An improved linear bound on the number of perfect matchings in cubic graphs*

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Abstract

We show that every cubic bridgeless graph with n vertices has at least $3n/4 - 10$ perfect matchings. This is the first bound that differs by more than a constant from the maximal dimension of the perfect matching polytope.

1 Introduction

We study the number of perfect matchings in cubic bridgeless graphs, in which parallel edges are allowed. By a classical theorem of Petersen [11], every such graph has a perfect matching. In fact, every edge of a cubic bridgeless graph is contained in a perfect matching, and thus every n-vertex cubic

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bridgless graph has at least three perfect matchings. Lovász and Plummer [8, Conjecture 8.1.8] conjectured that the number of perfect matchings in cubic bridgeless graphs should grow exponentially with n:

Conjecture 1 (Lovász and Plummer, 1970s). *Every cubic bridgeless graph with n vertices has at least $2^{\Omega(n)}$ perfect matchings.*

Conjecture 1 has been verified for several special classes of graphs, one of them being bipartite graphs. The first non-trivial lower bound on the number of perfect matchings in cubic bridgeless bipartite graphs was obtained in 1969 by Sinkhorn [14] who proved a bound of $\frac{n}{2}$, thereby establishing a conjecture of Hall. The same year, Minc [9] increased this lower bound by 2. Then, a bound of $\frac{3n}{2} - 3$ was proven by Hartfiel and Crosby [5]. The first exponential bound, $6 \cdot \left(\frac{4}{3}\right)^{n/2-3}$, was obtained in 1979 by Voorhoeve [15]. This was generalized to all regular bipartite graphs in 1998 by Schrijver [13] who thereby proved a conjecture of himself and Valiant [12].

Recently, an important step towards a proof of Conjecture 1 was achieved by Chudnovsky and Seymour [2] who proved the conjecture for planar graphs.

Theorem 1 (Chudnovsky and Seymour, 2008). *Every cubic bridgeless planar graph with n vertices has at least $2^{n/655978752}$ perfect matchings.*

Until recently, the only known lower bound on the number of perfect matchings of a general cubic bridgeless graph was an estimate given by the dimension of the perfect matching polytope. Edmonds, Lovász, and Pulleyblank [4], inspired by Naddef [10], proved that the dimension of the perfect matching polytope of a cubic bridgeless n-vertex graph is at least $n/4 + 1$ which implies:

Theorem 2 (Edmonds, Lovász, and Pulleyblank, 1982). *Every cubic bridgeless graph with n vertices has at least $n/4 + 2$ perfect matchings.*

An argument based on the dimension of the perfect matching polytope cannot yield a bound exceeding $n/2 + 2$, since the dimension of the perfect matching polytope is always between $n/4 + 1$ and $n/2 + 1$ (the upper bound is achieved by cubic bipartite graphs). In [6], the authors presented an argument based on the brick and brace decomposition of matching covered graphs, showing that every n-vertex cubic bridgeless graph G has at least $n/2$ perfect matchings. They also characterized those graphs G with exactly $n/2$ or $n/2 + 1$ perfect matchings. Their argument is inductive and uses the

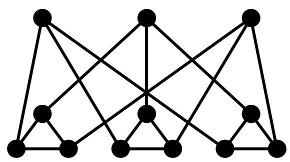

Figure 1: The only n-vertex cubic bridgeless graph with $n/2$ perfect matchings.

characterization of so-called extremal cubic bricks by de Carvalho *et al.* [1]. Let us state the result of [6] precisely:

Theorem 3. *Every cubic bridgeless graph G of order n contains at least $n/2+1$ perfect matchings unless G is the graph obtained from $K_{3,3}$ by replacing all three vertices of one of the two color classes with triangles (see Figure 1). This exceptional graph contains $n/2$ perfect matchings. Moreover, there are only 17 non-isomorphic cubic bridgeless graphs with at most $n/2+1$ perfect matchings.*

In this paper, we show that every n-vertex cubic bridgeless graph has at least $3n/4-10$ perfect matchings. We think that the main significance of the bound lies in the fact that it is the first result asserting that the number of vertices of the perfect matching polytope of a cyclically 4-edge-connected cubic graph exceeds its dimension by more than a constant.

One of our tools, similarly as in [6], is the machinery of brick and brace decompositions of matching covered graphs, which we introduce in the next section. However, unlike in [6], we have to show that the number of perfect matchings of cyclically 4-edge-connected graphs exceeds the dimension of the perfect matching polytope by a linear factor. This is done in Sections 3 and 4. In Sections 5 and 6, the bound is then extended to 3-edge-connected and eventually to all cubic bridgeless graphs.

2 Notation

In this section, we introduce notation used throughout the paper. If G is a graph, $V(G)$ denotes the vertex set of G and $E(G)$ denotes its edge set. $\mathbb{R}^{E(G)}$ is an $|E(G)|$-dimensional vector space with coordinates corresponding to the edges of G. If $A \subseteq V(G)$, $G[A]$ stands for the subgraph of G induced by the vertices of A.

A graph G is *k-vertex-connected* if G has at least $k+1$ vertices, and remains connected after removing any set of at most $k-1$ vertices. If $\{A, B\}$ is a partition of $V(G)$, the set $E(A, B)$ of edges with one end in A and the other in B is called an *edge-cut* or a *k-edge-cut* of G, where k is the size of $E(A, B)$. A graph is *k-edge-connected* if it has no edge-cuts of size less than k. Graphs that are 2-edge-connected are also called *bridgeless*. Finally, an edge-cut $E(A, B)$ is *cyclic* if the subgraphs induced by A and B both contain a cycle. A graph G is *cyclically k-edge-connected* if G has no cyclic edge-cuts of size less than k. The following is a useful observation that we implicitly use in our further considerations:

Observation 4. *If G is a graph with minimum degree three, in particular G can be a cubic graph, then a k-edge-cut $E(A, B)$ such that $|A| \geq k-1$ and $|B| \geq k-1$ must be cyclic.*

We say that a graph G is *X-near cubic* for a multiset X of positive integers, if the multiset of degrees of G not equal to three is X. For example, the graph obtained from a cubic graph by removing an edge is $\{2, 2\}$-near cubic.

If v is a vertex of G, then $G \setminus v$ is the graph obtained by removing the vertex v together with all its incident edges. If e is an edge of G, $G - e$ is the graph obtained from G by removing the edge e and keeping its end vertices. We also use this notation with e replaced by a set of edges and v replaced by a set of vertices. If H is a connected subgraph of G, G/H is the graph obtained by contracting all the vertices of H to a single vertex, removing arising loops and preserving all parallel edges. An *odd minor* of G is a graph obtained by contracting connected subgraphs of G, each having an odd number of vertices. Observe that if all the degrees of G are odd, then all the degrees of an odd minor of G are also odd.

A perfect matching of G is a spanning subgraph with all vertices of degree one. A theorem of Tutte (1947) asserts that G has a perfect matching if and only if the number of components of $G \setminus S$ with an odd number of vertices (also called *odd* components) is at most $|S|$ for every $S \subseteq V(G)$. One of the consequences of Tutte's theorem is that for every edge e of a cubic bridgeless graph, there is a perfect matching containing e and for every two edges e and f, there is a perfect matching avoiding both e and f.

2.1 Brick and brace decomposition of graphs

The brick and brace decomposition plays a crucial role in the study of the structure of perfect matchings in graphs. A graph G is said to be *matching covered* if every edge is contained in a perfect matching of G, and it is *matching double-covered* if every edge is contained in at least two perfect matchings of G. A theorem of Kotzig (see [8, Section 8.6]) asserts that if a graph G has a unique perfect matching, then G has a bridge. An immediate consequence of this theorem is the following proposition:

Proposition 5. *Every cyclically* 4*-edge-connected cubic graph different from* K_4 *is matching double-covered.*

An edge-cut $E(A, B)$ is *tight* if every perfect matching contains precisely one edge of $E(A, B)$. If G is a connected matching covered graph with a tight edge-cut $E(A, B)$, then $G[A]$ and $G[B]$ are also connected. Moreover, every perfect matching of G corresponds to a pair of perfect matchings in the graphs G/A and G/B. Hence, both G/A and G/B are also matching covered. We say that we have decomposed G into G/A and G/B. If any of these graphs still have a tight edge-cut, we can keep decomposing it until no graph in the decomposition has a tight edge-cut. Matching covered graphs without tight edge-cuts are called *braces* if they are bipartite and *bricks* otherwise, and the decomposition of a graph G obtained this way is known as the *brick and brace decomposition* of G.

Lovász [7] showed that the collection of graphs obtained from G in any brick and brace decomposition is unique up to the multiplicity of edges. This allows us to speak of *the* brick and brace decomposition of G, as well as *the* number of bricks and *the* number of braces in the decomposition of G.

A graph is said to be *bicritical* if $G \setminus \{u, v\}$ has a perfect matching for any two vertices u and v. Edmonds *et al.* [4] gave the following characterization of bricks:

Theorem 6 (Edmonds *et al.*, 1982). *A graph G is a brick if and only if it is* 3*-vertex-connected and bicritical.*

It can also be proven that a brace is a bipartite graph such that for any two vertices u and u' from the same color class and any two vertices v and v' from the other color class, the graph $G \setminus \{u, u', v, v'\}$ has a perfect matching, see [8].

We finish this subsection with an observation that the brick and brace decomposition of a bipartite graph contains braces only; we include the proof of this fact as a demonstration of the just introduced notation.

Proposition 7. *If H is a bipartite matching covered graph, then its brick and brace decomposition consists of braces only.*

Proof. We proceed by induction on the size of H. Let U and V be the two color classes of H. If H has no tight edge-cut, then H is a brace and there is nothing to prove. Otherwise, let $E(A, B)$ be a tight edge-cut of H. Let e be an edge of $E(A, B)$. By symmetry, we can assume that e is incident with a vertex of $A \cap U$. Since H contains a perfect matching such that e is the only edge of $E(A, B)$ in the matching, $|A \cap U| = |A \cap V| + 1$ and $|B \cap V| = |B \cap U| + 1$. Hence, any matching containing a single edge of the cut $E(A, B)$, say f, must satisfy that f is incident with a vertex of $A \cap U$. Since $E(A, B)$ is a tight edge-cut, all the edges of $E(A, B)$ join vertices of $A \cap U$ and $B \cap V$, and so both graphs G/A and G/B are bipartite. The claim follows by applying the induction to G/A and G/B. □

2.2 Perfect matching polytope

Some of our arguments also involve the perfect matching polytopes of graphs. The *perfect matching polytope* of a graph G is the convex hull of characteristic vectors of perfect matchings of G. The sufficient and necessary conditions for a vector $w \in \mathbb{R}^{E(G)}$ to lie in the perfect matching polytope are known [3]:

Theorem 8 (Edmonds 1965). *If G is a graph, then a vector $w \in \mathbb{R}^{E(G)}$ lies in the perfect matching polytope of G if and only if the following holds:*

(i) w is non-negative,

(ii) for every vertex v of G the sum of the entries of w corresponding to the edges incident with v is equal to one, and

(iii) for every set $S \subseteq V(G)$, $|S|$ odd, the sum of the entries corresponding to edges having exactly one vertex in S is at least one.

It is also well-known that conditions (i) and (ii) are necessary and sufficient for a vector to lie in the perfect matching polytope of a bipartite graph G.

The dimension of the perfect matching polytope of a matching covered graph G can be computed from the brick and brace decomposition of G: Edmonds, Lovász, and Pulleyblank [4], using some ideas from Naddef [10], showed that it is equal to $|E(G)| - |V(G)| + 1 - b(G)$ where $b(G)$ denotes the number of bricks in the decomposition.

Let w be a vector lying in the perfect matching polytope of G and $E(A, B)$ be an edge-cut of G. If the sum of the entries of w corresponding to edges of $E(A, B)$ is not equal to one, then at least one of the matchings whose characteristic vectors convexly combine to w does not contain exactly one edge of the cut. Hence, $E(A, B)$ cannot be tight. Conversely, if an edge-cut is tight, the entries corresponding to the edges of the cut of every vector lying in the perfect matching polytope sum to one. Let us formulate this observation as a propostion.

Proposition 9. *An edge-cut of G is tight if and only if the sum of the entries corresponding to the edges of the cut is equal to one for every vector lying in the perfect matching polytope of G.*

If G is a cubic bridgeless graph, it is easy to infer from Theorem 8 that the vector with all entries equal to $1/3$ lies in the perfect matching polytope of G. Hence, every tight cut of a cubic bridgeless graph must have size three by Proposition 9. In particular, the brick and brace decomposition of a cubic bridgeless graph only contains cubic (bridgeless) graphs.

3 Cyclically 5-edge-connected graphs

Our aim in this section is to show that if G is a cyclically 5-edge-connected cubic graph, and e is an edge of G, then $G - e$ has few bricks in its brick and brace decomposition, or there exists an edge f so that $G - \{e, f\}$ is bipartite and matching covered. This will imply that G has at least $3|V(G)|/4 - 3/2$ perfect matchings.

Lemma 10. *Let G be a cyclically 5-edge-connected cubic graph, and let $E(U, U')$ be a 5-edge-cut of G. If G/U is matching covered, then it is cyclically 5-edge-connected and 3-vertex-connected.*

Proof. Since G is cyclically 5-edge-connected, $G[U]$ is connected, and so $H = G/U$ is well-defined. Observe that any cyclic edge-cut of H corresponds to a cyclic edge-cut of G. Hence, H is cyclically 5-edge-connected. Moreover, it is a $\{5\}$-near cubic graph, and since the minimum degree of H is three, any edge-cut of size at most two is cyclic. This implies that H is 3-edge-connected. Also note that H is 2-vertex-connected, otherwise it would contain an edge-cut of size at most two since the maximum degree of H is five.

We now show that H is 3-vertex-connected, which will establish the lemma. For the sake of contradiction, assume that H has a vertex-cut of size two formed by vertices x and y, and let A and B be the components of $H \setminus \{x, y\}$. If both x and y have degree three, one easily infer a 2-edge-cut. Hence, we may assume that x has degree five and y has degree three. By the 3-edge-connectivity of H, the graph $H \setminus \{x, y\}$ cannot have more than two components.

A simple check shows that the only $\{5\}$-near cubic graph of order at most four is the graph obtained from K_4 by removing an edge, say uv, and doubling the edges uw and vw, where w is one of the two vertices distinct from u and v. However, this graph is not matching covered. Since the number of vertices of H is even, we can assume that H has at least six vertices.

If x and y are joined by an edge, then the number of edges between A or B and $\{x, y\}$ must be three. At least one these two edge-cuts is however cyclic; otherwise, both A and B have order one and the order of H is four. Hence, the number of edges leaving $\{x, y\}$ is eight and x and y are non-adjacent.

Neither x nor y is incident with a bigon (an edge with multiplicity two); otherwise the edges leaving the bigon form a cyclic edge-cut of H of size at most four. Since the number of edges between A or B and $\{x, y\}$ must be at least three and neither x nor y is incident with a bigon, it follows that both A and B contain at least two vertices. Hence, the number of edges between A or B and $\{x, y\}$ must be at least four since otherwise these edges would form a cyclic edge-cut of size three in H. Consequently, there are exactly four edges between A or B and $\{x, y\}$, and the sets A and B both contain exactly two vertices. Since x has degree five and is neither adjacent to y nor incident to a bigon, this is impossible. □

We now prove that under the same assumptions as in the previous lemma, the brick and brace decomposition of G/U contains exactly one brick.

Lemma 11. *Let G be a cyclically 5-edge-connected cubic graph, and let $E(U, U')$ be a 5-edge-cut of G. If G/U is matching covered, then $b(G/U) = 1$.*

Proof. The proof proceeds by induction on the order of $H = G/U$ (G is fixed). Since H is a $\{5\}$-near cubic graph, H is not bipartite. By Lemma 10, H is cyclically 5-edge-connected and 3-vertex-connected. By Theorem 6, H is a either a brick (in which case $b(H) = 1$) or is not bicritical. So we can focus on the latter case.

Let x and y be the vertices of H such that $H \setminus \{x, y\}$ has no perfect matching. According to Tutte's Theorem, there exists a set of vertices S of $H \setminus \{x, y\}$ such that $H \setminus (S \cup \{x, y\})$ has at least $|S| + 1$ odd components. Let $S' = S \cup \{x, y\}$. Since the number of vertices of H is even, $H \setminus S'$ has at least $|S| + 2 = |S'|$ odd components. As H is $\{5\}$-near cubic, the number of edges leaving S' is at most $3|S'| + 2$. In what follows, we distinguish two cases regarding the sizes of the components in $H \setminus S'$.

Suppose first that all the components of $H \setminus S'$ are single vertices of degree three in H. Then the number of edges between S' and $H \setminus S'$ is exactly $3|S'|$. In this case, the vertex of degree five is in S' and S' contains two vertices joined by an edge. Observe that H has no matching containing this edge which contradicts our assumption that H is matching covered.

Suppose now that at least one of the components of $H \setminus S'$ is not a single vertex whose degree is three in H, then the number of edges leaving the odd components of $H \setminus S'$ is at least $3|S'|+2$: there are at least five edges leaving every odd component that is not a single vertex since H is cyclically 5-edge-connected and there are five edges leaving a vertex of degree five in case this vertex were one of the components of $H \setminus S'$. We conclude that the number of edges between S' and $H \setminus S'$ is exactly $3|S'| + 2$ (and thus S' is a stable set and contains the vertex of degree five), and $H \setminus S'$ contains exactly $|S'|$ components, $|S'| - 1$ of them being isolated vertices and the remaining one having odd size.

Let B be the set of vertices of the only component of $H \setminus S'$ that is not an isolated vertex and set $A = V(H) \setminus B$. As $H \setminus S'$ contains exactly $|S'|$ components and S' is a stable set, the 5-edge-cut $E(A, B)$ is tight. In particular, H/B is a bipartite matching covered graph, so $b(H/B) = 0$ by Proposition 7. Let A' be the set of vertices of G corresponding to A, i.e. $H/A = G/A'$. The graph $H[A]$ is connected and contains the vertex of degree five, so $H/A = G/A'$ is a matching covered graph that satisfies the induction hypothesis. Since the order of $G/A' = H/A$ is smaller than that of $G/U = H$, the induction yields that $b(H/A) = 1$. Consequently, $b(H) = b(H/A) + b(H/B) = 1 + 0 = 1$. □

Using the same approach as in Lemma 10, we now study the connectivity of a matching covered $\{4, 4\}$-near cubic graph obtained from $G - e$ by

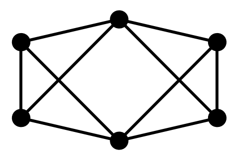

Figure 2: The only possible odd minor of $G-e$ (G being a cyclically 5-edge-connected cubic graph) that is a matching covered $\{4,4\}$-near cubic graph and that is not 3-vertex-connected.

contracting some odd components.

Lemma 12. *Let G be a cyclically 5-edge-connected cubic graph, e an edge of G and H an odd minor of $G-e$. If H is a $\{4,4\}$-near cubic graph, then H is 2-vertex-connected. Moreover, if H has a 2-vertex-cut and is matching covered, then H is isomorphic to the graph depicted in Figure 2.*

Proof. Since G is cyclically 5-edge-connected, the graph H is cyclically 4-edge-connected. We first show that we can focus on graphs H of order six or more. The only $\{4,4\}$-near cubic graphs of order at most four that are matching covered but not 3-vertex-connected have the cycle C_4 as an underlying simple graph. In that case, H must be either

- the graph obtained from C_4 by doubling three distinct edges, or
- the graph obtained from C_4 by tripling an edge and doubling the opposite one.

Since both these graphs contain a cyclic edge-cut of size at most three, the order of H is at least six.

The argument that H is 2-vertex-connected is analogous to that in the proof of Lemma 10, so we leave the details to the reader. Assume that H contains a bigon. Since H has order at least six and is cyclically 4-edge-connected, exactly four edges leave this bigon. Observe that e is not contained in the corresponding edge-cut in G, since the ends of the bigon are the two vertices of degree four of H. Hence, the four edges leaving the bigon correspond to a cyclic 4-edge-cut of G, which is impossible. So we can assume that H is a simple graph.

Finally, we focus on analyzing vertex-cuts of size two. Let $\{x,y\}$ be a 2-vertex-cut of H and let $C_1,\dots,C_k$ be the $k \geq 2$ components of $H \setminus \{x,y\}$.

Since H has no bigons, each of the sets C_i contains at least two vertices. Hence, the number of edges between C_i and $\{x, y\}$ is at least four (otherwise, they would form a cyclic edge-cut of H of size at most three). Consequently, $k = 2$ and x and y are non-adjacent vertices of degree four. This implies that the number of edges between each C_i $(i = 1, 2)$ and $\{x, y\}$ is precisely four. Since the edge e corresponds to an edge joining x and y, each of the cuts $E(C_1, C_2 \cup \{x, y\})$ and $E(C_1 \cup \{x, y\}, C_2)$ has the same size in H and G. As G is cyclically 5-edge-connected, both C_1 and C_2 must contain exactly two vertices. We conclude that H must be the graph depicted in Figure 2. □

In the next lemma, we show that graphs satisfying the assumptions of Lemma 12 have few bricks in their decomposition.

Lemma 13. *Let G be a cyclically 5-edge-connected cubic graph, e an edge of G and H an odd minor of $G - e$. If H is a matching covered $\{4, 4\}$-near cubic graph, then $b(H) \leq 2$.*

Proof. The proof proceeds by induction on the order of the odd minor H of $G - e$ (G and e are fixed). If H is bipartite, then $b(H) = 0$ by Proposition 7. If H is not 3-vertex-connected, then by Lemma 12 it is isomorphic to the graph depicted in Figure 2 and its brick and brace decomposition consists of two graphs isomorphic to K_4 with a single parallel edge.

Hence, we can assume that H is a 3-vertex-connected non-bipartite graph. If H is a brick, then $b(H) = 1$. By Theorem 6, we may assume that H is not bicritical. Let x and y be two vertices of H such that $H \setminus \{x, y\}$ has no perfect matching. Let S be a set of vertices of $H \setminus \{x, y\}$ such that $H \setminus (S \cup \{x, y\})$ has at least $|S| + 1$ odd components, and let $S' = S \cup \{x, y\}$. Since the number of vertices of H is even, $H \setminus S'$ has at least $|S| + 2 = |S'|$ odd components. Based on the degree distribution of H and the fact that G is cyclically 5-edge-connected, the number of edges leaving S' is $3|S'|$, $3|S'| + 1$ or $3|S'| + 2$ and $H \setminus S'$ contains precisely $|S'|$ components (which are all odd) and at most one of these components is not an isolated vertex. Notice that if all the odd components of $H \setminus S'$ were isolated vertices, then either H would be bipartite (which case has already been considered) or S' would contain both vertices of degree four. In the latter case, there would be an edge joining two vertices of S' but such an edge cannot be contained in a perfect matching of H contrary to our assumption that H is matching covered. We conclude that $H \setminus S'$ contains precisely one non-trivial odd component B.

Let $A = V(H) \setminus B$. We consider three possibilities, regarding whether the vertices of degree four belong to S'. If S' only contains vertices of degree three, then there are $3|S'|$ edges leaving S'. In this case, the two vertices of degree four are in B and $E(A, B)$ is a cyclic edge-cut of size three, which is impossible. Depending whether S' contains one or both vertices of degree four of H, the number of edges between A and B is four or five. Observe that in both cases, H/B is bipartite, and hence the edge-cut $E(A, B)$ is tight. By Proposition 7, this also implies that $b(H/B) = 0$. Let A' the set of vertices of G corresponding to A, i.e. $H/A = G/A'$. Since $E(A, B)$ is tight, the graph $H/A = (G - e)/A'$ is matching covered. If S' contains a single vertex of degree four, then H/A is a $\{4, 4\}$-near cubic graph. In this case we apply induction on H/A. If S' contains two vertices of degree four, then $E(A, B)$ is a cyclic 5-edge-cut and we can apply Lemma 11 on H/A. In both cases, $b(H) = b(H/A) + b(H/B) = b(H/A) \leq 2$. □

Lemma 13 has the following corollary:

Lemma 14. *Let G be a cyclically 5-edge-connected cubic graph and e an edge of G. If $G - e$ is matching covered, then $b(G - e) \leq 2$.*

Proof. Since G is a cyclically 5-edge-connected cubic graph and $G - e$ is matching covered, we infer that G is not isomorphic to K_4. This implies that G is triangle-free. Hence, the two vertices of degree two of $G - e$, say u and u', have no common neighbor.

Let A be comprised of the vertex u and its two neighbors in $G - e$ and $B = V(G) \setminus A$. Similarly, let A' be comprised of the vertex u' and its two neighbors in $G - e$ and $B' = V(G) \setminus A'$. The cuts $E(A, B)$ and $E(A', B')$ are tight in $G - e$. Since the sets A and A' are disjoint, after reducing the tight edge-cuts $E(A, B)$ and $E(A', B')$ of $G - e$, we obtain two bipartite graphs of order four and a $\{4, 4\}$-near cubic graph. The statement follows from Proposition 7 and Lemma 13. □

We now study the structure of a graph G such that the graph $G - e$ is not matching covered for some edge e.

Lemma 15. *Let G be a cyclically 5-edge-connected cubic graph and e an edge of G. If $G - e$ is not matching covered, then G contains an edge f such that $G - \{e, f\}$ is matching covered and bipartite.*

Proof. Let $e = uu'$ and $H = G - e$, and assume that H contains an edge $f = vv'$ that is not contained in any perfect matching of H. Hence, $H \setminus \{v, v'\}$

contains a set S of vertices such that the number of odd components of $H \setminus S'$ where $S' = S \cup \{v, v'\}$ is at least $|S| + 1$. Since the number of vertices of H is even, the number of the odd components is at least $|S| + 2 = |S'|$. Since v and v' are both contained in S', the number of edges leaving S' is at most $3|S'| - 2$. Since G is cyclically 5-edge-connected, all the components of $H \setminus S'$ are isolated vertices and neither u nor u' is contained in S'. This implies that $H' = G \setminus \{e, f\}$ is a $\{2, 2, 2, 2\}$-near cubic bipartite graph. Denote by U and V the two color classes of H', in such way that $\{u, u'\} \subseteq U$ and $\{v, v'\} \subseteq V$.

We now show that H' is matching covered. Let H'' be a graph obtained from H' by adding a vertex v_e (resp. v_f) and joining it by two parallel edges to each of the end-vertices of e (resp. f). We claim that H'' has no edge-cut of size at most three separating v_e and v_f. Assume the opposite and let $E(A, B)$ be such an edge-cut. By symmetry, $v_e \in A$ and $v_f \in B$.

If A contains both end-vertices of e and B contains both end-vertices of f, then $E(A, B)$ corresponds to a non-trivial edge-cut of size at most three of G which violates our assumption that G is cyclically 5-edge-connected. Hence, we can assume by symmetry that A contains u but not u'. As the size of $E(A, B)$ is at most three, both v and v' must be contained in B. Let us estimate the size of the edge-cut of G corresponding to $E(A, B)$: the two edges between v_e and u' are not present anymore and but the edge e is now present. Hence, the size of the corresponding edge-cut of G is at most two. Since G is cubic, this is also a cyclic edge-cut of size at most two, which contradicts our assumption that G is cyclically 5-edge-connected.

Since there is no edge-cut of size at most three separating v_e and v_f in H'', there are four edge-disjoint paths connecting v_e and v_f by Menger's theorem. Consequently, H' contains four edge-disjoint paths P_1, P_2, P_3 and P_4 joining the vertices u and u' to the vertices v and v'. Direct the paths P_i from u and u' to v and v', and consider now the following vector $w \in \mathbb{R}^{E(H')}$:

$$w_e = \begin{cases} 1/2 & \text{if } e \text{ is directed from } U \text{ to } V, \\ 1/6 & \text{if } e \text{ is directed from } V \text{ to } U, \text{ and} \\ 1/3 & \text{otherwise.} \end{cases}$$

Observe that H' is bipartite and for every vertex x of H', the sum of the entries of w corresponding to the edges incident with x is equal to one. Hence, w lies in the perfect matching polytope of H'. Since all the entries of w are non-zero, the graph H' is matching covered. □

We now apply Lemmas 14 and 15 to prove the main result of this section.

Theorem 16. *Let G be a cyclically 5-edge-connected cubic graph of order n. For every edge e of G, the graph $G - e$ has at least $n/2 - 1$ perfect matchings.*

Proof. Let e be an arbitrary edge of G. If $G - e$ is matching covered, then $b(G - e) \leq 2$ by Lemma 14. Hence, the dimension of the perfect matching polytope of $G - e$ is at least $(3n/2 - 1) - n + 1 - 2 = n/2 - 2$. Consequently, $G - e$ has at least $n/2 - 1$ perfect matchings.

If $G - e$ is not matching covered, then Lemma 15 guarantees the existence of an edge f such that $G \setminus \{e, f\}$ is matching covered and bipartite, in which case $b(G \setminus \{e, f\}) = 0$ by Proposition 7. Hence, the dimension of the perfect matching polytope of $G \setminus \{e, f\}$ is at least $(3n/2 - 2) - n + 1 = n/2 - 1$ and $G - e$ contains at least $n/2$ perfect matchings. □

This theorem has the following easy consequence on the number of perfect matchings of cyclically 5-edge-connected cubic graphs.

Corollary 17. *Let G be a cubic graph of order n. If G is cyclically 5-edge-connected, then the number of perfect matchings of G is at least $3n/4 - 3/2$.*

Proof. Let e, e' and e'' be the edges incident with an arbitrary vertex v. By Theorem 16, each of the graphs $G - e$, $G - e'$ and $G - e''$ has at least $n/2 - 1$ perfect matchings. Since a perfect matching of G is a perfect matching of exactly two of these three graphs, G has at least $3n/4 - 3/2$ perfect matchings. □

4 Cyclically 4-edge-connected graphs

In this section, we prove that cyclically 4-edge-connected cubic graphs have at least $3n/4 - 9$ perfect matchings. Actually, we prove a slightly stronger version of this result that will be used in the next section.

Theorem 18. *Let H be a cyclically 4-edge-connected cubic graph that is not cyclically 5-edge-connected. If G is a graph of order n obtained from H by replacing some of its vertices with triangles (possibly, $G = H$), then G contains at least $3n/4 - 9$ perfect matchings.*

Proof. Let $E(A', B') = \{e'_1, e'_2, e'_3, e'_4\}$ be a cyclic 4-edge-cut of H. Let a'_i be the end-vertex of the edge e'_i lying in A'. Observe that all the vertices a'_i

are distinct, since otherwise there would be a cyclic edge-cut of size at most three in H. We claim that the graph $H[A']$ is connected and bridgeless: If $H[A']$ were disconnected, then a proper subset of $\{e'_1, e'_2, e'_3, e'_4\}$ would also be a cyclic edge-cut which is impossible by our assumption that H is cyclically 4-edge-connected. If $H[A']$ has a bridge e', this bridge must separate in A' two of the vertices a'_1, a'_2, a'_3, a'_4 from the other two (otherwise, H would contain an edge-cut of size two). Assume that the bridge e' separates $\{a'_1, a'_2\}$ from $\{a'_3, a'_4\}$. As $\{e', e'_1, e'_2\}$ is an edge-cut of H of size three, a'_1 and a'_2 must coincide (otherwise, this edge-cut is cyclic). Similarly, we infer that $a'_3 = a'_4$. This implies that the subgraph $H[A']$ is just an edge contrary to the fact that $E(A', B')$ is a cyclic edge-cut. Hence, $H[A']$ and $H[B']$ are 2-edge-connected.

Observe that $E(A', B')$ corresponds to a cyclic 4-edge-cut $E(A, B) = \{e_1, e_2, e_3, e_4\}$ of G. Let a_i and b_i be the end-vertex of the edge e_i lying in A and B, respectively. Now, let m_X^A, $X \subseteq \{1, 2, 3, 4\}$, be the number of matchings of $G[A]$ that cover all the vertices of $G[A]$ except the vertices a_i, $i \in X$. We use m_X^B in an analogous way. To simplify our notation, we further write m_{13}^A instead of $m_{\{1,3\}}^A$, etc. Clearly, if $|X|$ is odd, then $m_X^A = m_X^B = 0$. As the number of matchings of G is equal to

$$\sum_{X \subseteq \{1,2,3,4\}} m_X^A \cdot m_X^B \ ,$$

we will estimate the summands to obtain the desired bound. Consider a permutation $\{i, j, k, l\}$ of $\{1, 2, 3, 4\}$ with $i < j$, and define G_{ij}^A as the graph obtained from $G[A]$ by adding the edges $a_i a_j$ and $a_k a_l$. $G_{(ij)}^A$ denotes the graph obtained from $G[A]$ by introducing two new adjacent vertices, joining one of them to the vertices a_i and a_j, and the other one to a_k and a_l. Observe that $G_{12}^A = G_{34}^A$ and $G_{(12)}^A = G_{(34)}^A$.

Since $H[A']$ is 2-edge-connected, so is the graph $G[A]$. Hence, the graphs G_{ij}^A and $G_{(ij)}^A$ are cubic and bridgeless. Consequently, they have a perfect matching containing any prescribed edge and a perfect matching avoiding any two prescribed edges. In particular, G_{12}^A has a matching avoiding the edges $a_1 a_2$ and $a_3 a_4$. Consequently, $G[A]$ has a perfect matching. Since $G[A]$ is bridgeless, it has at least two perfect matchings by Kotzig's theorem. We conclude that $m_\varnothing^A \geq 2$. Also by Theorem 3, the graphs G_{ij}^A have at least $|A|/2$ perfect matchings and the graphs $G_{(ij)}^A$ have at least $|A|/2 + 1$ perfect matchings.

If $m_{1234}^A = 0$, then the fact that G_{ij}^A has a perfect matching containing the edge a_ia_j implies that $m^A_{ij} \geq 1$ for every i,j. On the other hand, if $m^A_{ij} = 0$ for some i,j and $k \notin \{i,j\}$, then the fact that $G^A_{(jk)}$ has a perfect matching containing the added edge incident with a_i implies that $m^A_{ik} \geq 1$. We conclude that at least one of the following two possibilities occurs:

Case A: *All the quantities m^A_{ij} are non-zero and $m^A_{\varnothing} \geq 2$.*

Case B: *There exist i and j such that the quantities m^A_{1234}, m^A_{ik} and m^A_{jk} are non-zero for any $k \notin \{i,j\}$, and $m^A_{\varnothing} \geq 2$.*

For every subset $X \subseteq \{1,2,3,4\}$ such that $m^A_X \geq 1$, fix a matching M^A_X avoiding the vertices a_i, $i \in X$. In addition, fix a second matching $M^{A*}_{\varnothing} \neq M^A_{\varnothing}$ covering all the four vertices a_i, $i \in \{1,2,3,4\}$ (such a matching exists as $m^A_{\varnothing} \geq 2$). The fixed matchings of $G[A]$ are referred to as *canonical* matchings of $G[A]$ and the other matchings of $G[A]$ are *non-canonical*. Consider also the analogous definitions for the matchings of $G[B]$.

Assume first that Case A applies. Consider a non-canonical matching of $G[B]$ that avoids vertices b_i and b_j for some $i,j \in \{1,2,3,4\}$. This matching can be completed by adding the canonical matching M^A_{ij} and the edges a_ib_i and a_jb_j to a perfect matching of G. Similarly, a non-canonical matching of $G[B]$ covering all the four vertices can be completed by one of the two canonical matchings $M^A_{\varnothing}$ and $M^{A*}_{\varnothing}$ of $G[A]$. We conclude that the number of perfect matchings of G that are canonical when restricted to $G[A]$ and non-canonical when restricted to $G[B]$ is at least

$$\overline{m}^B_{12} + \overline{m}^B_{13} + \overline{m}^B_{14} + \overline{m}^B_{23} + \overline{m}^B_{24} + \overline{m}^B_{34} + 2\overline{m}^B_{\varnothing}, \tag{1}$$

where $\overline{m}^B_X$ denotes the number of non-canonical matchings of $G[B]$ avoiding $\{b_i, i \in X\}$. On the other hand, if $\{i,j,k,l\}$ is a permutation of $\{1,2,3,4\}$, the number of perfect matchings of $G^B_{(ij)}$ is equal to

$$m^B_{ik} + m^B_{il} + m^B_{jk} + m^B_{jl} + m^B_{\varnothing}. \tag{2}$$

Every graph $G^B_{(ij)}$ has order $|B|+2$, so the number of perfect matchings of $G^B_{(ij)}$ is at least $|B|/2+1$ by Theorem 3 (and thus the number of non-canonical matchings of $G[B]$ is at least $|B|/2-5$). Summing (2) for $(i,j) \in \{(1,2),(1,3),(1,4)\}$ yields the following estimate:

$$2\overline{m}^B_{12} + 2\overline{m}^B_{13} + 2\overline{m}^B_{14} + 2\overline{m}^B_{23} + 2\overline{m}^B_{24} + 2\overline{m}^B_{34} + 3\overline{m}^B_{\varnothing} \geq 3|B|/2 - 15. \tag{3}$$

Comparing (1) and (3), we see that the number of perfect matchings of G that are canonical in $G[A]$ and non-canonical in $G[B]$ is at least $3|B|/4-7.5$.

Assume now that Case B applies for $i = 1$ and $j = 2$. The number of matchings of G that are canonical in $G[A]$ and non-canonical in $G[B]$ is at least

$$\overline{m}^B_{1234} + \overline{m}^B_{13} + \overline{m}^B_{14} + \overline{m}^B_{23} + \overline{m}^B_{24} + 2\overline{m}^B_{\varnothing}. \tag{4}$$

The number of perfect matching of G^B_{13} is equal to the following quantity which must be at least $|B|/2$ as argued before:

$$m^B_{1234} + m^B_{13} + m^B_{24} + m^B_{\varnothing} \geq |B|/2. \tag{5}$$

Similarly, we bound the number of perfect matchings of G^B_{14}:

$$m^B_{1234} + m^B_{14} + m^B_{23} + m^B_{\varnothing} \geq |B|/2. \tag{6}$$

Finally, we estimate the number of perfect matchings of $G^B_{(12)}$:

$$m^B_{13} + m^B_{14} + m^B_{23} + m^B_{24} + m^B_{\varnothing} \geq |B|/2 + 1. \tag{7}$$

Summing (5), (6) and (7) and subtracting the maximum possible number of canonical matchings, we obtain

$$2\overline{m}^B_{1234} + 2\overline{m}^B_{13} + 2\overline{m}^B_{14} + 2\overline{m}^B_{23} + 2\overline{m}^B_{24} + 3\overline{m}^B_{\varnothing} \geq 3|B|/2 - 15. \tag{8}$$

Comparing (4) and (8), we see that the number of perfect matchings of G that are canonical in $G[A]$ and non-canonical in $G[B]$ is at least $3|B|/4-7.5$.

A completely symmetric argument yields that the number of perfect matchings of G that are non-canonical in $G[A]$ and canonical in $G[B]$ is at least $3|A|/4 - 7.5$. We now consider matchings of G that are canonical when restricted to both $G[A]$ and $G[B]$. If Case A applies to both $G[A]$ and $G[B]$, there are at least $6 + 2 \cdot 2 = 10$ such perfect matchings of G. If Case A only applies to one of these two subgraphs, there are at least $4 + 2 \cdot 2 = 8$ such perfect matchings. Finally, if Case B applies to both $G[A]$ and $G[B]$, there are at least $2 + 2 \cdot 2 = 6$ such perfect matchings. In total, the number of perfect matchings of G is at least $3|A|/4 - 7.5 + 3|B|/4 - 7.5 + 6 = 3n/4 - 9$. □

5 Cyclically 3-edge-connected graphs

A *klee-graph* is inductively defined as being either K_4, or the graph obtained from a klee-graph by replacing a vertex by a triangle. Every klee-graph is a cubic planar brick. Moreover, if G is a graph with an edge-cut $E(A,B)$ such that both G/A and G/B are klee-graphs, then G is also a klee-graph.

Recall that every edge of a cubic bridgeless graph is contained in at least one perfect matching. We now prove that if an edge of a 3-edge-connected cubic graph is contained in only one perfect matching, then the graph is a klee-graph.

Lemma 19. *A 3-edge-connected cubic graph G that is not a klee-graph is matching double-covered.*

Proof. The proof proceeds by induction on the order of G. If G has no cyclic 3-edge-cuts, then it is matching double-covered by Proposition 5 (as G is not a klee-graph, it is different from K_4). Otherwise, let $E(A,B)$ be a cyclic 3-edge-cut of G. Since G is not a klee-graph, at least one of the graphs G/A and G/B, say G/A, is not a klee-graph. By induction, G/A is matching double-covered. Since G/B is cubic and bridgeless, it is matching covered. Hence, every perfect matching of G/A extends to G, and so every edge with at least one end-vertex in B is contained in at least two perfect matchings of G.

If e is an edge with both end-vertices in A, then there exists a perfect matching of G/B containing e. Since G/A is matching double-covered, this matching extends in two different ways to a matching of G. Hence, G is matching double-covered. □

In this section, our general strategy to prove that a cyclically 3-edge-connected cubic graph has many matchings is to split the graph along a 3-edge-cut and then use an inductive argument. If the smaller graphs are not klee-graphs, every edge of such graphs is in at least two perfect matchings and those can be combined to form many different matchings in the original graph.

Lemma 20. *Every n-vertex 3-edge-connected cubic graph G with a 3-edge-cut $E(A,B)$ such that neither G/A nor G/B is a klee-graph, has at least $3n/4-6$ perfect matchings.*

Proof. Let $E(A,B)=\{e_1,e_2,e_3\}$, and let m_i^A (resp. m_i^B) be the number of perfect matchings of G/A (resp. G/B) containing the edge e_i. By Lemma 19, each of m_i^A and m_i^B is at least two. By Theorem 3, unless G/A is the exceptional graph from Figure 1,

$$m_1^A+m_2^A+m_3^A \geq |B|/2+3/2 \quad \text{and} \quad m_1^B+m_2^B+m_3^B \geq |A|/2+1/2\ .$$

Since any perfect matching of G/A containing e_i combines with a perfect matching of G/B containing e_i to form a perfect matchings of G containing e_i, the number of perfect matchings of G is at least

$$\sum_{i=1}^{3} m_i^A m_i^B \geq 2\,(|B|/2-5/2)+2\,(|A|/2-7/2)+2\cdot 2 = |A|+|B|-8 = n-8\ .$$

Since neither G/A nor G/B is a klee-graph, and both A and B have odd size, $|A| \geq 5$ and $|B| \geq 5$. Consequently, $n = |A|+|B| \geq 10$ and thus G has at least $n-8 \geq 3n/4-5.5$ perfect matchings.

If G/A is the exceptional graph, then $|B| = 11$ and $m_1^A = m_2^A = m_3^A = 2$. The bound on the number of perfect matchings of G is now

$$\sum_{i=1}^{3} m_i^A m_i^B \geq 2\,(|A|/2+1/2) = |A|+1 = n-10\ .$$

Since $|B| = 11$ and $|A| \geq 5$, the number n of vertices of G is at least 16, and so G has at least $n-10 \geq 3n/4-6$ perfect matchings. □

We say that a 3-edge-cut $E(A,B)$ of a cubic graph G is *nice*, if G/A is not a klee-graph and at least one of the following holds:

(i) G/B is not a klee-graph;

(ii) $|A| \geq 9$;

(iii) $|A| \geq 5$ and $E(A,B)$ is not tight;

(iv) $|A| = 3$, and there are at least two perfect matchings of G containing all the three edges of $E(A,B)$.

The next lemma shows that if we split the graph along a nice 3-edge-cut, the general induction will run smoothly.

Lemma 21. *Let n be a positive integer, and assume that every 3-edge-connected cubic graph of order $n' < n$ has at least $3n'/4-9$ perfect matchings. If G is an n-vertex 3-edge-connected cubic graph with a nice 3-edge-cut $E(A,B)$, then G also has at least $3n/4-9$ perfect matchings.*

Proof. By the assumption of the lemma, G/A is not a klee-graph. If G/B is also not a klee-graph, the bound follows from Lemma 20. We now focus on the remaining three cases and assume that G/B is a klee-graph. By Lemma 19, the graph G/A is matching double-covered. Since G/A has fewer vertices than G, by the assumption of the lemma G/A has at least $3|B|/4+3/4-9$ perfect matchings. Since G/B is a klee-graph, we conclude that it is not the exceptional graph from Figure 1, and thus it has at least $|A|/2+3/2$ perfect matchings.

Let $E(A,B) = \{e_1, e_2, e_3\}$, and let m_i^A (resp. m_i^B) be the number of perfect matchings of G/A (resp. G/B) containing e_i, $i = 1,2,3$. The number of perfect matchings of G containing exactly one edge of the edge-cut $E(A,B)$ is at least

$$m_1^A \cdot m_1^B + m_2^A \cdot m_2^B + m_3^A \cdot m_3^B. \tag{9}$$

As every m_i^A is at least two and every m_i^B is at least one, the expression above is at least

$$(3|B|/4+3/4-13)\cdot 1+2\cdot(|A|/2-1/2)+2\cdot 1 = 3n/4+|A|/4+3/4-12 \tag{10}$$

If $|A| \geq 9$, then $3n/4+|A|/4+3/4-12 \geq 3n/4+12/4-12 = 3n/4-9$. If $|A| \geq 5$ and the edge-cut $E(A,B)$ is not tight, then there exists a perfect matching not counted in the estimate (10) and thus the number of perfect matchings is at least $3n/4+|A|/4+3/4-11 \geq 3n/4-9$. Finally, assume that $|A| = 3$ and there are at least two perfect matchings containing all the three edges of $E(A,B)$, i.e., at least two matchings are not counted in (10). Then the number of perfect matchings of G is at least $3n/4+|A|/4+3/4-10 > 3n/4-9$. □

Let G and H be two disjoint cubic graphs, u a vertex of G incident with three edges e_1, e_2, e_3, and v a vertex of H incident with three edges f_1, f_2, f_3. Consider the graph obtained from the union of $G \backslash u$ and $H \backslash v$ by adding an edge between the end-vertices of e_i and f_i ($1 \leq i \leq 3$) distinct from u and v. We say that this graph is obtained by *gluing* G and H through u and v. Note that gluing a graph G and K_4 through a vertex v of G is the same as replacing v by a triangle.

In the next lemma, we characterize the graphs that do not contain nice 3-edge-cuts.

Lemma 22. *Let G be a 3-edge-connected cubic graph that is not cyclically 4-edge-connected and that has no nice 3-edge-cut. If G is neither a klee-graph nor bipartite, then G must be of one of the following forms:*

(1) G can be obtained from a cubic brace H by gluing klee-graphs on 4, 6 or 8 vertices through some of the vertices of one of the two color classes of H;

(2) G has no tight edge-cuts and can be obtained from a cyclically 4-edge-connected cubic graph by replacing some of its vertices with triangles.

Proof. We assume that G is neither a klee-graph nor a bipartite graph and distinguish two cases depending whether G has a tight edge-cut or not.

If G has a tight edge-cut, then its brick and brace decomposition is non-trivial. Every non-trivial brick and brace decomposition of a cubic bridgeless graph contains a brace (see [6]). If the brick and brace decomposition of G contains two or more braces, then G has a tight 3-edge-cut $E(A, B)$ such that neither G/A nor G/B is a brick (again, see [6]). In particular, neither G/A nor G/B is a klee-graph, and so $E(A, B)$ is a nice edge-cut, which violates the assumption of the lemma.

We conclude that the brick and brace decomposition of G contains a single brace H, and that for any tight edge-cut $E(A, B)$ of G, exactly one of the graphs G/A and G/B is a brick. Observe that all the bricks are glued through the vertices of the same color class of H. To see this, assume that for two vertices u and v in different color classes of H, and two bricks H_1 containing a vertex u' and H_2 containing a vertex v', G is obtained from H by gluing H_1 through u and u' and H_2 to v and v'. Let u_1, u_2, u_3 (resp. v_1, v_2, v_3) be the neighbors of u (resp. v) in H, and let u'_1, u'_2, u'_3 (resp. v'_1, v'_2, v'_3) be the neighbors of u' (resp. v') in H_1 (resp. H_2). By definition, both $\{u_i u'_i, 1 \leq i \leq 3\}$ and $\{v_i v'_i, 1 \leq i \leq 3\}$ are tight edge-cuts of G. Since H_1 and H_2 are bricks, $H_1 \backslash \{u'_1, u'_2\}$ and $H_2 \backslash \{v'_1, v'_2\}$ both have a perfect matching. Since H is a brace, $H \backslash \{u_1, u_2, v_1, v_2\}$ also has a perfect matching. These three matchings combine to a perfect matching of G containing all the edges $u_i u'_i$ and $v_i v'_i$ for $1 \leq i \leq 3$ which contradicts the fact that the two edge-cuts were tight.

As for every 3-edge-cut $E(A, B)$, G/A or G/B is a klee-graph, all bricks of G are klee-graphs. Since $E(A, B)$ is not nice, the "klee-graph" side of the cut has at most 8 vertices. Hence, all bricks of G are klee-graphs with 4, 6 or 8 vertices, and G is exactly of the first form described in the lemma.

It remains to consider the case that G has no tight 3-edge-cuts. Consider a 3-edge-cut $E(A, B)$ of G. Since G is not a klee-graph, G/A or G/B, say G/A, is not a klee-graph. Since G has no nice 3-edge-cut, $|A| = 3$ and so $G[A]$ is a triangle. Now observe that every 3-edge-cut in G/A corresponds to a 3-edge-cut in G, and hence, separates a triangle. So we can keep contracting the original triangles of G to obtain a cyclically 4-edge-connected graph (no new 3-edge-cut, and hence no triangle, will be created during the process). We have observed that G can be obtained from a cyclically 4-edge-connected cubic graph by replacing some of its vertices by triangles. □

Let G be a 3-edge-connected cubic graph that is not a klee-graph, such that every cyclic 3-edge-cut $E(A,B)$ of G separates a triangle (in other words $|A|=3$ or $|B|=3$). The *core* of G, denoted by $\mathcal{C}(G)$, is the graph obtained by contracting every triangle of G. Since all cyclic 3-edge-cuts of G separate triangles, the graph G can be obtained from its core by replacing some of its vertices with triangles.

Lemma 23. *Let G be a 3-edge-connected cubic graph different from K_4 with no nice 3-edge-cut. Assume G was obtained from a cyclically 4-edge-connected cubic graph by replacing some of its vertices (at least one) by triangles. In particular, G is not a klee-graph. If $\mathcal{C}(G)$ is not bipartite, then $\mathcal{C}(G)$ has a cyclic 4-edge-cut, and G has no tight cyclic 3-edge-cut.*

Proof. Let $H=\mathcal{C}(G)$ and let v be any vertex of H. By the assumption, H is not bipartite. If the graph H' obtained from H by removing v and its three neighbors has no perfect matching, then there exists $S'\subseteq V(H')$ such that $H\setminus S'$ has at least $|S'|+2$ odd components. Let S be the set S' enhanced with the three neighbors of v. Clearly, $H\setminus S$ has at least $|S|=|S'|+3$ odd components. Since H is cyclically 4-edge-connected, this implies that all the odd components of $H\setminus S$ are isolated vertices and H is bipartite which is impossible. Hence, H' has a perfect matching.

Let u be a vertex of H that is replaced by a triangle T in G and let U be the set containing u and its three neighbors u_1,u_2,u_3 in H. As proven in the previous paragraph, $H\setminus U$ contains a perfect matching and the cut separating the triangle T is not tight. Hence, no cyclic 3-edge-cut of G is tight.

We now show that H has a cyclic 4-edge-cut. If $H\setminus U$ contains two perfect matchings, then G has two perfect matchings containing all the three edges of the cut separating T. Since G has no nice 3-edge-cut, this is impossible, so by Kotzig's theorem the graph $H\setminus U$ has a bridge. Let $E(A,B)$ be the cut of $H\setminus U$, that corresponds to this bridge.

Since H is cyclically 4-edge-connected, the set $\{u_1,u_2,u_3\}$ is a stable set. If A is comprised of a single vertex, say $A=\{v\}$, then v has two common neighbors with u, say u_1 and u_2. In particular, H contains the cycle of length four uu_1vu_2 which is disjoint from B. If B induces a forest it is easy to see that $|B|=3$ and B induces a path of length two, which together with u_3 forms a cycle of length four. Otherwise, B has a cycle. In both cases, H has a cyclic edge-cut of size four. Since the case $|B|=1$ is symmetric, we can assume that both A and B contain at least two vertices. Since H is cyclically 4-edge-connected, the sizes of the cuts $E(A, B \cup U)$ and $E(A \cup U, B)$ are at least four. Since the number of edges between U and $A \cup B$ is six, there are three edges joining U and A and three edges joining U and B.

If $|A| \geq 3$, then $E(A, B \cup U)$ is a cyclic edge-cut of size four. If $|A| = 2$, then one of the two vertices of A has two common neighbors with u and H has a cycle of length four. Again, H has a cyclic edge-cut of size four. □

As mentioned in the introduction, Chudnovsky and Seymour [2] proved that planar cubic bridgeless graphs (and consequently, klee-graphs) have exponentially many perfect matchings. However, their bound is not too good for graphs with small number of vertices. In the next lemma, we use the inductive structure of klee-graphs to provide a better lower bound on their number of perfect matchings.

Lemma 24. *Every n-vertex klee-graph has at least $3n/4 - 6$ perfect matchings.*

Proof. If $n \leq 8$, then there is nothing to prove. Hence, we can focus on klee-graphs of order at least ten.

Let G be a klee-graph and v a vertex of G with neighbors v_1, v_2 and v_3. The *type* of v is the 4-tuple $(\omega; \mu_1, \mu_2, \mu_3)$ such that the graph $G \setminus \{v, v_1, v_2, v_3\}$ contains ω perfect matchings and the graph $G \setminus \{v, v_i\}$ contains μ_i perfect matchings for $1 \leq i \leq 3$. Observe that there are exactly three non-isomorphic klee-graphs of order ten; these graphs are depicted in Figure 3(a)–(c), where the label of each edge represents the number of perfect matchings containing that edge and the label of a vertex v is the number of perfect matchings in the graph obtained by removing v and its three neighbors. In particular, the type of a vertex v is formed by its label and the labels of the three incident edges.

Let v be a vertex of type $(\omega; \mu_1, \mu_2, \mu_3)$ in the klee-graph G. The vertex v is said to be an *A-vertex* if $\omega = 1$ and $\mu_i = 1$ for a single index $i \in \{1, 2, 3\}$; v is a *B-vertex* if $\omega = 1$ and $\mu_i > 1$ for every $i \in \{1, 2, 3\}$ and v is a *C-vertex* if $\omega > 1$ and $\mu_i = 1$ for exactly two indices $i \in \{1, 2, 3\}$. A vertex is *dangerous* if at least three of the values ω, μ_1, μ_2 and μ_3 are equal to one. A vertex v is *good* if it is neither a A-, B-, C-vertex nor a dangerous vertex. In the following, $G \triangle v$ denotes the graph obtained from G by replacing v with a triangle. The number of perfect matchings in G is denoted by $m(G)$.

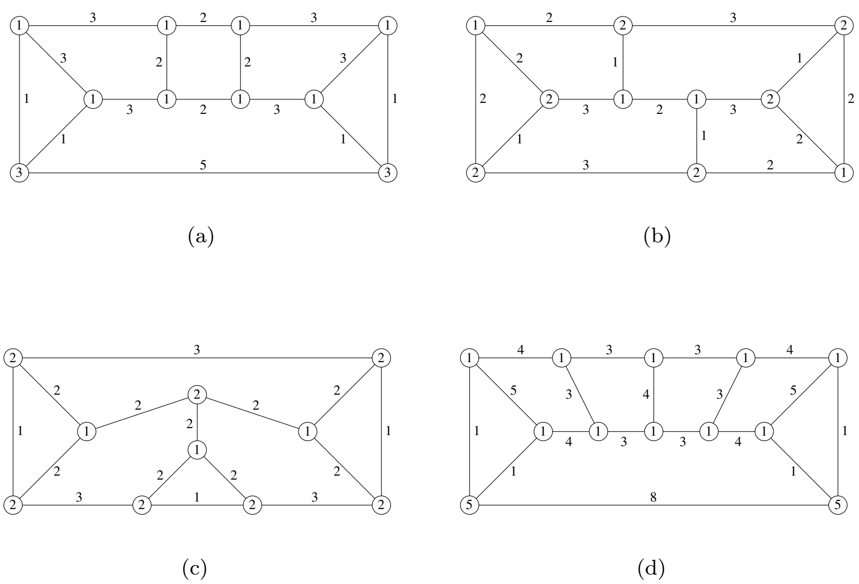

Figure 3: (a)–(c) The three non-isomorphic klee-graphs of order ten. (d) The only 12-vertex klee-graph that cannot be obtained by replacing a vertex by a triangle in (b) or (c).

Let G be a klee-graph and v a vertex of G of type $(\omega;\mu_1,\mu_2,\mu_3)$. As illustrated in Figure 4, the types of the three new vertices in $G\triangle v$ are

$$(\mu_1;\mu_1+\omega,\mu_2,\mu_3),\ (\mu_2;\mu_1,\mu_2+\omega,\mu_3), \text{ and } (\mu_3;\mu_1,\mu_2,\mu_3+\omega)\ .$$

In particular, $m(G\triangle v) = m(G)+\omega$. Finally, consider a vertex $v'\neq v$ and observe that if the type of v' in G is $(\omega';\mu'_1,\mu'_2,\mu'_3)$ and its type in $G\triangle v$ is $(\omega'';\mu''_1,\mu''_2,\mu''_3)$, then $\omega''\geq\omega'$ and $\mu''_i\geq\mu'_i$ for every $i\in\{1,2,3\}$. Hence, if v' is an A-vertex in G, it is an A-vertex, a B-vertex or a good vertex in $G\triangle v$. If v' is a B-vertex in G, it is a B-vertex or a good vertex in $G\triangle v$. If v' is a C-vertex in G, then it is a C-vertex or a good vertex in $G\triangle v$. Finally, if v' is a good vertex in G, it remains good in $G\triangle v$. This implies that a vertex is dangerous in $G\triangle v$ only if it was dangerous in G. Since no graph in Figure 3(a)–(c) contains a dangerous vertex, no klee-graph of order at least 12 contains a dangerous vertex.

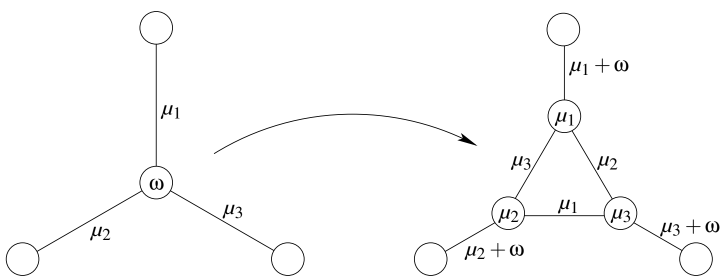

Figure 4: The types of the three new vertices in $G\triangle v$.

For any klee-graph G with α A-vertices and β B-vertices, let $M(G) = m(G)-\alpha-\beta/2$. The core of our proof is the following claim proven by induction on n.

Claim. *For any n-vertex klee-graph G, $n\geq 10$, distinct from the one in Figure 3(a), it holds $M(G)\geq 3n/4-6$.*

If $n=10$, then G is one of the graphs depicted in Figures 3(b) and 3(c), and

$$M(G) = \left\{\begin{array}{lcc} 6-2-2/2 & = & 3 \\ 6-0-3/2 & = & 4.5 \end{array}\right\} \geq 3\cdot 10/4-6\ .$$

The only 12-vertex klee-graph that cannot be obtained by replacing a vertex with a triangle in one of the graphs depicted in Figures 3(b) and 3(c) is the graph in Figure 3(d). For this graph, we have

$$M(G) = 10 - 4 - 6/2 = 3 \geq 3 \cdot 12/4 - 6 \; .$$

All other n-vertex klee-graphs G with $n \geq 12$ can be obtained by replacing a vertex v by a triangle $w_1w_2w_3$ in a klee-graph G' that satisfies the assumptions of the claim. Clearly, the number n' of vertices of G' is $n-2$. By the induction, we assume that $M(G') \geq 3n'/4 - 6$.

We now distinguish four cases based on the type of v; note that v cannot be dangerous as argued earlier. Observe that if an A- or B-vertex becomes good, or if an A-vertex becomes a B-vertex, then $-\alpha - \beta/2$ increases. So we can assume without loss of generality that every A-vertex and B-vertex distinct from v remains an A-vertex and B-vertex, respectively.

- *v is an A-vertex:* Since v is an A-vertex, $m(G) = m(G') + 1$. One of the vertices w_1, w_2 and w_3 is a B-vertex, and the other two vertices are good. Hence, α decreases by 1 and β increases by 1, and so $-\alpha - \beta/2$ increases by 1/2. We conclude that

$$M(G) \;=\; M(G') + 1 + 1/2 \;\geq\; 3n'/4 - 6 + 3/2 \;=\; 3n/4 - 6 \; .$$

- *v is a B-vertex:* Since v is a B-vertex, it holds that $m(G) = m(G') + 1$. All the vertices w_1, w_2 and w_3 are good, so β decreases by one and $-\alpha - \beta/2$ increases by 1/2. We conclude that

$$M(G) \;=\; M(G') + 1 + 1/2 \;\geq\; 3n'/4 - 6 + 3/2 \;=\; 3n/4 - 6 \; .$$

- *v is a C-vertex:* It is easy to see that in any klee-graph of order at least 12, any C-vertex has type $(\omega, \mu, 1, 1)$, where both ω and μ are at least five. Hence it holds that $m(G) \geq m(G') + 5$. Two vertices among w_1, w_2 and w_3 are A-vertices and the last one is a C-vertex. Hence, $-\alpha - \beta/2$ decreases by two. We again conclude that

$$M(G) \;\geq\; M(G') + 5 - 2 \;\geq\; 3n'/4 - 6 + 3 \;\geq\; 3n/4 - 6 \; .$$

- *v is good:* At most one of the vertices w_1, w_2 and w_3 is a B-vertex and the remaining vertices are good. Hence, $-\alpha - \beta/2$ decreases by at most 1/2. Since $m(G) \geq m(G') + 2$, it holds that

$$M(G) \;\geq\; M(G') + 2 - 1/2 \;\geq\; 3n'/4 - 6 + 3/2 \;=\; 3n/4 - 6 \; .$$

$n = 2k$	6	8	10	12	14	16
$g(k)$	4	6	8	11	15	20
$f(k)$	6	9	12	17	23	30
$3n/2 - 9$	0	3	6	9	12	15

Table 1: The minimum number $f(k)$ of distinct perfect matchings of a cubic bipartite with $2k$ vertices and the claimed bound $3n/2 - 9$.

This finishes the proof of the claim.

We have shown that $M(G) \geq 3n/4 - 6$ for every n-vertex klee-graph G with $n \geq 10$ distinct from the graph in Figure 3(a) which has $7 \geq 3 \cdot 10/4 - 6$ perfect matchings. In particular, the number of perfect matchings of any n-vertex klee-graph is at least $3n/4 - 6$. □

As mentioned in the introduction, cubic bridgeless bipartite graphs are known to have an exponential number of perfect matchings. We can derive the following more modest result, which will be sufficient for our purpose.

Lemma 25. *Every n-vertex cubic bipartite graph has at least $3n/2 - 9$ perfect matchings.*

Proof. Let $g(3) = 4$, and set $g(k) = \lceil 4g(k-1)/3 \rceil$ for any $k \geq 4$. Also, let $f(k) = \lceil 3g(k)/2 \rceil$. It can be shown that every cubic bridgeless graph with $2k$ vertices has at least $f(k)$ perfect matchings, see [6, 8]. The values of $f(k)$ for small k can be found in Table 1. If $n \leq 12$, the statement of the lemma holds by inspecting the values of $f(k)$. For $k = 7$, $g(k) \geq 2k$. Using the definition of $g(k)$, an easy argument by induction on k shows that $g(k) \geq 2k$ for all $k \geq 7$. Hence, $f(k) \geq 3g(k)/2 \geq 3k = 3n/2$ and the statement of the lemma follows. □

We are know ready to prove the main result of this section.

Theorem 26. *Every n-vertex 3-edge-connected cubic graph has at least $3n/4 - 9$ perfect matchings.*

Proof. The proof proceeds by induction on the order n of G. If $n \leq 12$, then there is nothing to prove since the bound claimed in the theorem is negative. Fix $n \geq 14$, and assume that we have proven the statement of the theorem for all $n' < n$. If G is cyclically 4-edge-connected, then G has at least $3n/4 - 9$ perfect matchings by Theorem 18. If G has a nice cyclic 3-edge-cut, then G has at least $3n/4-9$ perfect matchings using Lemma 21. If G is a klee-graph or a bipartite graph, Lemmas 24 and 25 yield the desired lower bound on the number of perfect matchings of G. Otherwise, G is of one of the two forms given in Lemma 22. We deal with each of these cases separately:

- *G can be obtained from a cubic brace H by gluing klee-graphs on 4, 6 or 8 vertices through some of the vertices of one of the two color classes of G:* Let N be the order of H. The number of perfect matchings of H is at least $3N/2 - 9$ by Lemma 25 and H is matching double-covered by Lemma 19. Let N_k be the number of vertices of H through which a klee-graph of order $k \in \{4, 6, 8\}$ is glued. Observe that

  $$N_4 + N_6 + N_8 \leq N/2 \text{ and } n = N + 2N_4 + 4N_6 + 6N_8 \; .$$

  Let us estimate the number of perfect matchings of G in more detail. We count in how many ways perfect matchings of H extend to the glued klee-graphs. There is a unique extension of each perfect matching of H to a glued klee-graph of order 4. Since the edges incident with every vertex of a klee-graph of order six are contained in 1, 1 and 2 perfect matchings respectively and H is matching double-covered, at least two perfect matchings extend to a glued klee-graph of order six in two different ways. Hence, any such gluing increases the number of perfect matchings by at least two. Similarly, the edges incident with every vertex of a klee-graph of order eight are contained in 1, 1 and 3 or 1, 2 and 2 perfect matchings which implies that at least two matchings of H extend to a glued klee-graph of order eight in three different ways or at least four matchings of H extend in two different ways. In both cases, the number of perfect matchings is increased by four.

  Using Lemma 25, we conclude that the number of perfect matchings of G is at least

  $$\begin{aligned} \tfrac{3}{2}N - 9 + 2N_6 + 4N_8 \;&\geq\; \tfrac{3}{4}N + 3\,(N_4 + N_6 + N_8)/2 + 2N_6 + 4N_8 - 9 \\ &\geq\; 3n/4 - 9 \; , \end{aligned}$$

  as desired.

- *G has no tight edge-cuts and it can be obtained from a cyclically 4-edge-connected cubic graph by replacing some of its vertices with triangles:*

If $H = \mathcal{C}(G)$ has a cyclic 4-edge-cut, Theorem 18 yields the desired bound. If H has no cyclic 4-edge-cut, then H is a bipartite cyclically 5-edge-connected cubic graph by Lemma 23. By Proposition 9, H is a brace. In particular, it is possible to remove two vertices from each of the two colors classes of H and the graph still has a perfect matching.

Let N be the number of vertices of H and N_i, $i = 1, 2$, be the number of vertices of each of the two color classes of H that are replaced by triangles in G. Observe that $n = N + 2N_1 + 2N_2$, $N_1 \leq N/2$ and $N_2 \leq N/2$. We can assume without loss of generality that $1 \leq N_1 \leq N_2$, since otherwise this would bring us to the previous case (replacing a vertex v by a triangle is the same as gluing a K_4 through v).

By Lemma 25, H has at least $3N/2 - 9$ perfect matchings and each of these matchings corresponds to a perfect matching of G which contains only one edge of each 3-edge-cut separating a triangle. Now, take two vertices u, v in different color classes of H, such that u and v are replaced by two triangles T_u and T_v in G. Let H' be the graph obtained from H by removing two neighbors of u and two neighbors of v. Since H is a brace, H' has a perfect matching. This perfect matching corresponds to a perfect matching of G containing the three edges leaving T_u, the three edges leaving T_v, and only one edge of each 3-edge-cut separating a different triangle. Hence, G contains at least $3N/2 - 9 + N_1N_2$ perfect matchings.

Since $n = N + 2N_1 + 2N_2$, proving that G has at least $3n/4 - 9$ perfect matchings is equivalent to proving that $N_1 + N_2 \leq \frac{N}{2} + \frac{2}{3}N_1N_2$. If $N_1 = 1$ then

$$N_1 + N_2 = N_2/3 + 1 + \tfrac{2}{3}N_1N_2 \leq N/2 + \tfrac{2}{3}N_1N_2$$

since $N \geq \lceil n/3 \rceil \geq 5$. On the other hand, if $N_1 \geq 2$ then

$$N_1 + N_2 \leq N/2 + (N_1 + N_2)/2 \leq N/2 + N_1N_2/2 \,.$$

This finishes the proof of Theorem 26. □

6 Bridgeless graphs

In this section, we prove our main result on the number of perfect matchings of cubic bridgeless graphs. Before we do so, we need an auxiliary lemma:

Lemma 27. *Let G be a cubic bridgeless graph with a 2-edge-cut. For every edge e of G, there are at least three perfect matchings avoiding e.*

Proof. Let $E(A, B)$ be an edge-cut of G of size two and let G^A and G^B be the cubic bridgeless graphs obtained from $G[A]$ and $G[B]$ by joining the two vertices of degree two with an edge. The added edges are denoted by e^A and e^B. If $e \in E(A, B)$, then G has at least four perfect matchings avoiding e as any of at least two perfect matchings of G^A avoiding e^A combines with any of at least two perfect matchings of G^B avoiding e^B to a perfect matching of G avoiding e.

We now assume that $e \notin E(A, B)$. By symmetry, let e be in $G[A]$. Recall that in a cubic bridgeless graph, it is possible to find a perfect matching avoiding any two given edges. Thus, the graph G^A contains at least two perfect matchings avoiding e and at least one such matching also avoids e^A. Any perfect matching of G^A avoiding both e and e^A can be extended to B in two different ways and any perfect matching of G^A avoiding e and containing e^A can be extended to B in at least one way. Altogether, G contains at least three perfect matchings avoiding e as desired. □

We are now ready to prove the main result:

Theorem 28. *Every cubic bridgeless graph G with n vertices has at least $3n/4 - 10$ perfect matchings.*

Proof. The proof proceeds by induction on the number of vertices of G. If G is 3-edge-connected, the bound follows from Theorem 26. Otherwise, take a 2-edge-cut $E(A, B)$ of G such that A is minimal with respect to inclusion. Let G^A and G^B be the cubic bridgeless graph obtained from $G[A]$ and $G[B]$ by adding edges e^A and e^B between the two vertices of degree two. Clearly, G^A is 3-edge-connected and contains at least $3|A|/4 - 9$ perfect matchings by Theorem 26. Also note that G^A contains at least two perfect matchings avoiding e^A, and similarly G^B contains at least two perfect matchings avoiding e^B.

Suppose first that the edge e^A is contained in two perfect matchings. Fix two perfect matchings of G^A containing e^A and two perfect matchings avoiding e^A. Each of $|B|/2$ perfect matchings of G^B can be extended to $G[A]$ in at least two different ways using the fixed matchings (note that, by Theorem 3, if $|B| \neq 12$, G^B has at least $|B|/2 + 1$ perfect matchings and if $|B| = 2$, G^B has $|B|/2 + 2 = 3$ perfect matchings). On the other hand, every of at least $3|A|/4 - 9 - 4 = 3|A|/4 - 13$ perfect matchings of

G^A distinct from the fixed ones can be extended to $G[B]$. Hence, unless $|B| = 2$ or $|B| = 12$ the number of perfect matchings of G is at least

$$3|A|/4 - 13 + 2 \cdot (|B|/2 + 1) \;=\; 3n/4 + |B|/4 - 11 \;\geq\; 3n/4 - 10 \;.$$

If $|B| = 2$, the number of perfect matchings of G is at least

$$3|A|/4 - 13 + 2 \cdot (|B|/2 + 2) \geq 3n/4 - 9 \;,$$

and if $|B| = 12$, the number of perfect matchings of G is at least

$$3|A|/4 - 13 + 2 \cdot |B|/2 \;=\; 3n/4 + |B|/4 - 13 \;=\; 3n/4 - 10 \;.$$

Suppose now that G^A has a single matching containing the edge e^A. We distinguish two cases regarding whether G^B is 3-edge-connected. If G^B is 3-edge-connected and e^B is contained in at least two perfect matchings, then we apply the same arguments as in the previous paragraph and the result follows. Hence, we can assume that e^B is contained in a single perfect matching of G^B. Consequently, by Theorem 3 there are at least $|A|/2 - 1$ perfect matchings of G^A avoiding e^A and at least $|B|/2 - 1$ perfect matchings of G^B avoiding e^B. Fix two matchings of G^A that avoid e^A and two matchings of G^B that avoid e^B, and call these four matchings *canonical*. Every non-canonical matching of G^A avoiding e^A combines with a canonical matching of G^B avoiding e^B, and vice-versa. Hence, the number of perfect matchings of G is at least

$$2(|A|/2 - 3) + 2(|B|/2 - 3) + 2 \cdot 2 \;=\; n - 8 \;\geq\; 3n/4 - 9 \;.$$

The only remaining case is when G^B is not 3-edge-connected and the edge e^A is contained in a single matching of G^A. By Lemma 27, G^B has at least three matchings avoiding e^B. Fix one matching of G^A containing e^A, one matching of G^A avoiding e^A and three matchings of G^B avoiding e^B. Again, we call these five perfect matchings *canonical*. By induction, G^B has at least $3|B|/4 - 10$ perfect matchings, each of which can be combined with a canonical perfect matching of G^A to form a perfect matching of G. Since e^A is contained in a single matching of G^A, there exist at least $|A|/2 - 2$ matchings of G^A (distinct from the canonical ones) avoiding e^A. Each of them can be combined with one of the three canonical matchings of G^B to form a perfect matching of G. Note that $|A|/2 - 2 \geq |A|/4$ if $|A| \geq 8$. If $|A| \in \{4, 6\}$, then by Theorem 3, G^A has at least $|A|/2 - 1$ matchings distinct from the two canonical ones, and again $|A|/2 - 1 \geq |A|/4$. Finally, if $|A| = 2$, then G^A has $|A|/2 = 1$ perfect matching distinct from the two canonical ones. In all cases, G^A has at least $|A|/4$ perfect matchings distinct from the two canonical matchings of G^A. We conclude that the number of perfect matchings of G is at least

$$3 \cdot |A|/4 + 3|B|/4 - 10 \ = \ 3n/4 - 10 \ .$$

This finishes the proof of the theorem. □

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