \in \mathbb{R}^{l\times n}$ be non-negative matrices and $c^{\Delta} \in \mathbb{R}^{l}$ a non-negative vector. They scale different intervals not to have the same width and they are set usually in the following manner. Put $a^{\Delta}_{ij} := 0$ if tolerance for a^0_{ij} is not in demand, put $a^{\Delta}_{ij} := 1$ for the absolute tolerance, and put $a^{\Delta}_{ij} := |a^0_{ij}|$ for the relative (percentage) tolerance. Likewise for B^{Δ}, C^{Δ} and c^{Δ}.

Formally, our problem states as follows. Denote $\boldsymbol{A}_{\delta} := [A^0 - \delta A^{\Delta}, A^0 + \delta A^{\Delta}]$, $\boldsymbol{B}_{\delta} := [B^0 - \delta B^{\Delta}, B^0 + \delta B^{\Delta}]$, $\boldsymbol{C}_{\delta} := [C^0 - \delta C^{\Delta}, C^0 + \delta C^{\Delta}]$, and $\boldsymbol{c}_{\delta} := [c^0 - \delta c^{\Delta}, c^0 + \delta c^{\Delta}]$. Find (possibly maximal) tolerance $\delta^* > 0$ such that $f(A,B,C,c) \in [\underline{f}, \overline{f}]$ for all $A \in \boldsymbol{A}_{\delta^*}$, $B \in \boldsymbol{B}_{\delta^*}$, $C \in \boldsymbol{C}_{\delta^*}$ and $c \in \boldsymbol{c}_{\delta^*}$. Any $\delta^* > 0$ with this property is called *admissible* tolerance.

We divide this problem onto two smaller sub-problems: Find tolerances $\delta_1, \delta_2 > 0$ such that $f(A,B,C,c) \geq \underline{f}$ for all $A \in \boldsymbol{A}_{\delta_1}$, $B \in \boldsymbol{B}_{\delta_1}$, $C \in \boldsymbol{C}_{\delta_1}$ and $c \in \boldsymbol{c}_{\delta_1}$, and $f(A,B,C,c) \leq \overline{f}$ for all $A \in \boldsymbol{A}_{\delta_2}$, $B \in \boldsymbol{B}_{\delta_2}$, $C \in \boldsymbol{C}_{\delta_2}$ and $c \in \boldsymbol{c}_{\delta_2}$. The overall tolerance then simply equals $\delta^* = \min(\delta_1, \delta_2)$. We call δ_1 a lower tolerance and δ_2 an upper tolerance.

First, we verify that under some assumptions the maximal admissible tolerance is well defined.

Lemma 1. *Let* $\delta_1^* := \sup\ \delta$ *subject to*

$$\underline{f} \leq f(A,B,C,c)\ \forall A \in \boldsymbol{A}_{\delta},\ B \in \boldsymbol{B}_{\delta},\ C \in \boldsymbol{C}_{\delta},\ c \in \boldsymbol{c}_{\delta},$$

and $\delta_2^* := \sup\ \delta$ *subject to*

$$\overline{f} \geq f(A,B,C,c)\ \forall A \in \boldsymbol{A}_{\delta},\ B \in \boldsymbol{B}_{\delta},\ C \in \boldsymbol{C}_{\delta},\ c \in \boldsymbol{c}_{\delta},$$

and denote $\delta^* = \min(\delta_1^*, \delta_2^*)$. *Assume that*

(A2) $(B^0 - \delta^* B^{\Delta})x > 0$ *for all solutions of* $(C^0 - \delta^* C^{\Delta})x \leq c + \delta^* c^{\Delta}$, $x \geq 0$.

(A3) $(C^0 + \delta^* C^{\Delta})x \leq c - \delta^* c^{\Delta}$, $x \geq 0$, $x \geq 0$ *is solvable.*

Then δ^* *is the maximal admissible tolerance.*

Proof. 1. (Lower tolerance) We prove that δ_1^* is an admissible tolerance for the lower bound. For contradiction suppose that δ_1^* is not an admissible tolerance, i.e.,

$$\underline{f} > \inf\ f(A,B,C,c),\ A \in \boldsymbol{A}_{\delta_1^*},\ B \in \boldsymbol{B}_{\delta_1^*},\ C \in \boldsymbol{C}_{\delta_1^*},\ c \in \boldsymbol{c}_{\delta_1^*}.$$

Thus there is some λ', x', $A \in \boldsymbol{A}_{\delta_1^*}$, $B \in \boldsymbol{B}_{\delta_1^*}$, $C \in \boldsymbol{C}_{\delta_1^*}$ and $c \in \boldsymbol{c}_{\delta_1^*}$ such that

$$\underline{f} > \lambda',\ Ax' \leq \lambda' Bx',\ Cx' \leq c,\ x' \geq 0.$$

Without loss of generality suppose that $\lambda' \geq 0$; the converse situation is dealt analogously. In this case, λ', x' solves also the system

$$\underline{f} > \lambda,\ (A^0 - \delta_1^* A^\Delta)x \leq \lambda(B^0 + \delta_1^* B^\Delta)x,$$
$$(C^0 - \delta_1^* C^\Delta)x \leq c^0 + \delta_1^* c^\Delta,\ x \geq 0.$$

Let $\eta > 0$ be arbitrarily small, and let x^0 be a point in the relative interior of polyhedra described by $C^0 x \leq c^0$. Define $x^\eta := \eta x^0 + (1-\eta)x'$. This point lies in the relative interior of $(C^0 - \delta_1^* C^\Delta)x \leq c^0 + \delta_1^* c^\Delta$, so for sufficiently small $\varepsilon > 0$ there is $\eta = \eta(\varepsilon)$ such that $x^{\eta(\varepsilon)}$ lies also in the relative interior of $(C^0 - \delta^\varepsilon C^\Delta)x \leq c^0 + \delta^\varepsilon c^\Delta$ with $\delta^\varepsilon := \delta_1^* - \varepsilon$.

Define λ^ε in this way:

$$\lambda^\varepsilon := \max_{i=1,\ldots,m} \left(\frac{(A^0_{i,\bullet} - \delta^\varepsilon A^\Delta_{i,\bullet})x^{\eta(\varepsilon)}}{(B^0_{i,\bullet} + \delta^\varepsilon B^\Delta_{i,\bullet})x^{\eta(\varepsilon)}} \right),$$

where $M_{i,\bullet}$ denotes the i-th row of a matrix M. The value λ^ε is well defined as the denominators are positive due to assumption (A2). This λ^ε goes to λ' as ε goes to zero. Hence $\underline{f} > \lambda^\varepsilon$ when $\varepsilon > 0$ is small enough. It means that δ^ε (and any larger value) is not admissible tolerance, which contradicts the definition of δ_1^*.

2. (Upper tolerance) By contradiction suppose that

$$\overline{f} < \sup\ f(A,B,C,c)\ A \in \boldsymbol{A}_{\delta_2^*},\ B \in \boldsymbol{B}_{\delta_2^*},\ C \in \boldsymbol{C}_{\delta_2^*},\ c \in \boldsymbol{c}_{\delta_2^*}.$$

That is,

$$\overline{f}^\varepsilon < \sup\ f(A,B,C,c)$$

for some $\varepsilon > 0$, $\overline{f}^\varepsilon := \overline{f} + \varepsilon$, and for some $A \in \boldsymbol{A}_{\delta_2^*}$, $B \in \boldsymbol{B}_{\delta_2^*}$, $C \in \boldsymbol{C}_{\delta_2^*}$, and $c \in \boldsymbol{c}_{\delta_2^*}$. Thus

$$\overline{f}^\varepsilon < \lambda$$

for any λ and x being a solution to $Ax \leq \lambda Bx$, $Cx \leq c$, $x \geq 0$. That is why the system

$$\overline{f}^\varepsilon \geq \lambda,\ Ax \leq \lambda Bx,\ Cx \leq c,\ x \geq 0$$

is not solvable. Without loss of generality assume that $\lambda \geq 0$. Then also the system

$$\overline{f}^{\varepsilon} \geq \lambda,\ (A^0 + \delta_2^* A^{\Delta})x \leq \lambda(B^0 - \delta_2^* B^{\Delta})x, \\ (C^0 + \delta_2^* C^{\Delta})x \leq c - \delta_2^* c^{\Delta},\ x \geq 0$$

has no solution. By assumptions (A2) and (A3), the sub-system

$$(A^0 + \delta_2^* A^{\Delta})x \leq \lambda(B^0 - \delta_2^* B^{\Delta})x,\ (C^0 + \delta_2^* C^{\Delta})x \leq c - \delta_2^* c^{\Delta},\ x \geq 0$$

is solvable; we just find a solution to the second and third inequality and then put λ large enough. Therefore

$$\overline{f} < \overline{f}^{\varepsilon} \leq \inf\ \lambda \ \text{ subject to } \ (A^0 + \delta_2^* A^{\Delta})x \leq \lambda(B^0 - \delta_2^* B^{\Delta})x, \\ (C^0 + \delta_2^* C^{\Delta})x \leq c - \delta_2^* c^{\Delta},\ x \geq 0.$$

For similar reasons as in the part 1. a sufficiently small decrease of δ_2^* implies that the above optimization problem still has optimal value greater that $\overline{f}$. This contradicts the definition of $\overline{f}$. □

Note that assumption (A2) in Lemma 1 is an analogy of assumption (A1) in Section 2. The strict inequality is necessary. For example, consider the problem

$$\inf\ \lambda \ \text{ subject to } \ x \leq \lambda x,\ x \geq 1,$$

i.e., $A = 1$, $B = 1$, $C = -1$, $c = -1$. Put tolerance rates $A^{\Delta} = 1$, $B^{\Delta} = 1$, $C^{\Delta} = 0$, $c^{\Delta} = 0$. For $\delta \in (0, 1)$ the optimal value ranges in $[\frac{1-\delta}{1+\delta}, \frac{1-\delta}{1+\delta}]$, but for $\delta = 1$ the optimal value can achieve $-\infty$.

Also assumption (A3) is necessary. Consider the example

$$\inf\ \lambda \ \text{ subject to } \ x_1 + x_2 \leq \lambda(x_1 + x_2),\ x_2 = 1,\ x_1 + x_2 \geq 2,$$

i.e.,

$$A = \begin{pmatrix} 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix},\ c = (1, -1, -2)^T.$$

Put tolerance rates as follows

$$A^{\Delta} = \begin{pmatrix} 0 & 0 \end{pmatrix},\ B^{\Delta} = \begin{pmatrix} 0 & 0 \end{pmatrix},\ C^{\Delta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},\ c^{\Delta} = (0, 0, 0)^T.$$

For $\delta \in (0, 1)$ the optimal value is constantly one, but for $\delta = 1$ the optimal value for any instance is either one or ∞.

Theorem 2. *Under assumption (A2) and (A3) the following procedure computes an admissible lower and upper tolerance:*

1. *(Lower tolerance) If $\underline{f} \geq 0$ then*

$$\delta_1 := \inf\ \delta \quad \textit{subject to} \quad (A^0 - \underline{f}B^0)x \leq \delta(A^\Delta + \underline{f}B^\Delta)x, \\ C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta),\ x \geq 0,$$

otherwise

$$\delta_1 := \inf\ \delta \quad \textit{subject to} \quad (A^0 - \underline{f}B^0)x \leq \delta(A^\Delta - \underline{f}B^\Delta)x, \\ C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta),\ x \geq 0.$$

2. *(Upper tolerance) If $\overline{f} \geq 0$ then*

$$\delta_2 := \sup\ \delta \quad \textit{subject to} \quad (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta + \overline{f}B^\Delta)x, \\ -C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta),\ x \geq 0,$$

otherwise

$$\delta_2 := \sup\ \delta \quad \textit{subject to} \quad (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta - \overline{f}B^\Delta)x, \\ -C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta),\ x \geq 0.$$

Proof. 1. (Lower tolerance) Due to Lemma 1 the maximal lower tolerance is computed by

$$\delta_1^* = \sup\ \delta \text{ subj. to } \underline{f} \leq f(A, B, C, c)\ \forall A \in \boldsymbol{A}_\delta,\ B \in \boldsymbol{B}_\delta,\ C \in \boldsymbol{C}_\delta,\ c \in \boldsymbol{c}_\delta. \tag{10}$$

Its optimal value is greater of equal to

$$\inf\ \delta \quad \text{subject to} \quad \underline{f} \geq f(A, B, C, c),\ A \in \boldsymbol{A}_\delta,\ B \in \boldsymbol{B}_\delta,\ C \in \boldsymbol{C}_\delta,\ c \in \boldsymbol{c}_\delta, \tag{11}$$

or

$$\inf\ \delta \quad \text{subject to} \quad \underline{f} \geq \lambda,\ Ax \leq \lambda Bx,\ Cx \leq c,\ x \geq 0, \\ A \in \boldsymbol{A}_\delta,\ B \in \boldsymbol{B}_\delta,\ C \in \boldsymbol{C}_\delta,\ c \in \boldsymbol{c}_\delta.$$

Due to our assumption, $Bx \geq 0$ for any feasible solution x. Thus the optimal solution will be achieved for $\lambda = \underline{f}$. It this setting the problem draws

$$\begin{aligned} \inf\ \delta \ \text{ subject to } \ & Ax \leq \underline{f}Bx,\ Cx \leq c,\ x \geq 0, \\ & A \in \boldsymbol{A}_\delta,\ B \in \boldsymbol{B}_\delta,\ C \in \boldsymbol{C}_\delta,\ c \in \boldsymbol{c}_\delta. \end{aligned}$$

Similarly, the can put $A := A^0 - \delta A^\Delta$, $C := C^0 - \delta C^\Delta$, $c := c^0 + \delta c^\Delta$ and $B := B^0 - \delta B^\Delta$ if $\underline{f} < 0$ and $B := B^0 + \delta B^\Delta$ if $\underline{f} \geq 0$. Therefore we conclude: If $\underline{f} \geq 0$ then

$$\begin{aligned} \delta_1^* \geq \delta_1 = \inf\ \delta \ \text{ subject to } \ & (A^0 - \delta A^\Delta)x \leq \underline{f}(B^0 + \delta B^\Delta)x, \\ & (C^0 - \delta C^\Delta)x \leq c^0 + \delta c^\Delta,\ x \geq 0, \\ = \inf\ \delta \ \text{ subject to } \ & (A^0 - \underline{f}B^0)x \leq \delta(A^\Delta + \underline{f}B^\Delta)x, \\ & C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta),\ x \geq 0. \end{aligned}$$

If $\underline{f} < 0$ then

$$\begin{aligned} \delta_1^* \geq \delta_1 = \inf\ \delta \ \text{ subject to } \ & (A^0 - \delta A^\Delta)x \leq \underline{f}(B^0 - \delta B^\Delta)x, \\ & (C^0 - \delta C^\Delta)x \leq c^0 + \delta c^\Delta,\ x \geq 0, \\ = \inf\ \delta \ \text{ subject to } \ & (A^0 - \underline{f}B^0)x \leq \delta(A^\Delta - \underline{f}B^\Delta)x, \\ & C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta),\ x \geq 0. \end{aligned}$$

2. (Upper tolerance) Due to Lemma 1,

$$\begin{aligned} \delta_2^* = \sup\ \delta \ \text{ subject to } \ & \overline{f} \geq f(A, B, C, c) \\ & \forall A \in \boldsymbol{A}_\delta,\ B \in \boldsymbol{B}_\delta,\ C \in \boldsymbol{C}_\delta,\ c \in \boldsymbol{c}_\delta, \\ \geq \sup\ \delta \ \text{ subject to } \ & \forall A \in \boldsymbol{A}_\delta,\ B \in \boldsymbol{B}_\delta,\ C \in \boldsymbol{C}_\delta,\ c \in \boldsymbol{c}_\delta\ \exists \lambda, x; \\ & \overline{f} \geq \lambda,\ Ax \leq \lambda Bx,\ Cx \leq c,\ x \geq 0. \end{aligned}$$

Again, we put $\lambda := \overline{f}$ yielding

$$\begin{aligned} \delta_2^* \geq \sup\ \delta \ \text{ subject to } \ & \forall A \in \boldsymbol{A}_\delta,\ B \in \boldsymbol{B}_\delta,\ C \in \boldsymbol{C}_\delta,\ c \in \boldsymbol{c}_\delta\ \exists x; \\ & Ax \leq \overline{f}Bx,\ Cx \leq c,\ x \geq 0. \end{aligned}$$

For the same reasons as in the proof of Theorem 2 it suffices to consider $A := A^0 + \delta A^\Delta$, $C := C^0 + \delta C^\Delta$, $c := c^0 - \delta c^\Delta$, and $B := B^0 - \delta B^\Delta$ if $\overline{f} \geq 0$

and $B := B^0 + \delta B^\Delta$ if $\overline{f} < 0$. Therefore, if $\underline{f} \geq 0$ then

$$\begin{aligned}\delta_2^* \geq \delta_2 = \sup\ \delta\ \ \text{subject to}\ \ & (A^0 + \delta A^\Delta)x \leq \overline{f}(B^0 - \delta B^\Delta)x,\\ & (C^0 + \delta C^\Delta)x \leq c^0 - \delta c^\Delta,\ x \geq 0,\\ = \sup\ \delta\ \ \text{subject to}\ \ & (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta + \overline{f}B^\Delta)x,\\ & -C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta),\ x \geq 0.\end{aligned}$$

If $\overline{f} < 0$ then

$$\begin{aligned}\delta_2 = \sup\ \delta\ \ \text{subject to}\ \ & (A^0 + \delta A^\Delta)x \leq \overline{f}(B^0 + \delta B^\Delta)x,\\ & (C^0 + \delta C^\Delta)x \leq c^0 - \delta c^\Delta,\ x \geq 0,\\ = \sup\ \delta\ \ \text{subject to}\ \ & (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta - \overline{f}B^\Delta)x,\\ & -C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta),\ x \geq 0.\end{aligned}$$

□

Note that for calculating the admissible tolerance along Theorem 2 we have to solve only two ordinary generalized linear fractional programming problems. The resulting tolerances δ_1 and δ_2 are maximal in the most of cases. It needn'd be maximal as (10) and (11) may differ. This happens rarely and arbitrarily small perturbation of $\underline{f}$ or $\overline{f}$ usually causes maximality of δ_1 and δ_2.

The next remark concerns assumptions (A2) and (A3). It is uncomfortable to check their validity as the maximal tolerance δ^* is unknown. However, an insight into proof of Theorem 2 reveals that—from the practical standpoint—it suffices to determine admissible tolerances δ_1 and δ_2 along the statement, and afterwards to check validity of assumptions (A2)–(A3) with $\delta^* = \min(\delta_1, \delta_2)$.

Note that our results are easily extendable to the non-symmetric case where interval matrices are defined as $\boldsymbol{A}_\delta := [A^0 - \delta A_1^\Delta, A^0 + \delta A_2^\Delta]$ and A_1^Δ, A_2^Δ are non-negative matrices. This case is more general, but for the sake of simplicity we don't discuss it here in detail.

Example 1. We illustrate our approach on a von Neumann economic growth model. Such a model reads

$$\max\ \lambda\ \ \text{subject to}\ \ \lambda Ax \leq Bx,\ x \geq 1,$$

where variables x_i, $i = 1, \dots, n$ denote activity of sector i. Matrix $A \in \mathbb{R}^{m\times n}$ consists of input coefficients and matrix $B \in \mathbb{R}^{m\times n}$ consists of output coefficients. This model corresponds to generalized linear fractional programming, and it is easily transformed into the standard form (1).

For concreteness, consider an example adapted from Li [16]:

$$A = \begin{pmatrix} 0.28 & 0.50 & 0.53 & 0 & 0 & 0 \\ 0.84 & 0 & 0 & 0 & 0 & 0.77 \\ 0 & 0.49 & 0.45 & 0.50 & 0.48 & 0 \\ 0 & 0 & 0 & 0.51 & 0.57 & 0.29 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0.25 & 1 & 1 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The optimal value is $\lambda^* = 1.049$, that is, the economy is slightly expanding. We compute tolerances for all coefficients of matrix A and B such that the growth rates for all admissible perturbations lie within the interval $[\underline{f}, \overline{f}] := [1, 1.2]$. We consider percentage tolerance, i.e., we put tolerance rates as $A^\Delta := |A|$ and $B^\Delta := |B|$. Calling Theorem 2 we obtain an admissible lower tolerance $\delta^1 = 0.024$ and upper tolerance $\delta^2 = 0.067$. Assumptions (A2) and (A3) are satisfied. We conclude that all entries of A and B may vary simultaneously and independently within 2.4% tolerance of their nominal value while the economy is guaranteed to be expanding in the growth rate [1, 1.2].

Naturally, larger tolerances can be obtained when we consider less parameters for perturbing. For instance, we are interested in tolerances for entries of B only. In this case, we put $A^\Delta := 0$ and $B^\Delta := |B|$ and call Theorem 2 again. We get an admissible lower tolerance $\delta^1 = 0.046$ and upper tolerance $\delta^2 = 0.143$ and the corresponding assumptions (A2)–(A3) are satisfied, too. Now, the resulting percentage tolerance is 4.6%.

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