ENUMERATION WORKSHOP

(Patejdlova bouda, Špindlerův Mlýn, November 17–23, 2007)

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Contents

Introduction

We decided to organize a workshop on enumeration with the aim to put together a small group of researchers to exchange information on different broad aspects of enumeration and its relations with graph theory and statistical physics. The workshop was held in Patejdlova bouda, a small but comfortable lodge located high up in the Krkonoše mountains above Špindlerův Mlýn resort. The lodge has the Internet, table-tennis, sauna, and a small ski lift: indeed, several participants started to ski during the week.

The idea from the beginning was to devote every day to a different topic, and to have two 90 minutes lectures (blocks of lectures) every day, one in the morning and the other in the late afternoon. The rest of the day was not planned.

Another key aspect was that thanks to generous funding from the Department of Applied Mathematics, Charles University, Prague, and from the Institute of Theoretical Computer Science, Prague, we could cover all the local expenses of the participants.

Speaking for us (the organizers), we liked the workshop extremely, both mathematically and socially. A sketch of the math activities during the workshop is presented in the following pages.

Martin Klazar and Martin Loebl

Chapter 1

Delia Garijo, Jaroslav Nešetřil and M. Pastora Revuelta: Homomorphisms and Polynomial Invariants of Graphs

This is a short version of the paper *Homomorphisms and polynomial invariants of graphs* submitted to the European Journal on Combinatorics, 2007.

Acknowledgments. We are indebted to Andrew Goodall for carefully reviewing the manuscript and making many valuable suggestions.

1.1 Introduction

Counting homomorphisms between graphs arise in many different areas including extremal graph theory, partition functions in statistical physics and property testing of large graphs. Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a *homomorphism of G to H*, written as $f : G \to H$, is a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. The number of homomorphisms of G to H is denoted by $\hom(G, H)$. This number, considered as a function of G with H fixed is a *graph parameter*, that is, a function of graphs invariant under isomorphisms. A more broader class of parameters related to homomorphisms has been recently intensively studied in the context of statistical physics, see for example [5].

The motivation of this paper is to show the usefulness of the homomorphism perspective in the study of polynomial invariants of graphs. Thus, our main contribution is to prove that there exists a strong connection between counting graph homomorphisms and evaluating polynomials associated with graphs. The importance of this approach lies on its applicability. For instance, it can put in a new context some well-known problems such as the uniqueness questions formulated by Bollobás, Peabody and Riordan in [3].

One of the most studied polynomial invariants in combinatorics is the *Tutte polynomial*, or *dichromate* of [26]. This is an isomorphism-invariant function from the set of finite multigraphs with loops allowed to $\mathbb{Z}[x, y]$ which can be defined in several ways, see for instance [4, 6, 26]. Throughout this paper, we shall consider its contraction-deletion formulae. Thus, the Tutte polynomial of a finite graph G, denoted by $T(G; x, y)$, can be defined by the following recurrence relations:

1. If G has no edges then $T(G; x, y) = 1$.
2. $T(G; x, y) = T(G - e; x, y) + T(G/e; x, y)$ provided that $e \in E(G)$ is neither a loop nor a bridge, $G - e$ and G/e denote, respectively, the result of deleting and contracting the edge e in G.
3. $T(G; x, y) = yT(G - e; x, y)$ whenever $e \in E(G)$ is a loop.
4. $T(G; x, y) = xT(G/e; x, y)$ whenever $e \in E(G)$ is a bridge.

It is well-known that homomorphisms of a graph G to the complete graph K_n are just the $n-$colorings of G (see [12]). In [14], Joyce showed that the number of homomorphisms of any graph G to a complete graph with non-multiple edges and p loops at each vertex, is an evaluation of the coboundary polynomial of G. This polynomial was first defined in [27] as a generalization of the chromatic polynomial. Since the Tutte polynomial can be regarded as an extension of the chromatic and the coboundary polynomials, a natural question arises: can we find other graphs H such that the number of homomorphisms of any graph G to H is given (up to a determined term) by an evaluation of the Tutte polynomial of G?

This paper contains two main results. We first prove that every complete graph with p loops at each vertex and constant multiplicity q at the non-loop edges can play the role of H, whenever p is different than q. As well as the Tutte polynomial is an extension of the chromatic and the coboundary polynomials, this complete graph which we denote by $K_n^{p,q}$, is a natural extension of both the complete graph K_n and the Joyce graph $K_n^{p,1}$. Figure 1.1 shows four instances of this family of graphs.

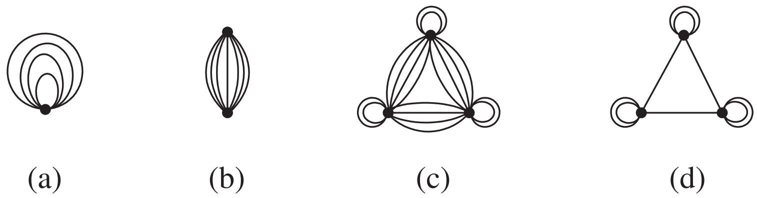

Figure 1.1: (a) $K_1^{4,0}$, (b) $K_2^{0,7}$, (c) $K_3^{2,4}$, (d) $K_3^{2,1}$.

Our second main result is the characterization, by assuming a local condition, of those graphs H such that the parameter hom($\_\_$, H) can be recovered from the Tutte polynomial. We prove that such graphs are necessarily isomorphic to graphs of the family $K_n^{p,q}$. The local condition is not too restrictive since it is satisfied by all the multiplicative invariants of graphs that can be deduced from the Tutte polynomial. This fact underlines the importance of the results proved in this paper.

The Tutte polynomial extends not only the chromatic and the coboundary polynomials but also, among others, the *flow*, the *boundary*, the *transition* and the *circuit partition* polynomials. Thus, our characterization leads to important connections between the homomorphism counting and these polynomials, which have a special role in the field of graph theory. Indeed, the boundary polynomial was introduced in [27] as a generalization of the flow polynomial, and along with the coboundary polynomial, has been recently used to obtain new evaluations of the Tutte polynomial at some points on the hyperbolae $H_\alpha = \{(x, y)|(x-1)(y-1) = \alpha\}$ for $\alpha \in \mathbb{N}$ (see [11]). The transition polynomial arose in [13] as a tool for summarizing and generalizing a number of results obtained by Martin [22, 23] and Las

Vergnas [18, 19, 20]. It has many interesting applications in knot theory, see for example [15]. The circuit partition polynomial was first defined in [7], and was so named in [2]. This polynomial is a simple transform of the original *Martin polynomial*, which was developed by Martin in [22] to study families of cycles in $4-$regular Eulerian graphs. Furthermore, the circuit partition polynomial has surprising applications to many areas including infrastructure networks and reconstruction of DNA sequences, see for instance [1].

As an application, we use our characterization to describe in terms of the homomorphism counting some important evaluations of the Tutte polynomial in abelian groups and statistical physics. Specifically, we sketch applications to difference sets in abelian groups, the Potts model, and the random cluster model in statistical mechanics.

We shall conclude the paper by introducing a new type of uniqueness of graphs related to the homomorphism counting, which we call *coloring uniqueness*, and by showing its relation with *Tutte uniqueness* and *chromatic uniqueness*.

1.2 Graph homomorphisms and the Tutte polynomial

In this section we establish a connection between counting graph homomorphisms and evaluating the Tutte polynomial.

We first state some notation that will be used throughout this paper. The graphs considered are finite and not necessarily simple. Thus, let us denote by Ω the set of finite multigraphs with loops allowed. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$. An edge $e \in E(G)$ can be either a loop uu or a non-loop edge uv with $u \neq v$. The multiplicity of a non-loop edge $e \in E(G)$ is written as $m(e)$. The set of homomorphisms of a graph G to a graph H is denoted by $\mathrm{Hom}(G, H)$, and its order is $\hom(G, H)$. Given $f \in \mathrm{Hom}(G, H)$ and $e = uv \in E(G)$, we write $f(e) = f(u)f(v) \in E(H)$. For a fixed $H \in \Omega$, a constant x that depends on H is written as x_H. The graphs $K_n^{p,q}$ defined in the previous section verify the following conditions: $p, q, n \in \mathbb{N}$, $n \geq 1$ and $p, q \geq 0$. When $n = 1$, we consider $q = 0$ and $p > 0$.

One of the most important properties of the Tutte polynomial is the existence of a contraction-deletion formula. In fact, this polynomial is an example of *Tutte-Grothendieck invariant* [6, 27], that is, a function f from the set of graphs to a fixed commutative ring satisfying the following:

• Contraction-Deletion Formula: $f(G) = f(G - e) + f(G/e)$ when G is connected and e is neither a loop nor a bridge.

• Multiplicity: the invariant of a graph is the product of the invariants of its connected components.

• Isomorphism Invariance: the invariants of two isomorphic graphs are the same.

In [6], Brylawski and Oxley showed that every Tutte-Grothendieck invariant is essentially an evaluation of the Tutte polynomial.

Theorem 1.2.1. [6] *Let f be any function from the set of graphs to a fixed commutative ring $\mathbb{Z}[x_0, y_0, n, a, b]$ which is multiplicative and isomorphism invariant. Further, let f verify the following recurrence relations:*

- *$f(G) = n^\lambda$ if G has no edges and λ vertices.*
- *$f(G) = af(G - e) + bf(G/e)$ provided that $e \in E(G)$ is neither a loop nor a bridge.*
- *$f(G) = x_0 f(G/e)$ whenever $e \in E(G)$ is a bridge.*
- *$f(G) = y_0 f(G - e)$ whenever $e \in E(G)$ is a loop.*

Then $f(G) = n^c a^{m-\lambda+c} b^{\lambda-c} T(G; \frac{x_0}{b}, \frac{y_0}{a})$ where G is a graph with λ vertices, m edges and c connected components.

Observe that the parameter $\hom(\_\_, K_n^{p,q})$ is multiplicative, and

$$\hom(G, K_n^{p,q}) = n^\lambda$$

if G is the graph with λ vertices and no edges. Thus, the following result is the key tool to relate this parameter to the Tutte polynomial. It defines $\hom(\_\_, K_n^{p,q})$ in terms of a contraction-deletion formulae.

Lemma 1.2.2. *The number of homomorphisms of any graph G to $K_n^{p,q}$ satisfies the following recurrence relations:*

(1) $\hom(G, K_n^{p,q}) = q \hom(G - e, K_n^{p,q}) + (p - q) \hom(G/e, K_n^{p,q})$ *provided that $e \in E(G)$ is neither a loop nor a bridge, $G - e$ and G/e denote, respectively, the result of deleting and contracting the edge e in G.*

(2) $\hom(G, K_n^{p,q}) = p \hom(G - e, K_n^{p,q})$ *whenever $e \in E(G)$ is a loop.*

(3) $\hom(G, K_n^{p,q}) = (p + q(n-1)) \hom(G/e, K_n^{p,q})$ *whenever* $e \in E(G)$ *is a bridge.*

Theorem 1.2.1 and Lemma 1.2.2 imply the following relationship for all $n \geq 1$ (for $n = 1$, the proof follows by induction on m and by only using Lemma 1.2.2).

Theorem 1.2.3. *For every graph G with λ vertices, m edges and c connected components, the following holds:*

1. $\hom(G, K_n^{p,q}) = n^c (p-q)^{\lambda - c} q^{m-\lambda+c} T\left(G; \frac{p+q(n-1)}{p-q}, \frac{p}{q}\right)$ *with* $q \geq 1$ *and* $p \neq q$.
2. $\hom(G, K_1^{p,0}) = (p/2)^m T(G; 2, 2)$ *with* $p > 0$.

Our next aim is to characterize the graphs H such that the parameter $\hom(\_\_, H)$ satisfies a contraction-deletion formula.

Definition 1.2.4. *Given $H \in \Omega$, a function $h_H : \Omega \to \mathbb{Q}$ depending on H is called local if for every graph $G \in \Omega$, and every type of edge $e \in E(G)$ (distinguishing between loops, bridges and neither of them) the fractions $h_H(G)/h_H(G-e)$ and $h_H(G)/h_H(G/e)$ do not depend on the choice of G and e.*

Theorem 1.2.5. *For every connected graph $H \in \Omega$, the following statements are equivalent:*

(1) *There exist two rational numbers x_H and y_H, and a local function h_H such that for every graph G,* $\hom(G, H) = h_H(G) T(G; x_H, y_H)$.

(2) *There exist $p, q, n \in \mathbb{N}$ with $p \neq q$ such that $H \cong K_n^{p,q}$.*

We conclude this section by establishing a relationship for homomorphisms of dual graphs.

Proposition 1.2.6. *Let G be a planar graph with λ vertices, m edges and c connected components, and G^* be its dual graph. Then the following holds:*

1. $\hom(G, K_n^{p,q}) = \left(\frac{p-q}{q}\right)^m n^{\lambda-m-1} \hom\left(G^*, K_n^{q+\frac{q^2 n}{p-q}, q}\right)$ *with* $n > 1$ *and* $q + \frac{q^2 n}{p-q} \in \mathbb{N}$.
2. $\hom(G, K_1^{p,0}) = \hom(G^*, K_1^{p,0})$ *with* $p > 0$.

We want to stress that the conditions $xq = q + \frac{q^2 n}{p-q} \in \mathbb{N}$ and $n > 1$ imply $q < p$ and $(x-1)(y-1) > 1$. Therefore the previous result provides a connection between the number of homomorphisms of G and G^* to $K_n^{yq,q}$ and $K_n^{xq,q}$ respectively, for an infinite number of points over the hyperbolae $(x-1)(y-1) = n$.

1.3 Homomorphisms and other polynomial invariants of graphs

There are many polynomial invariants that can be recovered from the Tutte polynomial. Among those, there are some (but not all) that can be related to the homomorphism counting. In this section, we establish the connections between the parameter $\hom(\_\_, H)$ and the *boundary*, the *coboundary*, the *transition*, and the *circuit-partition* polynomials. We start by considering the boundary and the coboundary polynomials which have a special role in the theory of the Tutte polynomial, see for example [11, 27].

Let G be a graph and ω a fixed orientation of its edges. For every $v \in V(G)$, we can divide the edges incident with v according to the orientation ω into two sets, $\omega^+(v)$ and $\omega^-(v)$, that is the edges directed *into* the vertex and the edges directed *out of* the vertex.

Given an abelian group A of order r, a function $f : E(G) \to A$ is called an $A-$*flow* of G with orientation ω if for each vertex $v \in V(G)$,

$$\sum_{e \in \omega^+(v)} f(e) = \sum_{e \in \omega^-(v)} f(e).$$

In particular, a *nowhere zero* $A-$*flow* is a $A \backslash \{0\}-$flow.

Let us denote by $\Theta_A(G)$ the set of $A-$flows of G. The *boundary* polynomial, or *bad flow* polynomial of [27], is defined as follows.

$$F(G; r, x) = \sum_{f \in \Theta_A(G)} x^{|f^{-1}(0)|}$$

where $|f^{-1}(0)|$ is the number of zero-edges in the $A-$flow f. Clearly this polynomial is an extension of the flow polynomial since it considers not only nowhere-zero $A-$flows of a graph G, but also $A-$flows in which there are i zero-edges with $1 \leq i \leq |E(G)|$. Thus, $F(G; r, 0)$ is the flow polynomial of G.

Similarly, the *coboundary* polynomial, or *monochrome* polynomial of [27], is defined for any abelian group A of order r by

$$P(G;r,y) = \sum_{g\in\mathcal{C}_r(G)} y^{|\Gamma_g(G)|}$$

where $\mathcal{C}_r(G)$ is the set of vertex $r-$colorings of G, and $\Gamma_g(G)$ is the set of *monochrome edges* in a given $g \in \mathcal{C}_r(G)$, that is, the edges which have endpoints of the same color. Since the chromatic polynomial only considers proper vertex $r-$colorings of a graph, it is clear that $P(G;r,0)$ is the chromatic polynomial of G.

The following relationships define the boundary and the coboundary polynomials as evaluations of the Tutte polynomial, up to local functions.

Theorem 1.3.1. [27] *For any graph G with λ vertices, m edges and c connected components the following holds:*

(1) $F(G;r,x) = (x-1)^{m-\lambda+c}T\left(G;x,\frac{x-1+r}{x-1}\right)$.

(2) $P(G;r,y) = r^c(y-1)^{\lambda-c}T\left(G;\frac{y-1+r}{y-1},y\right)$.

We now relate these polynomials to the parameter $\hom(\_\_, H)$.

Proposition 1.3.2. *For every graph G with λ vertices, m edges and c connected components, the following holds:*

(1) $\hom(G,K_n^{p,q}) = n^{\lambda-m}(p-q)^m F\left(G;n,\frac{p+q(n-1)}{p-q}\right)$ *with* $n>1$ *and* $p\neq q$.

(2) $\hom(G,K_n^{p,q}) = q^m P(G;n,p/q)$ *with* $n>1$ *and* $p\neq q$.

Theorem 1.3.3. *For every connected graph $H \in \Omega$, the two following statements are equivalent:*

(1) There exist a rational number x_H, a positive integer number $r_H > 1$, and a local function h_H satisfying that for every graph G, $\hom(G,H) = h_H(G)F(G;r_H,x_H)$.

(2) There exist $p,q,n \in \mathbb{N}$ with $p \neq q$ such that $H \cong K_n^{p,q}$.

Observe that Theorem 1.3.1 and the connection between the parameter hom($\_\_$, H) and the Tutte polynomial, enable us to state and prove a similar characterization for the coboundary polynomial.

Remark 1. The transition polynomial was first defined in [13] on 4−regular planar graphs in terms of a weight function A. One can find many interesting applications of this polynomial mostly in knot theory, see for instance [15]. The Tutte polynomial of a connected planar graph G with set of faces $R(G)$ is related to the transition polynomial $Q(M(G); A, \tau)$ of its medial graph $M(G)$. This relationship is proved in [13] for special values of μ and δ, and a special weight function A. It is then given by the following expression,

$$Q(M(G); A, \tau) = \delta^{1-|V(G)|}\mu^{1-|R(G)|}T\left(G; 1 + \tfrac{\delta\tau}{\mu}, 1 + \tfrac{\mu\tau}{\delta}\right).$$

The *medial graph* $M(G)$ is the planar connected 4−regular graph obtained from G as follows: The vertices of $M(G)$ correspond to the edges of G and two vertices of $M(G)$ are joined by an edge if the corresponding edges of G are neighbors in the cyclic order around the vertex (see Figure 1.2). We do not go into more details and just state the connection between the transition polynomial and the parameter hom($\_\_$, H). This connection is given by the two following results.

Proposition 1.3.4. *Let G be a connected planar graph with λ vertices and m edges. The number of homomorphisms of G to $K_n^{p,q}$ with $p \neq q$ and $n > 1$ is given by*

$$\begin{array}{lcll} \hom(G, K_n^{p,q}) & = & n^{m-\lambda+1}(p-q)^m\delta^m Q(M(G), A, \sqrt{n}) & \textit{if} \quad p - q \neq q\sqrt{n}\,. \\ \hom(G, K_n^{p,q}) & = & (\sqrt{n})^{\lambda+1}q^m\delta^m Q(M(G), A, \sqrt{n}) & \textit{if} \quad p - q = q\sqrt{n}\,. \end{array}$$

Theorem 1.3.5. *For every connected graph $H \in \Omega$, the following statements are equivalent:*

1. *There exist a constant τ_H, and a local function h_H such that for every connected planar graph G,* $\hom(G, H) = h_H(G)Q(M(G); A, \tau_H)$.
2. *There exist $p, q, n \in \mathbb{N}$ with $p \neq q$ and $n > 1$ such that $H \cong K_n^{p,q}$.*

Remark 2. We obtain similar results for the circuit partition polynomial which was first defined in [7] as a generating function for the number of Eulerian partitions of an Eulerian graph or digraph into s components.

This polynomial is a generalization, for a specific weight function, of one of Jaeger's transition polynomials (see [7]). It has many applications to several areas, including non-mathematical fields (see for instance [1, 2]).

The Tutte polynomial of a planar graph G with c connected components is related to the circuit partition polynomial of its directed medial graph $\overrightarrow{M(G)}$ (see [22, 23]). This relationship is given by the following expression,

$$j(\overrightarrow{M(G)}; x) = x^c T(G; x+1, y+1)$$

The *directed medial graph* $\overrightarrow{M(G)}$ results from directing the edges of $M(G)$ as follows. We first color the faces of $M(G)$ black or white, depending on whether they contain or do not contain, respectively, a vertex of the original graph G. The edges of $M(G)$ are directed so that the black face is on the left of an incident edge (see Figure 1.2). As in Remark 1, we just state the connection between the circuit partition polynomial and the parameter hom(__, H), without going into more details.

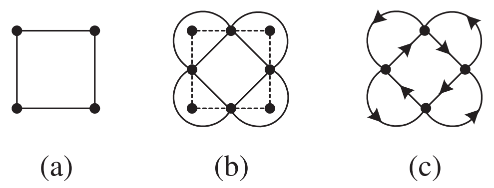

Figure 1.2: (a) The cycle C_4, (b) the medial graph of C_4, (c) the directed medial graph of C_4.

Proposition 1.3.6. *Let G be a planar graph with λ vertices, m edges and c connected components. For every $q, n \in \mathbb{N}$ with $q \geq 1$ and $n > 1$ such that $\sqrt{n} \in \mathbb{N}$, the following holds:*

$$\hom\left(G, K_n^{(1+\sqrt{n})q,q}\right) = (\sqrt{n})^\lambda q^m j(\overrightarrow{M(G)}; \sqrt{n}).$$

Theorem 1.2.5 and the connection between the Tutte polynomial and the circuit partition polynomial, let us prove the following result.

Theorem 1.3.7. *Let $H \in \Omega$ be a connected graph. Suppose that there exist a constant x_H, and a local function h_H such that for every planar graph G,*

$\hom(G,H) = h_H(G) j(\overrightarrow{M(G)}; x_H)$. *Then there exist* $p, q, n \in \mathbb{N}$ *with* $p \neq q$ *and* $n > 1$ *such that* $H \cong K_n^{p,q}$.

Remark 3. The Penrose polynomial is an instance of graph polynomial that can not be related to the homomorphism counting by using our technique. Indeed, this polynomial can be described in terms of a transition polynomial for specific values of δ, μ and τ (see [13] for more details). Thus, the Penrose polynomial is defined for $\tau = -2$ and Proposition 1.3.4 considers $\tau = \sqrt{n} > 0$.

1.4 Applications to abelian groups and statistical physics

The aim of this section is to sketch connections between the homomorphism counting and fundamental evaluations of the Tutte polynomial in abelian groups and statistical physics. Concretely, we focus on difference sets in abelian groups, the Potts model and the random cluster model in statistical mechanics.

1.4.1 Difference sets in abelian groups

Let G be a graph with a fixed orientation ω of its edges, and a uniform probability space of pairs (f_1, f_2) of functions from $E(G)$ into a subset B of an abelian group A of order r. A $(r,k,l)-$*difference set* in A is a subset $B \subseteq A$ of k elements with $2 \leq k \leq r$ such that, for all $0 \neq a \in A$ there exist l pairs $(b_1, b_2) \in B \times B$ with $b_1 - b_2 = a$.

Let $A^{V(G)}$ denote the set of all functions $g : V(G) \to A$, and $A^{E(G)}$ the set of all functions $f : E(G) \to A$. The *boundary operator* $d^* : A^{E(G)} \to A^{V(G)}$ is defined for a given function $f : E(G) \to A$ and each vertex $v \in V(G)$ by,

$$d^* f(v) = \sum_{e \in \omega^+(v)} f(e) - \sum_{e \in \omega^-(v)} f(e).$$

Observe that the kernel of d^* is the space of the $A-$flows of G.

In this subsection, we prove that for $k > l$ when B has the property of being a difference set in A, the event that f_1 and f_2 have the same boundary has probability equal, up to a factor, to $\hom(G, K_n^{p,q})$ for some values of

n, p and q. The relationship between such probability and the boundary polynomial is first recalled from [11].

Lemma 1.4.1. [11] *If B is a $(r,k,l)-$difference set then,*

$$Pr(d^*f_1 = d^*f_2) = k^{-2m}l^mF\left(G; r, \frac{k}{l}\right)$$

where G is a graph with m edges.

This lemma and Proposition 1.3.2 imply the following result.

Proposition 1.4.2. *Let A be an abelian group on r elements, $B \subseteq A$ a* $(r,k,l)-$difference set *in A, and q any positive integer number such that $(\frac{rl}{k-l}+1)q \in \mathbb{N}$. If two functions $f_1, f_2 : E(G) \to B$ are chosen uniformly at random then:*

$$Pr(d^*f_1 = d^*f_2) = k^{-2m}(k-l)^m r^{-\lambda} q^{-m} \hom\left(G, K_r^{(\frac{rl}{k-l}+1)q,q}\right)$$

where G is a graph with λ vertices and m edges.

Observe that the condition $(\frac{rl}{k-l}+1)q \in \mathbb{N}$ implies $k > l$. Hence, we can consider the value $q = k-l$ and the following result is a particular case of the above-stated proposition.

Corollary 1.4.3. *Let B be a* $(r,k,l)-$difference set *in A with $k > l$. If two functions $f_1, f_2 : E(G) \to B$ are chosen uniformly at random then,*

$$Pr(d^*f_1 = d^*f_2) = k^{-2m} r^{-\lambda} \hom\left(G, K_r^{(r-1)l+k,k-l}\right)$$

where G is a graph with λ vertices and m edges.

Note also that $B = A\backslash\{0\}$ form a $(r, r-1, r-2)-$difference set. Thus, we can state the following corollary.

Corollary 1.4.4. *Let q be any positive integer number. If two functions $f_1, f_2 : E(G) \to A\backslash\{0\}$ are chosen uniformly at random then,*

$$Pr(d^*f_1 = d^*f_2) = (r-1)^{-2m} r^{-\lambda} q^{-m} \hom\left(G, K_r^{(r-1)^2q,q}\right)$$

where G is a graph with λ vertices and m edges.

1.4.2 The Potts model and the Gibbs probability

For the combinatorial analysis of the Potts model on a finite graph G, it is assumed that the interaction energy, which measures the strength of the interaction between neighbourings pairs of vertices, is constant and equal to J. Consider that each atom can be in S different states, and denote $K = 2\beta J$ where β is a parameter of the model determined by the temperature. The following relationship between the partition function Z of the Potts model and the coboundary polynomial of G is proved in [27],

$$Z(G) = e^{-K|E(G)|}P(G; S, e^K).$$

This relationship and Proposition 1.3.2 lead to the connection between counting graph homomorphisms and the partition function Z of the Potts model.

Proposition 1.4.5. *Let q be any positive integer number such that $e^K q \in \mathbb{N}$. Then,*

$$Z(G) = e^{-Km}q^{-m}\hom\left(G, K_S^{e^K q,q}\right)$$

where G is a graph with m edges.

The random cluster model on a finite graph G can be regarded as the analytic continuation of the Potts model to non-integer S (see [25]). This model is a correlated bond percolation model in statistical mechanics, introduced by Fortuin and Kasteleyn in [8] (see also [25, 27]) and defined by a probability distribution, called the *Gibbs probability*, as follows. For every subset $A \subseteq E(G)$,

$$\mu(A) = N^{-1}\left(\prod_{e\in A} t_e\right)\left(\prod_{e\notin A}(1-t_e)\right)S^{k(A)}$$

where $k(A)$ is the number of connected components of the graph $(V(G), A)$, the value t_e is a probability assigned to every edge $e \in E(G)$, $S \geq 0$ is a parameter of the model, and N is the normalizing constant so that $\sum_{A\subseteq E(G)} \mu(A) = 1$.

When each of the t_e are made equal, the Gibbs probability appears as a two parameter family of probability measure $\mu = \mu(t, S)$ where $0 \leq t \leq 1$ and $S > 0$. In this case, this probability is essentially an evaluation of the

Tutte polynomial of G (see [27]). Indeed,

$$\mu(A) = \frac{\left(\frac{t}{1-t}\right)^{|A|} S^{-r(A)}}{\left(\frac{t}{S(1-t)}\right)^{r(E(G))} T\left(G; 1 + \frac{S(1-t)}{t}, \frac{1}{1-t}\right)}$$

where $r(A) = |V| - k(A)$ is the *rank* of A.

We now reformulate this relationship in terms of the homomorphism counting.

Proposition 1.4.6. *Let G be a finite graph, and $A \subseteq E(G)$. For every $\ell \in \mathbb{N}$ such that $(1-t)\ell$ is a positive integer number, the Gibbs probability is given by*

$$\mu(A) = \frac{\left(\frac{t}{1-t}\right)^{|A|} S^{-r(A)+|V(G)|}(1-t)^{|E(G)|}\ell^{|E(G)|}}{\hom\left(G, K_S^{\ell,(1-t)\ell}\right)}.$$

1.5 Coloring Uniqueness

Since the Tutte polynomial contains a great deal of information about the graph to which it is associated, a question that arises naturally is whether a graph can be recovered up to isomorphism from its Tutte polynomial. A graph G is said to be *Tutte-unique* if $T(G; x, y) = T(H; x, y)$ implies $H \cong G$, for every other graph H. Tutte uniqueness has been studied for several families of graphs, such as complete multipartite graphs, wheels, hypercubes (see [24]), locally grid graphs [9, 21], and hexagonal tilings [10]. In 2000, Bollobás, Peabody and Riordan [3] conjectured that almost all graphs are Tutte-unique. Since then there has been little progress on this conjecture.

The problem of finding graphs determined by polynomial invariants has been studied also for other polynomials such as the chromatic polynomial [16, 17]. Since the chromatic polynomial of a 2-connected graph can be recovered from its Tutte polynomial, we obtain that 2−connected chromatically-unique graphs are Tutte-unique. It is also conjectured that almost all graphs are chromatically-unique [3]. Following this line of research, we introduce the concept of *coloring-uniqueness*.

Definition 1.5.1. *A finite graph G is coloring-unique if* $\hom(G, K_n^{p,q}) = \hom(H, K_n^{p,q})$ *for all $n \geq 1$, $p, q \geq 0$ and $p \neq q$ implies $H \cong G$, for every other graph H.*

Observe that chromatically-unique graphs are coloring-unique.

Theorem 1.5.2. *Let G be a simple, $2-$connected graph. If G is coloring-unique then G is Tutte-unique.*

Theorem 1.5.3. *If almost all graphs are coloring-unique then almost all graphs are Tutte-unique.*

1.6 Concluding Remarks

1. Some of the connections provided in this paper are summarized in the following table. In particular, those between counting graph homomorphisms and evaluating polynomials associated with graphs.

Homomorphisms-Tutte polynomial $\hom(G, K_n^{p,q}) = n^c(p-q)^{\lambda-c}q^{m-\lambda+c}T\big(G; \frac{p+q(n-1)}{p-q}, \frac{p}{q}\big)$ $p \geq 0,\ q \geq 1,\ p \neq q$ $\hom(G, K_1^{p,0}) = (p/2)^m T(G, 2, 2)$ with $p > 0$	G any graph with λ vertices, m edges, c connected components
Homomorphisms-Transition polynomial $\hom(G, K_n^{p,q}) = n^{m-\lambda+1}(p-q)^m\delta^m Q(M(G), A, \sqrt{n})$ if $p - q \neq q\sqrt{n}$ $\hom(G, K_n^{p,q}) = (\sqrt{n})^{\lambda+1}q^m\delta^m Q(M(G), A, \sqrt{n})$ if $p - q = q\sqrt{n}$	G connected planar graph $M(G)$ medial graph
Homomorphisms-Circuit partition polynomial $\hom\left(G, K_n^{(1+\sqrt{n})q,q}\right) = (\sqrt{n})^\lambda q^m j(\overrightarrow{M(G)}; \sqrt{n})$ $q \geq 1,\ n > 1,\ \sqrt{n} \in \mathbb{N}$	G planar graph. $\overrightarrow{M(G)}$ directed medial graph
Homomorphisms-Boundary polynomial $\hom(G, K_n^{p,q}) = n^{\lambda-m}(p-q)^m F\left(G; n, \frac{p+q(n-1)}{p-q}\right)$ $p \geq 0,\ q \geq 1,\ p \neq q$	G any graph with λ vertices, m edges, c connected components
Homomorphisms-Coboundary polynomial $\hom(G, K_n^{p,q}) = q^m P(G; n, p/q)$ $p \geq 0,\ q \geq 1,\ p \neq q$	G any graph with λ vertices, m edges, c connected components

2. The connection between graph homomorphisms and graph invariants is well-known and useful. One can here quote not only classical (and not so classical) results such as those covered in [27] and [12], but also recent works for reaching an algebraical approach of this connection, see for example [5]. It is perhaps surprising how tight (in certain very concrete instances) this connection is. This paper shows both the connection in the case of polynomial invariants, and also its limitations. But perhaps this approach can put in a new context some well-known problems such as uniqueness questions. However, Andrew Goodall very recently showed the converse of Theorem 1.5.2: Every Tutte unique graph is coloring unique. A bit surprisingly, the concepts of Tutte- and coloring-uniqueness coincide.

3. In proving Theorem 1.2.5 we use the properties of the homomorphism function $\hom(\_\_, H)$ for very special graphs only: multiple edges and cycles (and their minors). Thus, it is sufficient to assume the locality of the function h_H for this small minor closed family.

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Chapter 2

Andrew Goodall: Graph polynomials and Tutte-Grothendieck invariants: an application of elementary finite Fourier analysis

The topics covered in this article are as follows. The graph polynomial, Tutte–Grothendieck invariants, an overview of relevant elementary finite Fourier analysis, the Tutte polynomial of a graph as a Hamming weight enumerator of tensions (or flows) of a graph, a family of polynomials containing the graph polynomial which yield Tutte–Grothendieck invariants in a similar way to it.

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2.1 Introduction

The graph polynomial is a generalization of the Vandermonde determinant (the graph polynomial of a complete graph) which was considered by Petersen in the early days of graph theory. Alon, Tarsi and Matiyasevich in more recent years have found that it contains a lot of information about the vertex colourings of a graph. Motivated by their results, which include the fact that the number of proper 3-colourings of a graph is a simply described function of the coefficients of its graph polynomial, in this article we consider a family of polynomials containing the graph polynomial and ask whether other Tutte–Grothendieck invariants can be obtained in a similar way as a function of their coefficients. Our results are obtained by expressing the relevant parameter as the partition function of a vertex colouring model (such as the Potts model) or, in different language, the graph parameter obtained from homomorphisms to a weighted graph.

After introducing the graph polynomial and its relation to proper vertex colourings, there follows a description of Tutte–Grothendieck invariants and an overview of the techniques of finite Fourier analysis which will be used to obtain our results. The notions of tensions and flows of a graph and the view of the Tutte polynomial as a Hamming weight enumerator are then adumbrated. We conclude with a characterization of those polynomials which share with the graph polynomial the property of yielding Tutte–Grothendieck invariants from their coefficients. More generally, the graph polynomial is seen to belong to a family of polynomials for which a simple function of their coefficients is equal to the complete weight enumerator of the set of tensions (or flows) of the graph.

2.2 The graph polynomial

Let $G = (V, E)$ be a graph with some fixed, arbitrary orientation of its edges, and denote its directed edge set by $\overrightarrow{E}$.

Let Q be a finite set of size q. A *proper vertex q-colouring* using colour set Q is an assignment of colours $(c_v : v \in V) \in Q^V$ such that $c_u \neq c_v$ whenever $\{u, v\} \in E$. The number of proper vertex q-colourings of G is denoted by $P(G; q)$ (an evaluation of the chromatic polynomial of G at q).

Let $\mathbf{x} = (x_v : v \in V)$ be a tuple of commuting indeterminates indexed by V and define the *graph polynomial*[1] $F(G)$ in $\mathbb{C}[\mathbf{x}]$ by

$$F(G;\mathbf{x}) = \prod_{(u,v)\in\overrightarrow{E}} (x_u - x_v).$$

Given an assignment of values $\mathbf{c} = (c_v : v \in V) \in \mathbb{C}^V$ to the indeterminates $\mathbf{x} = (x_v : v \in V)$, the graph polynomial takes a non-zero value if and only if $\mathbf{c}$ corresponds to a proper vertex colouring with colour set $Q = \{c_v : v \in V\}$. Set $\zeta = e^{2\pi i/q}$. By restricting c_v to one of the q points $1, \zeta, \ldots, \zeta^{q-1}$ on the unit circle a criterion emerges for the existence of a proper vertex q-colouring of G in terms of the polynomial $F(G;\mathbf{x})$.

The algebraic variety of points $\{(c_v : v \in V) : c_v \in \{1, \zeta, \ldots, \zeta^{q-1}\}\}$ corresponds to the ideal $(x_v^q - 1 : v \in V)$ of the ring $\mathbb{C}[\mathbf{x}]$. Denote the graph polynomial modulo the ideal generated by the polynomials $x_v^q - 1$ by

$$F^{(q)}(G;\mathbf{x}) = F(G;\mathbf{x}) \bmod (x_v^q - 1 : v \in V).$$

By Lagrange interpolation, with sums over $(a_v : v \in V) \in \{0, 1, \ldots, q-1\}^V$,

$$\begin{aligned} F^{(q)}(G;\mathbf{x}) &= \sum_{(a_v : v\in V)} \prod_{v\in V} \prod_{a\neq a_v} \frac{x_v - \zeta^a}{\zeta^{a_v} - \zeta^a}\, F(G; (\zeta^{a_v} : v \in V)\,) \\ &= q^{-|V|} \sum_{(a_v : v\in V)} \prod_{v\in V} \frac{x_v^q - 1}{\zeta^{-a_v} x_v - 1} F(G; (\zeta^{a_v} : v \in V)\,), \end{aligned}$$

the last line since $\prod_{a\neq a_v}(\zeta^{a_v} - \zeta^a) = \zeta^{(q-1)a_v}\prod_{b\neq 0}(1-\zeta^b) = \zeta^{-a_v} q$. The relationship between coefficients of the polynomial $F^{(q)}(G;\mathbf{x})$ and its evaluations at points $(\zeta^{a_v} : v \in V)$ is exhibited here as a basis change between the basis of monomials $\prod_{v\in V} x_v^{a_v}$ and the basis of polynomials $\prod_{v\in V} \frac{x_v^q - 1}{\zeta^{-a_v} x_v - 1}$. The connection is the Fourier transform. This article is an elaboration of this remark.

[1] The graph polynomial has not yet acquired the qualification of a proper name. The 'Sylvester–Petersen graph polynomial' might be a candidate [18, 17]. Matiyasevich analyses the graph polynomial of the line graph of a cubic plane graph in order to obtain reformulations of the Four Colour Theorem [15]. Alon and Tarsi [2, 3, 20] interpret its coefficients in terms of orientations; their interpretations in terms of proper vertex colourings will be described in this section. Ellingham and Goddyn [8] call the graph polynomial the *graph monomial* averring that the latter has a less anonynmous character than the former.

Alon and Tarsi [3] use the "Combinatorial Nullstellensatz" [1] to prove that $F^{(q)}(G;\mathbf{x}) \neq 0$ if and only if $P(G;q) > 0$, and also show that more can be said.

The (squared) ℓ_2-norm $\|F^{(q)}(\mathbf{x})\|_2^2$ of the polynomial $F^{(q)}(\mathbf{x})$ is defined to be the sum of the absolute squares of its coefficients. That this is a norm includes the fact that $F^{(q)}(G;\mathbf{x}) \neq 0$ if and only if $\|F^{(q)}(G;\mathbf{x})\|_2^2 \neq 0$.

Theorem 2.2.1. [3] For each $q \in \mathbb{N}$,

$$\|F^{(q)}(G;\mathbf{x})\|_2^2 = q^{-|V|}4^{|E|} \sum_{\mathbf{c}\in\{0,1,\ldots,q-1\}^V} \prod_{uv\in E} \sin^2 \frac{\pi(c_v - c_u)}{q},$$

the sum being over all vertex colourings of G with colours $\{0, 1, \ldots, q-1\}$. In particular, for $q = 3$ this is equal to $3^{|E|-|V|}P(G;3)$.

For the next theorem we require a further definition. A $(q,1)$-*flow* of G is a partial orientation of G with the property that at each vertex the number of edges directed out of v is congruent to the number of edges directed into v modulo q. (A partial orientation is obtained when some edges of G are assigned an orientation while the other edges remain undirected.) By referring to the fixed orientation $\overrightarrow{E}$ of G, it is possible to use the equivalent definition as an assignment of values $(b_e : e \in E)$ to the edges of G with the properties that $b_e \in \{0, 1, -1\}$ and the net flow (incoming minus outgoing values) at each vertex is equal to zero modulo q.

Theorem 2.2.2. [20] For each $q \in \mathbb{N}$,

$$\|F^{(q)}(G;\mathbf{x})\|_2^2 = (-1)^{|E|} \sum_{(q,1)\text{-flows } \mathbf{b}} (-2)^{|E|-|\mathbf{b}|},$$

where $|\mathbf{b}| = \#\{e \in E : b_e \neq 0\}$.

One aim of this article is to reveal the underlying relationship between Theorems 2.2.1 and 2.2.2 in a more general context. The other is to characterize those polynomials of the form

$$\prod_{(u,v)\in\overrightarrow{E}} \sum_{a,b\in\{0,1,\ldots,q-1\}} f(a,b)x_u^a x_v^b \qquad \bmod (x_v^q - 1 : v \in V)$$

whose ℓ_2-norm is a Tutte–Grothendieck invariant (such as $P(G;q)$). The graph polynomial is the case $f(1,0) = 1, f(0,1) = -1$ and $f(a,b) = 0$

otherwise, and Theorem 2.2.1 says that for $q = 3$ its ℓ_2-norm is the Tutte–Grothendieck invariant $3^{|E|-|V|}P(G; 3)$.

We begin in Section 2.3 by defining Tutte–Grothendieck invariants and motivating the search for them.

In Section 2.4 a potted account is given of Fourier analysis on finite Abelian groups and preparations made for Section 2.5 in which questions are anwered and our stated aims (more or less) achieved. An expanded version of Section 2.4 can be found in [12], and an even more fulsome presentation is given in [11]. The book [21] is recommended for an introduction to finite Fourier analysis and its wide range of applications.

2.3 Tutte–Grothendieck invariants

Let $G = (V, E)$ be a graph, loops and parallel edges permitted, with $k(G)$ components, rank $r(G) = |V| - k(G)$ and nullity $n(G) = |E| - r(G)$.

Deleting an edge $e \in E$ gives a graph $G\backslash e$ with one fewer edge than G. *Contracting* e gives a graph G/e with one fewer vertex and one fewer edge than G. Many graph parameters may be recursively defined via contraction-deletion recurrences.

Definition 2.3.1. A function F from graphs into $\mathbb{C}[\alpha, \beta, \gamma, x, y]$ is a *Tutte–Grothendieck invariant* if it satisfies, for each graph $G = (V, E)$ and any edge $e \in E$,

$$F(G) = \begin{cases} \gamma^{|V|} & E = \emptyset, \\ xF(G/e) & e \text{ a bridge}, \\ yF(G\backslash e) & e \text{ a loop}, \\ \alpha F(G/e) + \beta F(G\backslash e) & e \text{ not a bridge or loop}. \end{cases} \tag{2.1}$$

See for example the accounts in [22, 5, 10] for an appreciation of the ubiquity of Tutte–Grothendieck invariants. For $A \subseteq E$, the subgraph (V, A) is obtained from G by deleting edges not in A. Given $G = (V, E)$, the rank of the graph (V, A) is denoted by $r(A)$. A Tutte–Grothendieck invariant is an evaluation of the *Tutte polynomial*, defined by

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \tag{2.2}$$

The Tutte polynomial is a rescaling of the Whitney rank polynomial of G (for which see for example [10]), a generating function for $(|A|, r(A))$ over all subgraphs (V, A) of G. The coefficients of the Tutte polynomial are non-negative integers (see for example [4, 5]), a fact while not evident from its definition in equation (2.2) is more readily seen in its alternative formulation as a Tutte–Grothendieck invariant with $\alpha = \beta = \gamma = 1$.

Theorem 2.3.2. If F is a Tutte–Grothendieck invariant satisfying the equations (2.1) then

$$F(G) = \gamma^{k(G)}\alpha^{r(G)}\beta^{n(G)}T(G; \frac{x}{\alpha}, \frac{y}{\beta}).$$

Example 2.3.3. The *monochrome polynomial* $P(G) = P(G; q, y)$ (monochromial, bad colouring polynomial, coboundary polynomial, partition function of the q-state Potts model) is defined by

$$P(G; q, y) = \sum_{\mathbf{c} \in Q^V} y^{\#\{(u,v) \in \overrightarrow{E} : c_u = c_v\}}, \tag{2.3}$$

where Q is a set of q colours (states) and $\mathbf{c} = (c_v : v \in V)$ is a vertex colouring of G using colours from Q. It is easily verified that the function P satisfies

$$P(G) = \begin{cases} q^{|V|} & E = \emptyset, \\ (y + q - 1)P(G/e) & e \text{ a bridge}, \\ yP(G\backslash e) & e \text{ a loop}, \\ (y - 1)P(G/e) + P(G\backslash e) & e \text{ not a bridge or loop}. \end{cases}$$

By Theorem 2.3.2,

$$P(G; q, y) = q^{k(G)}(y - 1)^{r(G)}T(G; \frac{y-1+q}{y-1}, y). \tag{2.4}$$

In particular, the *chromatic polynomial* $P(G; q)$, counting the number of proper vertex q-colourings of G, is given by

$$P(G; q) = q^{k(G)}(-1)^{r(G)}T(G; 1 - q, 0).$$

Let Q be a set of size q (later to be an additive Abelian group of order q) and $\mathbf{w} = (w_{a,b})$ a tuple of complex numbers indexed by $(a, b) \in Q \times Q$. Assume that the edges $\{u, v\}$ of $G = (V, E)$ have been given an arbitrary, fixed orientation (u, v). Denote by $\overrightarrow{E}$ the resulting set of directed edges.

Consider the partition function for a vertex Q-colouring model that assigns a weight $w_{a,b}$ to a directed edge (u,v) coloured (a,b):

$$F(G;\mathbf{w}) = \sum_{\mathbf{c}\in Q^V} \prod_{(u,v)\in \overrightarrow{E}} w_{c_u,c_v} = \sum_{\mathbf{c}\in Q^V} \prod_{(a,b)\in Q\times Q} {w_{a,b}}^{\#\{(u,v)\in\overrightarrow{E}\,:\,(c_u,c_v)=(a,b)\}}. \tag{2.5}$$

This partition function may be interpreted as the weight of a graph homomorphism $G \to H$, where H is the directed graph on vertex set Q and edge set $\{(a,b) : w_{a,b} \neq 0\}$, with edge weights $w_{a,b}$, i.e., the weighted graph H has adjacency matrix $(w_{a,b})_{a,b\in Q}$. (It is possible to also have vertex weights for H in addition to its edge weights, but this will not be considered here. See for example [9, 19] and [13, 7] for more on vertex colouring models and on graph homomorphisms.)

Theorem 2.3.4. The graphical invariant $F(G;\mathbf{w})$ defined by equation (2.5) is a Tutte–Grothendieck invariant if and only if there are constants y, w such that

$$w_{a,b} = \begin{cases} w & a \neq b, \\ y & a = b. \end{cases}$$

In this case $F(G;\mathbf{w}) = F(G;w,y) = q^{k(G)}w^{n(G)}(y-w)^{r(G)}T(G;\frac{y-(q-1)w}{y-w},\frac{y}{w})$. (If $w=0$ then $F(G;0,y) = y^{|E|}$ and if $w = y$ then $F(G;y,y) = q^{|V|}y^{|E|}$.)

A sketch proof only of Theorem 2.3.4 is given.[2] The following lemma is the main tool.

Lemma 2.3.5. If two multisets of complex numbers $\{u_1, u_2, \ldots, u_r\}$ and $\{v_1, v_2, \ldots, v_r\}$ satisfy

$$u_1^m + u_2^m + \cdots + u_r^m = v_1^m + v_2^m + \cdots + v_r^m, \tag{2.6}$$

for integers $1 \leq m \leq r$, then $\{u_1, u_2, \ldots, u_r\} = \{v_1, v_2, \ldots, v_r\}$.

Proof. By Newton's relations the first r elementary symmetric functions in r arguments are determined by the first r power sum symmetric functions. Equation (2.6) implies that the first r elementary symmetric functions in $u_1, u_2, \ldots, u_r$ coincide with the first r elementary symmetric functions in $v_1, v_2, \ldots, v_r$. This determines $\{u_1, u_2 \ldots, u_r\}$ and $\{v_1, v_2, \ldots, v_r\}$ as the

[2] I am grateful to Delia Garijo for alerting me to the fact that I was assuming the truth of something that required proof (upon which my "iff" was curtailed to an "if"), and also for her description of how she has been tackling a related, stronger result.

multiset of roots of the same polynomial over $\mathbb{C}$ and hence as equal multisets. $\square$

Proof. (of Theorem 2.3.4.) In one direction, given that $x_{a,b} = w$ for $a \neq b$ and $x_{a,a} = y$, the evaluation of the Tutte polynomial follows from that of the monochrome polynomial given in equation (2.4) in Example 2.3.3 above with $x = y/w$.

In the other direction, suppose that there are constants $\alpha, \beta, \gamma, x, y$ such that $F(G; \mathbf{w}) = F(G)$ satisfies the relations (2.1) for a Tutte–Grothendieck invariant. By checking that this is indeed the case for the three families of graphs X_m, Y_m, Z_m itemized below the desired conclusion is reached. Each of these families of graphs possess the following virtues: (i) the graphs obtained by contracting or deleting an edge are of the same form or belong to one of the other families, and (ii) it is possible to write down F as given by the partition function (2.5) as a succinct formula, thereupon to substitute this formula into the contraction-deletion recurrence of equation (2.1), and finally to use Lemma 2.3.5.

(1) Y_m, $1 \leq m \leq q$, the graph on one vertex with m loops. $F(Y_1) = \sum_{a \in Q} x_{a,a} = qy$. The relation $F(Y_m) = yF(Y_{m-1}) = qy^m$ is used to show that $x_{a,a} = y$ for each $a \in Q$.

(2a) X_m, $1 \leq m \leq q^2$, the graph on two vertices connected by m parallel edges. $F(X_1) = \sum_{a,b \in Q} x_{a,b} = qx$. The relation $F(X_m) = \alpha F(Y_{m-1}) + \beta F(X_{m-1})$ for $m \geq 2$ is used to show first that $\{x_{a,b} : a, b \in Q\}$ contains at most two distinct values y, w: there is S with $\{(a,a) : a \in Q\} \subseteq S \subseteq Q \times Q$ such that $x_{a,b} = y$ for $(a,b) \in S$, and $x_{a,b} = w$ otherwise.

(2b) X_m^n, $1 \leq m \leq q^2$, the graph X_m with n edges oriented in one direction, $m - n$ in the other. That $F(X_m)$ is independent of any orientation of the edges of X_m (giving a graph X_m^n) is used to show that $x_{a,b} = x_{b,a}$ for all $a, b \in Q$, i.e., the set S defined in (2a) is closed under the involution $(a,b) \mapsto (b,a)$.

(3) Z_m, $1 \leq m \leq q$, the star graph with m edges (one vertex degree m, and m vertices degree 1). The relation $F(Z_m) = xF(Z_{m-1}) = qx^m$ is used to show that $\#\{b \in Q : (a,b) \in S\}$ is independent of a, whereby it follows from (2b) that either $S = \{(a,a) : a \in Q\}$ or $S = Q \times Q$.

$\square$

Note. The author in retrospect recognizes that a happier choice in the proof of Theorem 2.3.4 would have been the family of cycles $\{C_1, \ldots, C_q\}$ instead of $\{X_1, \ldots, X_{q^2}\} \cup \{Y_1, \ldots, Y_q\} \cup \{Z_1, \ldots, Z_q\}$. It transpires that the cycles together with the various possible orientations of their edges are sufficient to determine that Tutte–Grothendieck invariants take the form given in the statement of Theorem 2.3.4.

We now know how to tell whether a partition function of a vertex Q-colouring model is a Tutte–Grothendieck invariant or not. In order to look for Tutte–Grothendieck invariants in graph polynomials of the sort defined in Section 2.2 we shall use instruments from Fourier analysis, a subject to which we now turn.

2.4 Fourier analysis on finite Abelian groups

2.4.1 The algebra $\mathbb{C}^Q$

Let Q be an additive Abelian group of order q. In later sections $Q = \mathbb{Z}_q$, the integers under addition modulo q.

The set $\mathbb{C}^Q$ of functions $f : Q \to \mathbb{C}$ forms a q-dimensional Hermitian inner product space. The inner product will be denoted by

$$\langle f, g \rangle = \sum_{a \in Q} f(a)\overline{g(a)}.$$

The ℓ_2*-norm* is defined by $\|f\|_2 = \langle f, f \rangle^{\frac{1}{2}}$ and defines a metric on the space $\mathbb{C}^Q$.

The space $\mathbb{C}^Q$ has an orthonormal basis of *indicator functions* $\{\delta_a : a \in Q\}$,

$$\delta_a(b) = \begin{cases} 1 & a = b, \\ 0 & a \neq b. \end{cases}$$

There are several definitions of multiplication that make $\mathbb{C}^Q$ an algebra:

- *Pointwise product*
$$f \cdot g(a) = f(a)g(a).$$
- *Convolution*
$$f * g(a) = \sum_{b \in Q} f(a)g(b - a).$$

- *Cross-correlation*

$$f \star g(a) = \sum_{b \in Q} \overline{f(a)} g(b + a).$$

The effect of these operations on the indicator functions is as follows:

$$\delta_a \cdot \delta_b = \delta_a(b)\delta_a, \qquad \delta_a * \delta_b = \delta_{a+b}, \qquad \delta_a \star \delta_b = \delta_{b-a}.$$

The Abelian group Q has *dual group* equal to the set of *characters* of Q under pointwise multiplication. For each $c \in Q$, the character $\chi_c : Q \to \mathbb{C}^\times$ is a group homomorphism: $\chi_c(a + b) = \chi_c(a)\chi_c(b)$ for all $a, b \in Q$. The multiplicative group of characters of Q is isomorphic to the additive group Q. (This is only true when Q is a finite Abelian group, and, for the applications later in this article, is the reason why only finite Abelian groups are considered.)

The set $\{\chi_c : c \in Q\}$ forms an orthogonal basis for $\mathbb{C}^Q$, with $\langle \chi_a, \chi_b \rangle = q\delta_a(b)$.

In the algebra $\mathbb{C}^Q$,

$$\chi_a \cdot \chi_b = \chi_{a+b}, \qquad \chi_a * \chi_b = q\delta_a(b)\chi_a = \chi_a \star \chi_b.$$

Supposing the additive group Q has the further structure of a ring (such as $\mathbb{Z}_q$ with addition and multiplication modulo q), a *generating character* χ satisfies $\chi_a(b) = \chi(ab)$ for all $a, b \in Q$. When $Q = \mathbb{Z}_q$, the character χ defined by $\chi(a) = e^{2\pi i a/q}$ (or $e^{2\pi i c a/q}$ for any fixed c coprime with q) is a generating character.

2.4.2 The Fourier transform

The evaluation of the Fourier transform of a function at a point is the projection of the function onto a character:

$$\widehat{f}(b) = \langle f, \chi_b \rangle = \sum_{a \in Q} f(a)\chi_b(-a),$$

i.e.,

$$\widehat{f} = \sum_{b \in Q} f(b)\chi_{-b}.$$

Orthogonality of the basis $\{\chi_c : c \in Q\}$ yields

(i) the Fourier inversion formula,

$$f(a) = q^{-1}\langle \widehat{f}, \chi_{-a}\rangle = q^{-1}\sum_{b\in Q}\widehat{f}(b)\chi_b(a),$$

i.e., the Fourier transform may be regarded as a change of basis from indicators to characters:

$$f = \sum_{a\in Q} f(a)\delta_a = q^{-1}\sum_{b\in Q}\widehat{f}(b)\chi_b.$$

(ii) Plancherel's formula,

$$\langle \widehat{f}, \widehat{g}\rangle = q\langle f, g\rangle,$$

(iii) Parseval's formula,

$$\|f\|_2^2 = q^{-1}\|\widehat{f}\|_2^2.$$

Thus the normalized Fourier transform $f \mapsto q^{-\frac{1}{2}}\widehat{f}$ is a unitary transformation, giving an isometry of the metric space $\mathbb{C}^Q$.

The Fourier transform is an isomorphism of the algebra $(\mathbb{C}^Q, *)$ with the algebra $(\mathbb{C}^Q, \cdot)$:

$$\widehat{f\cdot g} = q^{-1}\widehat{f} * \widehat{g}, \qquad \widehat{f * g} = \widehat{f}\cdot\widehat{g}, \qquad \widehat{f\star g} = \overline{\widehat{f}}\cdot\widehat{g}, \tag{2.7}$$

and in particular

$$\widehat{f\star f} = |\widehat{f}|^2$$

(the finite version of the Wiener-Khintchine formula). That the Fourier transform is an isometry carrying convolution to pointwise multiplication makes it useful in the analysis of random walks on Cayley graphs on Q, where steps on the graph correspond to addition of group elements – see for example [21] and the references therein. To prove the formulae in (2.7) it suffices to determine the effect of the Fourier transform on basis functions and then appeal to linearity and distributivity. For example,

$$\widehat{\delta_a\star\delta_b} = \widehat{\delta_{b-a}} = \chi_{a-b} = \overline{\widehat{\delta_a}}\cdot\widehat{\delta_b}.$$

For an additive subgroup P of Q, the *annihilator* of P is defined by

$$P^\sharp = \{b\in Q : \forall_{a\in P}\ \chi_b(a) = 1\}.$$

and is isomorphic to the quotient group Q/P.

Extend the indicator function notation from elements to subsets $P \subseteq Q$ by setting $\delta_P = \sum_{a \in P} \delta_a$.

For our purposes, a key property of the Fourier transform is that

$$\widehat{\delta_P} = |P| \delta_{P^\sharp}.$$

By Fourier inversion,

$$\delta_P \star f(b) = q^{-1} \langle \widehat{\delta_P} \cdot \widehat{f}, \chi_{-b} \rangle,$$

giving the *Poisson summation formula*

$$\sum_{a \in P} f(a+b) = |P^\sharp|^{-1} \sum_{a \in P^\sharp} \widehat{f}(a) \chi_b(a).$$

2.4.3 The algebra $\mathbb{C}^{Q^n}$ and the ring $\mathbb{C}[\mathbf{x}]/(x_i^q - 1 : 1 \leq i \leq n)$

Let $\mathbf{Q} = Q^n$ denote the n-fold direct product of Q, which is an Abelian group of order q^n and a module over Q. Given that Q is also a ring, put a ring structure on Q^n by defining componentwise multiplication of $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in Q^n$,

$$\mathbf{ab} = (a_1 b_1, \ldots, a_n b_n).$$

The Hermitian inner product space $\mathbb{C}^{Q^n}$ is the n-fold tensor product of $\mathbb{C}^Q$: given $f_1, \ldots, f_n \in \mathbb{C}^Q$ define

$$f_1 \otimes \cdots \otimes f_n(a_1, \ldots, a_n) = f_1(a_1) \cdots f(a_n),$$

and in particular $f^{\otimes n}(\mathbf{a}) = f(a_1) \cdots f(a_n)$.

The characters of Q^n are the functions defined by $\chi_{\mathbf{a}} = \chi_{a_1} \otimes \cdots \otimes \chi_{a_n}$.

Define the Euclidean (dot) product by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n.$$

If χ a generating character for Q, then $\chi^{\otimes n}$ is a generating character for Q^n:

$$\chi_{\mathbf{a}}(\mathbf{b}) = \chi^{\otimes n}(\mathbf{ab}) = \chi(a_1 b_1) \cdots \chi(a_n b_n) = \chi(\mathbf{a} \cdot \mathbf{b}).$$

Given that Q has a generating character, for a submodule $\mathbf{P}$ of Q^n the annihilator

$$\mathbf{P}^\sharp = \{\mathbf{b} \in Q^n : \forall_{\mathbf{a}\in\mathbf{P}} \quad \chi_\mathbf{a}(\mathbf{b}) = 1\}$$

is equal to the *orthogonal* submodule

$$\mathbf{P}^\perp = \{\mathbf{b} \in Q^n : \forall_{\mathbf{a}\in\mathbf{P}} \quad \mathbf{a}\cdot\mathbf{b} = 0\}.$$

The Fourier transform on Q^n is given by

$$\widehat{f_1 \otimes \cdots \otimes f_n} = \widehat{f_1} \otimes \cdots \otimes \widehat{f_n},$$

and in particular $\widehat{f^{\otimes n}} = \widehat{f}^{\otimes n}$.

It may be helpful to spell out the relationship between polynomials in the ring $\mathbb{C}[\mathbf{x}]/(x_i^q - 1 : 1 \le i \le n)$ (where $\mathbf{x} = (x_i : 1 \le i \le n)$ is an n-tuple of commuting indeterminates) and functions in the space $\mathbb{C}^{\mathbb{Z}_q^n}$. The aim of course is to translate statements about the reduced graph polynomial $F^{(q)}(G;\mathbf{x})$, which belongs to $\mathbb{C}[\mathbf{x}]/(x_v^q - 1 : v \in V)$, into statements about functions in $\mathbb{C}^{\mathbb{Z}_q^V}$. The latter space has now the advantage of familiarity and the accoutrements of a succinct notation.

Take $Q = \mathbb{Z}_q$, which has generating character $\chi(a) = \zeta^a$ for primitive qth root of unity ζ.

The algebra $\mathbb{C}^{\mathbb{Z}_q^n}$ is isomorphic to $\mathbb{C}[x_1, \ldots, x_n]/(x_1^q - 1, \ldots, x_n^q - 1)$ and the following correspondences obtain:

$$\delta_\mathbf{a} = \delta_{a_1} \otimes \cdots \otimes \delta_{a_n} \qquad \text{with} \qquad \mathbf{x}^\mathbf{a} = \prod_{i=1}^n x_i^{a_i},$$

$$\chi_\mathbf{a} = \chi_{a_1} \otimes \cdots \otimes \chi_{a_n} \qquad \text{with} \qquad \prod_{i=1}^n \frac{x_i^q - 1}{\zeta^{-a_i} x_i - 1},$$

$$\mathbf{f} = \sum_\mathbf{a} \mathbf{f}(\mathbf{a})\delta_\mathbf{a} \qquad \text{with} \qquad F(\mathbf{x}) = \sum_{\mathbf{a}\in\mathbb{Z}_q^n} \mathbf{f}(\mathbf{a})\mathbf{x}^\mathbf{a}.$$

Finally,

$$F(\zeta^{a_1}, \ldots, \zeta^{a_n}) = \widehat{\mathbf{f}}(\mathbf{a}),$$

and Lagrange interpolation on points $\{(\zeta^{a_1}, \ldots, \zeta^{a_n}) : (a_1, \ldots, a_n) \in \mathbb{Z}_q^n\}$ is the Fourier basis change

$$\sum_{\mathbf{a}\in\mathbb{Z}_q^n} \mathbf{f}(\mathbf{a})\mathbf{x}^\mathbf{a} = q^{-n} \sum_{\mathbf{a}\in\mathbb{Z}_q^n} \widehat{f}(\mathbf{a}) \prod_{i=1}^n \frac{x_i^q - 1}{\zeta^{-a_i} x_i - 1}.$$

2.4.4 Weight enumerators and the Tutte polynomial

We finish this section on Fourier analysis with a discussion of the Tutte polynomial as a weight enumerator that gives us the opportuity at the same time to define *flows* and *tensions* of a graph, which definitions are needed for the next section.

It will be convenient to extend the domain of a function $\mathbf{f}$ on elements $\mathbf{a} \in Q^n$ to subsets $\mathbf{P} \subseteq Q^n$, setting

$$\mathbf{f}(\mathbf{P}) = \sum_{\mathbf{a} \in \mathbf{P}} \mathbf{f}(\mathbf{a}).$$

The *Hamming weight* of $\mathbf{a} = (a_1, \ldots, a_n)$ is $|\mathbf{a}| = \#\{i : a_i \neq 0\}$. The *Hamming weight enumerator* of $\mathbf{P}$ is defined to be the the generating function for vectors in $\mathbf{P}$ counted according to their number of zero entries:

$$\sum_{\mathbf{a} \in \mathbf{P}} x^{n-|\mathbf{a}|} = (x\delta_0 + \delta_{Q\setminus 0})^{\otimes n}(\mathbf{P}).$$

The *complete weight enumerator* of $\mathbf{P}$ keeps account of the number of entries equal to a given element of Q:

$$\sum_{\mathbf{a} \in \mathbf{P}} \prod_{a \in Q} x_a^{\#\{1 \leq i \leq n : a_i = a\}} = \big(\sum_{a \in Q} x_a \delta_a\big)^{\otimes E}(\mathbf{P}).$$

The Hamming weight enumerator is the specialization $x_0 = x$ and $x_a = 1$ for $0 \neq a \in Q$.

For a submodule $\mathbf{P}$ of Q^n,

$$\widehat{\delta_{\mathbf{P}}} = |\mathbf{P}| \delta_{\mathbf{P}^\perp}.$$

The Poisson summation formula

$$\mathbf{f}(\mathbf{P} + \mathbf{b}) = \frac{1}{|P^\perp|} \widehat{\mathbf{f}} \cdot \chi_{\mathbf{b}}(\mathbf{P}^\perp),$$

with $\mathbf{b} = \mathbf{0}$ and $\mathbf{f} = f^{\otimes n}$ gives the duality formula between the complete weight enumerator of $\mathbf{P}$ (with $x_a = f(a)$) and the complete weight enumerator of $\mathbf{P}^\perp$ (with $x_a = \widehat{f}(a)$). When $f = x\delta_0 + \delta_{Q\setminus 0}$ it yields MacWilliams duality formula for Hamming weight enumerators.

Recall that the graph $G = (V, E)$ has a fixed orientation of its edges, with directed edge set denoted by $\overrightarrow{E}$. Represent this ground orientation as a matrix γ indexed by $V \times E$, setting

$$\gamma_{v,e} = \begin{cases} +1 & \text{if } e = (u, v) \text{ in } \overrightarrow{E}, \\ -1 & \text{if } e = (v, u) \text{ in } \overrightarrow{E}, \\ 0 & \text{if } e \text{ is not incident with } v. \end{cases}$$

A *Q-tension* of G is a vector $\mathbf{a} \in Q^E$ comprising the differences between endpoints in a vertex colouring $\mathbf{c} \in Q^V$, i.e., if $e = (u, v)$ then the Q-tension $\mathbf{a}$ associated with $\mathbf{c}$ is defined by

$$a_e = \sum_{v \in V} \gamma_{v,e} c_v = c_v - c_u.$$

A *Q-flow* of G is a vector $\mathbf{b} \in Q^E$ such that, for each vertex v,

$$\sum_{e \in E} \gamma_{v,e} b_e = 0.$$

When G is planar, Q-flows of G correspond to Q-tensions of the planar dual graph G^*. In particular, when $Q = \mathbb{F}_2$, the $\mathbb{F}_2$-flows of G (cycles/ Eulerian subgraphs) correspond to $\mathbb{F}_2$-tensions (cutsets) of G^*.

A *nowhere-zero* Q-tension is one that takes non-zero values only and arises from a proper vertex Q-colouring; similarly, a nowhere-zero Q-flow is a flow takes non-zero values only (and for plane graphs corresponds to a proper face Q-colouring of the embedded graph).

If $\mathbf{P}$ is the set of Q-tensions of G (of which there are $q^{r(G)}$) then $\mathbf{P}^\perp$ is the set of Q-flows of G (of which there are $q^{n(G)}$). With this notation, the monochrome polynomial is given by

$$\sum_{\mathbf{c} \in Q^V} y^{\#\{(u,v) \in \overrightarrow{E} : c_u = c_v\}} = q^{k(G)} \sum_{\mathbf{a} \in \mathbf{P}} y^{|E| - |\mathbf{a}|},$$

since there are $q^{k(G)}$ vertex Q-colourings yielding any given Q-tension. Consequently, by Example 2.3.3, the Hamming weight enumerator of the set $\mathbf{P}$ of Q-tensions of G is a specialization of the Tutte polynomial:

$$\sum_{\mathbf{a} \in \mathbf{P}} y^{|E| - |\mathbf{a}|} = (x-1)^{r(G)} T(G; \frac{x - 1 + q}{x - 1}, x).$$

By the Poisson summation formula (MacWilliams duality),

$$(y\delta_0 + \delta_{Q\setminus 0})^{\otimes E}(\mathbf{P}) = q^{-n(G)}[(y-1+q)\delta_0 + (y-1)\delta_{Q\setminus 0}]^{\otimes E}(\mathbf{P}^{\perp}).$$

Putting $x = \frac{y-1+q}{y-1}$, the Hamming weight enumerator of the set $\mathbf{P}^{\perp}$ of Q-flows of G is given by

$$\sum_{\mathbf{b}\in\mathbf{P}^{\perp}} x^{|E|-|\mathbf{b}|} = (x-1)^{n(G)}T(G;x,\frac{x-1+q}{x-1}).$$

A corollary of Theorem 2.3.4 is that if a complete weight enumerator of Q-tensions (or Q-flows) is a Tutte–Grothendieck invariant (an evaluation of the Tutte polynomial with a certain simple type of prefactor) then it is in fact a Hamming weight enumerator. In fact, the proof of Theorem 2.3.4 says the same is true of any class of graphs that contains multiple loops on one vertex, multiple parallel edges between two vertices, and stars whose central vertex is of arbitrary degree. This notably includes the class of planar graphs.

There are nevertheless (infinite) classes of graphs for which an evaluation of the complete weight enumerator of Q-tensions of G coincides with the value of a Tutte–Grothendieck invariant and yet is not an evaluation of the Hamming weight enumerator.

For example, if $G = (V, E)$ is the line graph of a plane cubic graph then a result ultimately due to Penrose [16] (but see [8] for a full account) is that

$$\sum_{\mathbf{c}\in\mathbb{Z}_3^V} 0^{\#\{(u,v)\in\overrightarrow{E}:c_u=c_v\}}(-1)^{\#\{(u,v)\in\overrightarrow{E}:c_v-c_u=-1\}} = (-1)^{|V|}P(G;3),$$

i.e., the complete weight enumerator of $\mathbb{Z}_3$-tensions of G with $x_0 = 0, x_1 = 1, x_{-1} = -1$ is an evaluation of the Tutte polynomial. However, since the class of line graphs of plane cubic graphs is not closed under deletion or contraction, one is prevented from calling this a Tutte–Grothendieck invariant.

2.5 Polynomials akin to the graph polynomial

Suppose $F^{(q)}(G;\mathbf{x}) \in \mathbb{C}[x_v : v \in V]/(x_v^q - 1 : v \in V)$ is a graph polynomial of the general form

$$\begin{aligned} F^{(q)}(G;\mathbf{x}) &= \prod_{(u,v)\in\overrightarrow{E}} \sum_{(a,b)\in\mathbb{Z}_q^2} f(a,b)x_u^a x_v^b \\ &= \sum_{\mathbf{c}\in(\mathbb{Z}_q^2)^E} f^{\otimes E}(\mathbf{c}) \prod_{uv\in E} x_u^{c_{u,e}} x_v^{c_{v,e}}, \end{aligned}$$

where $\mathbf{c} = (c_e : e \in E)$, $c_e = (c_{u,e}, c_{v,e})$ and $f^{\otimes E}(\mathbf{c}) = \bigotimes_{e\in E} f(c_{u,e}, c_{v,e})$. The graph polynomial of Petersen et al. introduced in Section 2.2 is the case $f(1,0) = 1, f(0,1) = -1$ and $f(a,b) = 0$ otherwise. (Henceforth the name "Petersen's graph polynomial" will be used when it needs to be distinguished.)

In this section we address the following questions:

(A) When is the partition function of the vertex colouring (states) model [3]

$$\sum_{\mathbf{d}\in\mathbb{Z}_q^V} F^{(q)}(G;(\zeta^{d_v} : v \in V)\,) = q^{|V|}[\mathbf{x}^{\mathbf{0}}]F^{(q)}(G;\mathbf{x})$$

a Tutte–Grothendieck invariant (an evaluation of the Tutte polynomial)?

(B) When is the squared ℓ_2-norm

$$\|F^{(q)}(G;\mathbf{x})\|_2^2 = \sum_{\mathbf{a}\in\mathbb{Z}_q^V} \left|[\mathbf{x}^{\mathbf{a}}]F^{(q)}(G;\mathbf{x})\right|^2$$

a Tutte–Grothendieck invariant?

[3] The vertex colouring model assigns weight $F^{(q)}(G;(\zeta^{d_v} : v \in V)\,)$ to a given vertex colouring $\mathbf{d} \in \mathbb{Z}_q^V$. In terms of graph homomorphisms, this vertex colouring model corresponds to considering $\mathbf{d}$ as a homomorphism from G to a weighted directed graph H on vertex set $\mathbb{Z}_q$, with an edge (c,d) having weight

$$\sum_{a,b} f(a,b)\zeta^{ac+bd} = \widehat{f}(c,d).$$

The total weight of the homomorphism $\mathbf{d} : G \to H$ is the product of all the weights on (d_u, d_v) for edges (u,v) of G, i.e., $\widehat{f}^{\otimes E}(\mathbf{c})$ where $\mathbf{c} \in (\mathbb{Z}_q^2)^E$ is defined by $(c_{u,e}, c_{v,e}) = (d_u, d_v)$. The partition function in question (1) is a sum over all homomorphisms [$\mathbf{d}$, encoded by $\mathbf{c} \in (\mathbb{Z}_q^2)^E$] weighted in this way.

(C) What are the equivalents of Theorems 2.2.1 and 2.2.2 in this more general case?

By Parseval's formula,

$$\|F^{(q)}(G;\mathbf{x})\|_2^2 = q^{-|V|} \sum_{\mathbf{d}\in\mathbb{Z}_q^V} |F^{(q)}(\zeta^{d_v} : v \in V)|^2.$$

where, writing $\mathbf{c}$ for the vector with entries $(c_{u,e}, c_{v,e}) = (d_u, d_v)$,

$$|F^{(q)}(G;\zeta^{d_v} : v \in V)|^2 = |\widehat{f}(\mathbf{c})|^2.$$

Since $|\widehat{f}|^2 = \widehat{f \star f}$, this implies that the ℓ_2-norm of $F^{(q)}(G;\mathbf{x})$ is also the constant term of the polynomial $\widetilde{F}^{(q)}(G;\mathbf{x})$ in $\mathbb{C}[\mathbf{x}]/(x_v^q - 1 : v \in V)$ defined by

$$\widetilde{F}^{(q)}(G;\mathbf{x}) = \prod_{(u,v)\in\overrightarrow{E}} f \star f(a,b) x_u^a x_v^b.$$

For example, the ℓ_2-norm of the reduced graph polynomial

$$F^{(q)}(G;\mathbf{x}) = \prod_{(u,v)\in\overrightarrow{E}} (x_u - x_v) \bmod (x_v^q - 1 : v \in V)$$

is the constant term of the polynomial

$$\begin{aligned}|F^{(q)}(G;\mathbf{x})|^2 &= \prod |x_u - x_v|^2 = \prod (x_u - x_v)(x_u^{-1} - x_v^{-1}),\\ &= \prod (2 - x_u x_v^{-1} - x_u^{-1} x_v) \bmod (x_v^q - 1 : v \in V).\end{aligned}$$

(This uses the correspondence of the ideal $(x_v^q - 1 : v \in V)$ with the algebraic variety of points $(\zeta^{d_v} : v \in V)$, i.e., indeterminates x_v are roots of unity, for which complex conjugation is the same as taking the multiplicative inverse.)

Let $M = \{(a,a) : a \in Q\}$ be the submodule of $Q \times Q$ comprising monochromatic pairs. The orthogonal submodule is $M^\perp = \{(a,-a) : a \in Q\}$. By Theorem 2.3.4, $[\mathbf{x}^\mathbf{0}]F^{(q)}(G;\mathbf{x})$ is a Tutte–Grothendieck invariant if and only if there are constants y, w such that $\widehat{f} = y\delta_M + w\delta_{Q\times Q\setminus M}$. By the above remarks, $\|F^{(q)}(G;\mathbf{x})\|_2^2$ is a Tutte–Grothendieck invariant if and only if

$$\widehat{f \star f} = y\delta_M + w\delta_{Q\times Q\setminus M}.$$

By Fourier inversion, this is the case if and only if $f \star f = (y + (q-1)w)\delta_0 + (y - w)\delta_{M^\perp\setminus 0}$.

Proposition 2.5.1. The constant term of $F^{(q)}(G;\mathbf{x})$ is a Tutte–Grothendieck invariant if and only if $F^{(q)}(G;\mathbf{x})$ equals

$$\prod_{(u,v)\in\overrightarrow{E}}\left[y+(q-1)w+(y-w)(x_u^{q-1}x_v+\cdots+x_ux_v^{q-1})\right] \bmod (x_v^q-1 : v\in V),$$

in which case

$$[\mathbf{x^0}]F^{(q)}(G;\mathbf{x}) = (qw)^{n(G)}(y-w)^{r(G)}T(G;\frac{y-(q-1)w}{y-w},\frac{y}{w}).$$

For example, when $y=0, w=1$ and $q=3$ this says that

$$\prod_{(u,v)\in\overrightarrow{E}}(2-x_ux_v^2-x_u^2x_v) \bmod (x_v^3-1 : v\in V)$$

has constant term $3^{n(G)}(-1)^{r(G)}T(G;-2,0)=3^{|E|-|V|}P(G;3)$.

That the constant term of the polynomial defined in Proposition 2.5.1 is a Tutte polynomial evaluation can be seen by inspection since, for $a,b\in\mathbb{Z}_q$,

$$\zeta^{(q-1)a+b}+\zeta^{(q-2)a+2b}+\cdots+\zeta^{a+(q-1)b} = \begin{cases} -1 & a\neq b \\ q-1 & a=b, \end{cases}$$

so that in this case

$$\begin{aligned} q^{|V|}[\mathbf{x^0}]F^{(q)}(G;\mathbf{x}) &= \sum_{\mathbf{a}\in\mathbb{Z}_q^V} F^{(q)}(G;(\zeta^{a_v} : v\in V)) \\ &= \sum_{\mathbf{c}\in\mathbb{Z}_q^V}(qy)^{\#\{(u,v)\in\overrightarrow{E}:c_u=c_v\}}(qw)^{\#\{(u,v)\in\overrightarrow{E}:c_u\neq c_v\}}. \end{aligned}$$

Whereas Proposition 2.5.1 limits the number of graph polynomials which have a coefficient equal to an evaluation of the Tutte polynomial to a single family – giving a rather dull answer to question (A) above – the possible choices for f defining $F^{(q)}(G;\mathbf{x})$ so that the ℓ_2-norm $\|F^{(q)}(G;\mathbf{x})\|_2^2$ is a Tutte–Grothendieck invariant are unlimited – making the answer to question (B) potentially equally as dull. The criterion $|\widehat{f}|^2 = y\delta_M + w\delta_{Q\times Q\setminus M}$ [or $f\star f = (y+(q-1)w)\delta_0 + (y-w)\delta_{M^\perp\setminus 0}$] can be satisfied by taking $\widehat{f}=\sum_{a\in Q}z_{a,b}\delta_{(a,b)}$ for any complex numbers $z_{a,b}$ that satisfy $|z_{a,a}|^2=y$ if $a=b$ and $|z_{a,b}|^2=w$ otherwise.

Nonetheless, it seems worth describing a family of polynomials which contains Petersen's graph polynomial as a special case and in some sense naturally generalizes it. In this family it is also possible to give a meaningful answer to question (C) asking for equivalents to Theorems 2.2.1 and 2.2.2.

2.5.1 A family of polynomials containing the graph polynomial

Suppose $\mathrm{supp}(f) \subseteq \{(a,b) : a + sb = t\}$ for some constants $s, t \in \mathbb{Z}_q$. Then

$$\begin{aligned} F^{(q)}(G; (\zeta^{d_v} : v \in V)) &= \prod_{(u,v)\in \overrightarrow{E}} \sum_{(a,b)\in\mathbb{Z}_q^2} f(a,b)\zeta^{ad_u+bd_v} \\ &= \sum_{\mathbf{c}\in(\mathbb{Z}_q^2)^E} f^{\otimes E}(\mathbf{c}) \prod_{uv\in E} \zeta^{c_{u,e}d_u+c_{v,e}d_v}. \end{aligned}$$

The equation $f(a,b) = f(t-sb,b) =: g(b)$ defines $g \in \mathbb{C}^{\mathbb{Z}_q}$ and the sum over $\mathbf{c} \in (\mathbb{Z}_q^2)^E$ can be rewritten as a sum over $\mathbf{b} \in \mathbb{Z}_q^E$. In particular, $s = 1$ when the polynomial $\sum_{a,b} f(a,b)x_u^a x_v^b$ is homogeneous.

Given that $a_e + sb_e = t$, we have $a_e d_u + b_e d_v = (t - sb_e)d_u + b_e d_v = b_e(d_v - sd_u) + td_u$. For $e = (u,v) \in \overrightarrow{E}$, define $S : \mathbb{Z}_q^V \to \mathbb{Z}_q^E$ by

$$(S\mathbf{d})_e = d_v - sd_u$$

and $T : \mathbb{Z}_q^V \to \mathbb{Z}_q^E$ by

$$(T\mathbf{d})_e = td_u.$$

For $\mathbf{b} \in \mathbb{Z}_q^E$, the transpose $S^\top$ is given by

$$(S^\top\mathbf{b})_v = \sum_{e=(u,v)\in\overrightarrow{E}} b_e \;-\; s\sum_{e=(v,u)\in\overrightarrow{E}} b_e$$

and

$$(T^\top\mathbf{b})_v = t\sum_{e=(v,u)\in\overrightarrow{E}} b_e.$$

An important example is when $s = 1$ (which is the case for Petersen's graph polynomial). Here the linear transformation S is the *coboundary* and $S^\top$ the *boundary*. The submodule $\ker(S^\top)$ comprises the $\mathbb{Z}_q$-flows of G and $\mathrm{im}(S)$ the $\mathbb{Z}_q$-tensions of G.

We have

$$\begin{aligned} F^{(q)}(G;\mathbf{x}) &= \prod_{(u,v)\in\overrightarrow{E}} \sum_{b\in\mathbb{Z}_q} g(b)x_u^{t-sb}x_v^b \\ &= \sum_{\mathbf{b}\in\mathbb{Z}_q^E} \prod_{e=(u,v)\in\overrightarrow{E}} g(b_e)x_u^{t-sb_e}x_v^{b_e} \\ &= \sum_{\mathbf{b}\in\mathbb{Z}_q^E} g^{\otimes E}(\mathbf{b}) \prod_{v\in V} x_v^{S^\top\mathbf{b}+T^\top\mathbf{1}}, \end{aligned}$$

where $\mathbf{1}$ is the all-one vector in $\mathbb{Z}_q^V$. ($T^\top\mathbf{1}$ is t times the outdegree score of $\overrightarrow{E}$.)

The following theorem provides an answer to the question (A) posed in the previous section, and more.

Theorem 2.5.2. If $S^\top\mathbf{b} = \mathbf{a} - T^\top\mathbf{1}$ then

$$[\mathbf{x}^\mathbf{a}]F^{(q)}(G;\mathbf{x}) = g^{\otimes E}(\ker(S^\top) + \mathbf{b}),$$

a complete coset weight enumerator of $\ker(S^\top)$.

In particular, the coefficient $[\mathbf{x}^{T^\top\mathbf{1}}]F^{(q)}(G;\mathbf{x})$ is an evaluation of the complete weight enumerator of $\ker(S^\top)$ (and of $\mathrm{im}(S)$).

For example, in Petersen's graph polynomial, where $g = \delta_0 - \delta_1$,

$$[\mathbf{x}^{T^\top\mathbf{1}}] \prod_{(u,v)\in\overrightarrow{E}} (x_u - x_v) \bmod (x_v^q - 1 : v \in V) =$$

$$= \sum_{(q,1)\text{-flows } \mathbf{b}} 0^{\#\{e\in E: b_e=-1\}}(-1)^{\#\{e\in E: b_e=1\}},$$

where a $(q,1)$-flow is a $\mathbb{Z}_q$-flow taking values only in $\{0,1,-1\}$ (and here the sum need only range over those taking values in $\{0,1\}$).

When $s=1$ (for which S is the coboundary, $\mathrm{im}(S)$ the set of $\mathbb{Z}_q$-tensions, $\ker(S^\top)$ the set of $\mathbb{Z}_q$-flows) and

$$F^{(q)}(G;\mathbf{x}) = \prod_{(u,v)\in\overrightarrow{E}} \sum_{b\in\mathbb{Z}_q} g(b)x_u^{t-b}x_v^b,$$

the coefficient $[\mathbf{x}^{T^\top\mathbf{1}}]F^{(q)}(G;\mathbf{x})$ is an evaluation of the Tutte polynomial if and only if $g = x\delta_0 + \delta_{\mathbb{Z}_q\setminus 0}$ (by Theorem 2.3.4). If g does not take this form then the coefficient $[\mathbf{x}^{T^\top \mathbf{1}}]\overline{F}(\mathbf{x})$ is not a Hamming weight enumerator of $\mathbb{Z}_q$-flows but some other specialization of the complete weight enumerator.

To find the ℓ_2-norm, observe that, for $\mathbf{d} \in \mathbb{Z}_q^V$,

$$\begin{aligned} F^{(q)}(G; (\zeta^{d_v} : v \in V)\,) &= \sum_{\mathbf{b} \in \mathbb{Z}_q^E} g^{\otimes E}(\mathbf{b}) \zeta^{(S^\top \mathbf{b})\cdot \mathbf{d} + T^\top \mathbf{1} \cdot \mathbf{d}} \\ &= \sum_{\mathbf{b} \in \mathbb{Z}_q^E} g^{\otimes E}(\mathbf{b}) \zeta^{\mathbf{b}\cdot S\mathbf{d} + \mathbf{1}\cdot T\mathbf{d}} \\ &= \zeta^{\mathbf{1}\cdot T\mathbf{d}} \widehat{g}^{\otimes E}(-S\mathbf{d}), \end{aligned}$$

and

$$|F^{(q)}(G; (\zeta^{d_v} : v \in V)\,)|^2 = |\widehat{g}^{\otimes E}(-S\mathbf{d})|^2 = |\widehat{g}^{\otimes E}(S\mathbf{d})|^2.$$

By Parseval's formula,

$$\begin{aligned} \|F^{(q)}(G; \mathbf{x})\|_2^2 &= q^{-|V|} \sum_{\mathbf{d} \in \mathbb{Z}_q^V} |\widehat{g}^{\otimes E}(S\mathbf{d})|^2. \\ &= q^{-|V|} |\ker(S)| \sum_{\mathbf{b} \in \mathrm{im}(S)} (|\widehat{g}|^2)^{\otimes E}(\mathbf{b}). \end{aligned}$$

By the Poisson summation formula, and using

$$\mathrm{im}(S)^\perp = \ker(S^\top),\ |\ker(S)| = q^{|V|}/|\mathrm{im(S)}|,$$

we deduce the following, which provides an answer to question (C).

Theorem 2.5.3. If

$$F^{(q)}(G; \mathbf{x}) = \prod_{(u,v) \in \overrightarrow{E}} \sum_{b \in \mathbb{Z}_q} g(b) x_u^{t-sb} x_v^b,$$

then

$$\begin{aligned} \|F^{(q)}(G; \mathbf{x})\|_2^2 &= \frac{1}{|\mathrm{im}(S)|} \sum_{\mathbf{b} \in \mathrm{im}(S)} |\widehat{g}^{\otimes E}|^2(\mathbf{b}) \\ &= \sum_{\mathbf{b} \in \ker(S^\top)} (g \star g)^{\otimes E}(\mathbf{b}), \end{aligned}$$

where as usual $S : \mathbb{Z}_q^V \to \mathbb{Z}_q^E$ is defined by $(S\mathbf{d})_e = d_v - sd_u$ for $e = (u,v) \in \overrightarrow{E}$.

Example 2.5.4. Petersen's graph polynomial modulo $(x_v^q - 1 : v \in V)$ has $s = 1 = t$, $g = \delta_0 - \delta_1$, $g \star g = 2\delta_0 - \delta_1 - \delta_{-1}$. The transformation $S : \mathbb{Z}_q^V \to \mathbb{Z}_q^E$ is the coboundary operator, $S^\top$ the boundary, $\ker(S^\top)$ the set of $\mathbb{Z}_q$-flows of G. This gives Tarsi's result, Theorem 2.2.2, that the ℓ_2-norm of Petersen's graph polynomial modulo $(x_v^q - 1 : v \in V)$ is equal to

$$(-1)^{|E|} \sum_{\mathbf{b} \in \{-1,0,1\}^E \cap \ker(S^\top)} (-2)^{\#\{e \in E : b_e = 0\}},$$

where the sum is over $(q,1)$-*flows* of G.

Example 2.5.5. The polynomial

$$\prod_{uv \in E} (x_u + x_v)$$

is a generating function for score vectors of orientations of G, and as such its number of non-zero coefficients turns out to be equal to $T(G;2,1)$, the number of forests of G. (See for example [6]). By Theorem 2.5.3 with $g = \delta_0 + \delta_1, g \star g = 2\delta_0 + \delta_1 + \delta_{-1}$, and the expression of the Tutte polynomial as a Hamming weight enumerator of flows, when this polynomial is reduced modulo $(x_v^3 - 1 : v \in V)$ it has ℓ_2-norm equal to $T(G;2,4)$. Determining how many non-zero coefficients the polynomial has (its ℓ_0-norm) when reduced modulo $(x_v^q - 1 : v \in V)$ is closely related to the notion of $\mathbb{Z}_q$-*connectedness* of a graph as defined in [14].

Theorem 2.3.4 applied to the result of Theorem 2.5.3 has the following consequence, answering question (B).

Corollary 2.5.6. The ℓ_2-norm $\|F^{(q)}(G;\mathbf{x})\|_2^2$ of the polynomial defined in Theorem 2.5.3 is an evaluation of the Tutte polynomial $T(G;x,y)$ with $(x-1)(y-1) = q$ if and only if $s = 1$ and $g \star g$, equivalently $|\widehat{g}|^2$, is constant on $\mathbb{Z}_q \setminus 0$.

We finish with three examples of functions g satisfying the conditions of Corollary 2.5.6, yielding families of polynomials that have ℓ_2-norm equal to a Tutte–Grothendieck invariant.

A (q,k,ℓ)-*difference set* in an Abelian group Q is a subset P of size k with the property that $\#\{a, b \in P : a - b = c\} = \ell$ for each $c \in Q \setminus 0$. For example, $Q \setminus 0$ is a $(q, q-1, q-2)$-difference set: all non-zero c have exactly $q-2$ ways of being written as $a - b$ for $a, b \in Q \setminus 0$. (Given $a \in Q \setminus \{0, c\}$ there is a unique $b \in Q \setminus \{0, c\}$ with $a - b = c$.)

Note that a function is constant on non-zero values if and only if the same is true of its Fourier transform: if $f = t\delta_0 + \delta_{Q\setminus 0}$ then $\widehat{f} = (t-1+q)\delta_0 + (t-1)\delta_{Q\setminus 0}$. This fact, together with the equation $\delta_P \star \delta_P = \sum_{c\in Q} \#\{a, b \in P : a - b = c\}\delta_c$, implies that the Fourier transform $\widehat{\delta_P \star \delta_P} = |\widehat{\delta_P}|^2$ is constant on $Q \setminus 0$ if and only if P is a (q, k, ℓ)-difference set in Q, i.e., $\delta_P \star \delta_P = k\delta_0 + \ell\delta_{Q\setminus 0}$.

Example 2.5.7. If $g = \delta_P$ for some $P \subseteq \mathbb{Z}_q$, or more generally $g = \delta_P + r\delta_{\mathbb{Z}_q\setminus P}$ for any constant r, then $|\widehat{g}|^2$ is constant on $\mathbb{Z}_q \setminus 0$ if and only if P is a difference set in $\mathbb{Z}_q$. When $P = \mathbb{Z}_q \setminus 0$ this is the family of polynomials described in Proposition 2.5.1 whose constant terms were also Tutte–Grothendieck invariants.

A (q, k, ℓ, m)-*partial difference set* in Q is a subset P of size k with the property that $\delta_P \star \delta_P = k\delta_0 + \ell\delta_{P\setminus 0} + m\delta_{Q\setminus(P\cup 0)}$. For example, a subgroup P of size k is a $(q, k, k, 0)$-partial difference set.

Example 2.5.8. If $P \subseteq \mathbb{Z}_q \setminus 0$ and $g = \delta_P - \delta_{\mathbb{Z}_q\setminus(P\cup 0)}$ then then $|\widehat{g}|^2$ is constant on $\mathbb{Z}_q \setminus 0$ iff q is odd and P is a Paley difference set or partial difference set, i.e. $|P| = (q-1)/2$ and

$$\delta_P \star \delta_P = \begin{cases} \frac{q-1}{2}\delta_0 + \frac{q-5}{4}\delta_P + \frac{q-1}{4}\delta_{\mathbb{Z}_q\setminus(P\cup 0)} \\ \frac{q-1}{2}\delta_0 + \frac{q-3}{4}\delta_{\mathbb{Z}_q\setminus 0}, \end{cases}$$

according as $q \equiv \pm 1 \pmod 4$. (For odd prime q, the set of non-zero squares in $\mathbb{Z}_q$ is an example of such a P.)

Example 2.5.9. When q is prime and $g = \sum_{a\in\mathbb{Z}_q} \psi(a)\delta_a$ for a multiplicative character ψ of $\mathbb{Z}_q^\times$, then $|\widehat{g}|^2 = q\delta_{\mathbb{Z}_q\setminus 0}$, i.e., the polynomial

$$F^{(q)}(G; \mathbf{x}) = \prod_{(u,v)\in\overrightarrow{E}} \sum_{b\in\mathbb{Z}_q} \psi(b) x_u^{t-b} x_v^b$$

has ℓ_2-norm $q^{|E|-|V|}P(G; q)$. (The case $q = 3$ is Petersen's graph polynomial reduced modulo $(x_v^3 - 1 : v \in V)$.)

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Chapter 3

Anna de Mier: Patterns (mostly crossings and nestings) in ordered graphs

Several results on the distribution of crossings, nestings, and related patterns in set partitions have been recently extended to ordered graphs. We survey these generalizations and explore further relationships between graphs and set partitions.

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3.1 Introduction and main definitions

An ordered graph is a graph whose set of vertices is $[n]$; unless otherwise stated, isolated vertices and multiple edges are allowed. Let $m_G(i, j)$ be the number of edges that join the vertices i and j in the graph G. Given two ordered graphs $G = ([n], E_G)$ and $H = ([k], E_H)$, we say that G *contains* H if there is an increasing function $\phi = [k] \to [n]$ such that $m_H(i, j) \leq m_G(\phi(i), \phi(j))$ for all $i, j \in [k]$. If we ask that equality holds, we say that G

contains H as an induced subgraph. The graph H is referred to as a *pattern*. If $\mathcal{G}$ is a set of graphs, we denote by $\mathcal{EX}(H;\mathcal{G})$ the set of graphs of $\mathcal{G}$ that do not contain H; its cardinality is denoted by $\mathrm{ex}(H;\mathcal{G})$. If two graphs H and H' are such that $\mathrm{ex}(H;\mathcal{G}) = \mathrm{ex}(H';\mathcal{G})$, we will sometimes say that they are *equally restrictive*.

The patterns we are mainly interested in here are crossings and nestings. A *crossing* (respectively, a *nesting*) is the graph X_2 (resp., N_2) on 4 vertices and having as edges $\{1,3\}$ and $\{2,4\}$ (resp., $\{1,4\}$ and $\{2,3\}$). For a graph G, let $\mathrm{cr}(G)$ and $\mathrm{ne}(G)$ be the number of crossings and nestings of G, respectively (counted taking edge multiplicities into account). A graph is *noncrossing* (resp., *nonnesting*) if $\mathrm{cr}(G) = 0$ (resp., $\mathrm{ne}(G) = 0$).

The *left-right degree sequence* of an ordered graph on $[n]$ is the sequence $\big((d_i^l, d_i^r)\big)_{1\le i\le n}$, where d_i^l (resp., d_i^r) is the left (resp., right) degree of vertex i, that is, the number of edges that join i to a vertex j with $j < i$ (resp., $j > i$). If a graph G has D as its left-right degree sequence, we say that G is a graph *on D*. We denote by $\mathcal{G}(D)$ the set of all graphs on D.

A *P-graph* is a graph where $d_i^r \le 1$ and $d_i^l \le 1$ for all i. The "P" stands for "partition", since many results on crossings and nestings in set partitions are cast in terms of P-graphs through a construction that we will review in Section 3.3. It is well known that the number of noncrossing P-graphs on n vertices equals the number of nonnesting P-graphs (see for instance items (pp) and (uu) in [18, Exercise 6.19]). Actually, it is not hard to prove that the same holds for any degree sequence D.

It turns out that the equality of the numbers of noncrossing and nonnesting P-graphs is just a special case of the fact that the joint distribution of the numbers of crossings and nestings is symmetric, as stated in the next theorem. A bijective proof of this result in terms of partitions appears in [7].

Theorem 3.1.1. For every degree sequence $D \in \{(0,0),(0,1),(1,0),(1,1)\}^*$ the polynomial

$$\sum_{G\in\mathcal{G}(D)} x^{\mathrm{cr}(G)} y^{\mathrm{ne}(G)}$$

is symmetric.

In Section 3.2 we explore if the previous theorem can be extended to arbitrary degree sequences; more concretely, we show that the theorem does not hold in general, but that it does for a class of degree sequences that include those of P-graphs. In Section 3.3 we consider several possible definitions of pattern containment for set partitions and discuss their relationships. In the last section we review results that have been extended from P-graphs to arbitrary graphs through the use of fillings of Ferrers diagrams.

3.2 The numbers of crossings and nestings

The equality between the numbers of noncrossing and nonnesting P-graphs with a given degree sequence holds also if we consider an arbitrary degree sequence (see Section 3.4). For simple graphs on n vertices, it is known that among them there are as many noncrossing as nonnesting (see [10]), but this is no longer true if we also want to fix the degree sequence (see [14]).

As for the joint distribution, the statistic $(\mathrm{cr}, \mathrm{ne})$ is not symmetrically distributed over the set of graphs $\mathcal{G}(D)$ for an arbitrary degree sequence D. Some examples are the sequences $(0,2),(0,1),(1,0),(2,0)$ and $(0,1),(0,2),(1,0),(0,1),(2,0),(1,0)$. The symmetry does not hold either over the set of all graphs with a given number of vertices and edges, as can be checked computationally for 5 vertices and 5 edges, for instance.

In this section we present a superclass of P-graphs over which the numbers of crossings and nestings have a symmetric joint distribution. We start with a simple observation that will also be used implicitly in Section 3.4.

Let G be a graph on n vertices and let $e_G(i,j)$ denote the number of edges that join vertices i and j. Let k be a vertex of G with $d_k^l \neq 0, d_k^r \neq 0$ and define G' as the graph on $[n+1]$ vertices where for $i<j$

$$e_{G'}(i,j) = \begin{cases} e_G(i,j) & \text{if } i<j\leq k; \\ e_G(i-1,j-1) & \text{if } k<i<j; \\ e_G(i,j-1) & \text{if } i<k<j-1; \\ 0 & \text{otherwise.} \end{cases}$$

Graphically, this corresponds to splitting the vertex k into two consecutive vertices, the first of them joined to the left neighbours of k and the second one joined to the right neighbours. It is clear that G and G' have the same numbers of crossings and nestings, and in fact the same number of occurrences of any pattern of a more general form, as stated next. We say that a graph is *split* if no vertex has simultaneously non-zero left and right degree and if all vertices with non-zero right degree are smaller than all vertices with non-zero left degree.

Lemma 3.2.1. Let H be a split graph. Then for every pair of graphs G, G' as defined above, the numbers of occurrences of H in G and in G' are equal.

We now show that the distribution of the numbers of crossings and nestings is symmetric over $\mathcal{G}(D)$ if D is a degree sequence where left degrees are at most one. Graphs with this kind of degree sequence will be called *S-graphs* (note that S-graphs are a superset of P-graphs).

Theorem 3.2.1. Let D be a degree sequence with left degrees equal to zero or one. Then the polynomial

$$\sum_{G \in \mathcal{G}(D)} x^{\mathrm{cr}(G)} y^{\mathrm{ne}(G)}$$

is symmetric.

Sketch of the proof. By symmetry we prove the statement for graphs with right degrees at most one. By Lemma 3.2.1 it is enough to consider degree sequences D whose elements are in $\{(d,0) : d > 0\} \cup \{(0,1)\}$ (isolated vertices are easy to handle).

We define an involution on the set of graphs on D that interchanges the numbers of crossings and nestings. We refer to a vertex with right degree one as an "opener" and to a vertex with non-zero left degree as a "closer". First we associate to each graph a sequence of subsets from which we can compute the numbers of crossings and nestings. We sweep the graph from left to right and every time we encounter a closer i, we compute how many unclosed openers are there to its left; that is $op_G(i) = \sum_{j<i} d^r_j - d^l_j$. Of the $op_G(i)$ openers available, there will be d^l_i that are joined to vertex i. Associate to i the subset $\chi_G(i)$ of $[op_G(i)]$ of size d^l_i corresponding to the positions of the vertices that are adjacent to i among the $op_G(i)$ openers available (the positions are counted from left to right). It is easy to determine the numbers of crossings and nestings of G from the knowledge of the sets $\chi_G(i)$. Then one can show that to find a graph $\varphi(G)$ where these numbers are interchanged it suffices to take $\chi_{\varphi(G)}(i) = \{op_G(i) - s + 1 : s \in \chi_G(i)\}$. □

The papers [11, 15] contain stronger results about the symmetry of the distribution of crossings and nestings in P-graphs. In [7] there is an expression as a continued fraction for the generating function of the numbers of crossings and nestings in P-graphs. We do not know of any results similar to these for the class of S-graphs.

3.3 Pattern avoidance in set partitions

In this section we use the S-graphs that appeared in the previous section to define a new notion of pattern containment for set partitions. We state without proofs (which are generally easy) some properties of this type of containment and its relationship with previously existing ones.

Let $\pi = B_1|B_2|\ldots|B_m$ be a partition of $[n]$ with blocks listed in increasing order of their smallest elements. For $i \in [n]$, denote by $B_\pi(i)$ the block of π containing i. Let σ be a partition of $[m]$. Our first definition of containment of partitions is perhaps the most natural one. We say that π *contains* σ if there exists an increasing function $\varphi : [m] \to [n]$ such that $B_\sigma(i) = B_\sigma(j)$ if and only if $B_\pi(\varphi(i)) = B_\pi(\varphi(j))$. This notion has been studied for instance in [8, 9, 17]. Another definition recently proposed by Sagan [17] asks that the relative order of the blocks is preserved. The partition π *R-contains* σ if there is an increasing function $\psi : [m] \to [n]$ such that $B_\sigma(i) = B_\sigma(j)$ if and only if $B_\pi(\psi(i)) = B_\pi(\psi(j))$ and $\min(B_\sigma(i)) < \min(B_\sigma(j))$ if and only if $\min(B_\pi(\psi(i))) < \min(B_\pi(\psi(j)))$. This type of containment is also studied in [4, 6].

The other two definitions are in terms of P-graphs and S-graphs. Given a partition π of $[n]$, let the graph $P(\pi)$ be the graph on $[n]$ where the subgraph induced by the elements of each block of π is a monotone path (see Figure 3.1). We say that π *P-contains* σ if $P(\pi)$ contains $P(\sigma)$ as an induced subgraph. Similarly, construct the graph $S(\pi)$ on $[n]$ by taking each block $i_1 < i_2 < \cdots < i_k$ of π and adding the edges $\{i_1, i_2\}, \{i_1, i_3\}, \ldots, \{i_1, i_k\}$ (see Figure 3.1). Then we say that π *S-contains* σ if $S(\pi)$ contains $S(\sigma)$ as an induced subgraph. In both definitions we require that the containment is induced to make sure that elements in different blocks are not mapped to the same block. The notion of P-containment is the one that has been used usually to define crossings and nestings in partitions. It is easy to show that being noncrossing is equivalent for all four kinds of containment, but being nonnesting is not (probably this is one of the reasons that the concept of nonnesting partition is less popular). A partition $\star$-contains a crossing (respectively, nesting) if it $\star$-contains the partition $13|24$ (resp., $14|23$), where $\star$ stands for any of $\{P, R, S\}$ or for nothing.

Proposition 3.3.1. (i) The partition π contains a crossing if and only if it $\star$-contains a crossing, for any $\star \in \{P, R, S\}$.

(ii) The numbers of P-nonnesting, R-nonnesting and S-nonnesting partitions of $[n]$ are all equal to C_n, the n-th Catalan number. The numbers

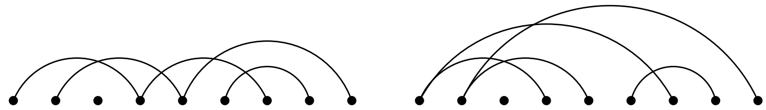

Figure 3.1: The graphs $P(\pi)$ and $S(\pi)$ for $\pi = 147|259|3|68$.

of P-nonnesting and nonnesting partitions are different.

Observe that part (ii) above does not state that the sets of P-nonnesting and S-nonnesting partitions are the same, as happens with crossings. Indeed, the partition $134|25$ is S-nonnesting but it does P-contain a nesting. It is not the case either that the number of partitions containing k crossings is the same as those P-containing k crossings. By Theorem 3.2.1 we know that for P-containment and S-containment the distribution of the numbers of crossings and nestings is symmetric, but the exact values are different; for instance, there are two partitions of $[5]$ that S-contain two nestings and no crossing but only one that P-contains two nestings and no crossing. Also, the knowledge of the degree sequence in the case of P-containment translates to knowing which are the smallest and largest elements of the blocks, and in the case of S-containment to knowing for each block the smallest element and the size of the block.

The relationships between the four kinds of containment are as follows.

Proposition 3.3.2. Let π and σ be partitions. Then:

(i) if π P-contains σ, then π contains σ;

(ii) if π S-contains σ, then π R-contains σ;

(iii) if π R-contains σ, then π contains σ.

All these implications are strict and no other implication holds for all π, σ.

The equivalence between R-containment and S-containment for crossings and nestings in Propositon 3.3.1 is a particular instance of the following result.

Proposition 3.3.3. Let σ be a partition with k blocks such that the first element of the i-th block is i, and let π be any partition. Then π R-contains σ if and only if π S-contains σ.

Therefore, for many patterns R-containment can be expressed as containment in S-graphs.

3.4 Generalizing from P-graphs

We survey in this section other instances of extensions of a result about P-graphs to arbitrary graphs. Again, the patterns that have received most attention are related to crossings and nestings. Consider the graphs X_k and N_k consisting of k pairs of mutually crossing and nested edges, respectively; they are usually called *k-crossings* and *k-nestings*. The following theorem of Chen et al. [3] has inspired much subsequent research; it was proved in the context of set partitions but we state it in terms of P-graphs.

Theorem 3.4.1. For every degree sequence $D \in \{(0,0),(0,1),(1,0),(1,1)\}^*$ and for every $k, l \geq 2$, $\text{ex}(\{X_k, N_l\}, \mathcal{G}(D)) = \text{ex}(\{X_l, N_k\}, \mathcal{G}(D))$.

This result cannot be generalized to arbitrary degree sequences (in [13] there is a counterexample for the case $k = 2$ and $l = 3$). It is conjectured also in [13] that the symmetry holds for S-graphs with n vertices fixing the subset of vertices that have left degree one (but not fixing right degrees). It seems that this conjecture has been recently proved [2].

Nevertheless, a weak version of Theorem 3.4.1 holds for all degree sequences. This result has been obtained by establishing a correspondence between graphs and fillings of Ferrers diagrams of integer partitions. We refer to [14, Section 3] for a detailed account of the following discussion. A filling of a Ferrers diagram consists of assigning a non-negative integer to each cell of the Ferrers diagram. In particular, matrices can be seen as fillings of rectangular diagrams, and one can define in a natural way what it means for a filling to contain a given matrix. Roughly speaking, a filling contains an $r \times s$ matrix of zeros and ones if there is a selection of r rows and s columns of the diagram such that for entries equal to one in the matrix the corresponding cells in the diagram are non-zero. There is a bijective correspondence that assigns a filling $F(G)$ of a Ferrers diagram to each graph G without vertices with both left and right degrees positive. Under this bijection left degrees (resp., right degrees) are mapped to row (resp., column) sums of the filling; also, the sequence of opening and closing vertices can be recovered from the shape of the diagram. The case where the diagram is rectangular corresponds to graphs where all opening vertices appear before all closing ones, that is, to split graphs. The following theorem says that avoiding split graphs in ordered graphs and avoiding matrices in fillings of diagrams is essentially the same problem.

Theorem 3.4.2. Let H and H' be two split graphs. Then $\text{ex}(H; \mathcal{G}(D)) = \text{ex}(H'; \mathcal{G}(D))$ for all degree sequences D if and only if for each diagram with prescribed row and column sums there are as many fillings avoiding $F(H)$ as avoiding $F(H')$.

We have that $F(N_k) = I_k$ and $F(X_k) = J_k$, where I_k and J_k are the identity and the antiidentity matrices, respectively. In [1] it is shown that I_k and J_k are equally restrictive for fillings whose row and column sums are equal to one. This gives a proof of Theorem 3.4.1 when we sum for all values of l. The papers [12, 14] contain two different proofs that the matrices I_k and J_k are as hard to avoid in fillings of diagrams with arbitrary but fixed row and column sums; this immediately gives the following theorem, which in the light of the previous remarks is the best possible extension of Theorem 3.4.1.

Theorem 3.4.3. For any k and any D, $\mathrm{ex}(X_k; \mathcal{G}(D)) = \mathrm{ex}(N_k; \mathcal{G}(D))$.

This result actually gives infinitely many pairs of graphs that are equally restrictive, in the following way. Let H be a split graph on $[h]$ and let k be a positive integer. The graph $X_k(H)$ is the graph on $[2k + h]$ such that the graph induced by the vertices $\{k + 1, \dots, k + h\}$ is H and the graph induced by $[k] \cup \{k + h + 1, \dots, 2k + h\}$ is X_k; the graph $N_k(H)$ is defined analogously. We have the following corollary (see [14] for details).

Corollary 3.4.4. For any split graph H, any positive integer k, and any degree sequence D, we have that $\mathrm{ex}(X_k(H), \mathcal{G}(D)) = \mathrm{ex}(N_k(H), \mathcal{G}(D))$.

In particular, all the graphs $X_t(N_k)$ are as restrictive as the graph N_{t+k}, and hence as X_{t+k}. However, the graph $N_1(X_2)$ is not as restrictive as the graph N_3 (see [5]). It would be interesting to classify the graphs $N_k(X_t)$ from less to more restrictive. To our knowledge, there are no other pairs of graphs that are known to be equally restrictive for all degree sequences.

The above results restricted to S-graphs can be translated as results about S-containment (and thereby R-containment) of large crossings and nestings in set partitions. The paper [6] also uses fillings of Ferrers diagrams (but in a different way) to prove some more general results about R-containment of patterns related to crossings and nestings.

It is not always the case that if two patterns are equirestrictive with respect to P-containment they are also for S-containment. An example are the partitions 15|24|36 and 14|26|35 (see [5] and the numbers in the appendix of [6]).

Finally, let us discuss what happens if we restrict to simple graphs. In this case, we can only fix left degrees (or right degrees, by symmetry) to have that k-crossings and k-nestings are equally restrictive. The following is a consequence of Theorem 5.2 of Rubey [16] on fillings of moon polyominoes.

Theorem 3.4.5. Let L be a sequence and let $\mathcal{G}_L^s$ be the set of simple graphs whose left degrees are given by L. Then for all $k \geq 2$, $\mathrm{ex}(X_k; \mathcal{G}_L^s) = \mathrm{ex}(N_k; \mathcal{G}_L^s)$.

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Chapter 4

Abstracts of lectures

4.1 Hong Van Le: Lower bounds for determinantal complexity

Abstract. Let P be a polynomial over a vector space F^n, where F is a field. According to a theorem of Valiant if the formula size of P equals m, then there exists an affine map $A : F^n \to Mat(F, 2m+2)$ such that $P = A^*(\det_{2m+2})$, where $\det_{2m+2}$ is the determinantal polynomial on $Mat(F, 2m+2)$. We define the determinantal complexity $c_d(P)$ to be the smallest number m such that there exists an affine map $A : F^n \to Mat(F, m)$ with the property $P = A^*(\det_m)$. In my talk I discuss several ways to obtain a lower bound for the determinantal complexity of a polynomial P on the vector space over a field F of characteristic 0. In particular I explain the Mignon-Ressayer lower bound $c_d(Perm_n) \geq n^2/2$, here $Perm_m$ denotes the permanent polynomial on $Mat(F, m)$. I also suggest a geometric way to reformulate the lower bound problem for $c_d(P)$ by associating P with an point $P^\phi \in S^m(Mat(F, m))$ and then to verify if this point P^ϕ lies in an algebraic variety in $S^m(Mat(F, m))$. Here $S^m(Mat(F, m))$ denotes the space of all homogeneous polynomials of degree m on $Mat(F, m)$. This geometric way is a version of the Mumuley-Sohoni approach to find a lower bound for the determinantal complexity.

4.2 Martin Klazar: Meanders

Abstract. Meanders are combinatorial structures which can be defined geometrically by closed plane curves intersecting in $2n$ points a fixed line or more combinatorially by noncrossing matchings. I will survey some results on enumeration of meanders. In particular, I will sketch proofs for lower and upper bounds on the exponentially growing numbers of meanders.

4.3 Mihyun Kang, Martin Loebl: The enumeration of planar graphs via Wick's theorem

Abstract. A seminal technique of theoretical physics called *Wick's theorem* interprets the Gaussian matrix integral of the products of the trace of powers of Hermitian matrices as the number of labelled *maps* with a given degree sequence, sorted by their Euler characteristics. This leads to the map enumeration results analogous to those obtained by combinatorial methods. We survey this classical theorems and present our new results: we show that the enumeration of the graphs *embeddable* on a given 2-dimensional surface (a main research topic of contemporary enumerative combinatorics) can also be formulated as the Gaussian matrix integral of an ice-type partition function. Some of the most puzzling conjectures of discrete mathematics are related to the notion of the *cycle double cover*. We express the number of the graphs with a fixed *directed cycle double cover* as the Gaussian matrix integral of an Ihara-Selberg-type function. (Preprint is available on our webpages.)

Chapter 5

Problems (and solutions)

5.1 Mihyun Kang: unlabelled planar graphs

Let p_n be the number of unlabelled planar graphs on n vertices. One can pose quite a few problems concerning these numbers.
1. What is the asymptotics of p_n as $n \to \infty$? If $p(z) = \sum_{n \geq 0} p_n z^n$, what is the radius of convergence of $p(z)$?
2. Is there a polynomial algorithm for computing numbers p_n?
3. Is there a polynomial algorithm to sample random unlabelled planar graph?

5.2 Martin Klazar: asymptotics of meanders

Recall that a meander of size n is a pair (M, N) of two noncrossing matchings on the vertex set $[2n] = \{1, 2, \dots, 2n\}$ such that $M \cup N$ is a connected 2-regular graph on $[2n]$ (i.e., a cycle of length $2n$). Let m_n be the numbers of meanders with size n. What is the limit $\lim_{n\to\infty} m_n^{1/n}$? What is the asymptotics on m_n as $n \to \infty$? Is there a polynomial (i.e., polynomial in n) algorithm for calculating the function $n \mapsto m_n$?

It is not hard to show that the limit $\mu = \lim_{n\to\infty} m_n^{1/n}$ exists and is finite. The best current bounds are $11.380 < \mu < 12.901$ by Albert and Paterson (*J. Combin. Theory, Ser. A* **112** (2005), 250–262).

5.3 Martin Loebl: A polynomial matrix related to partitions of n

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \lambda_k \geq 1)$ of a number $n \in \mathbb{N} = \{1, 2, \dots\}$ (so $\lambda_i \in \mathbb{N}$ and $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$) and a number $l \in \mathbb{N}$, we define the polynomial

$$v_{\lambda,l}(q) := \prod_{i=1}^{k}(1 + q^{\lambda_i} + q^{2\lambda_i} + \dots + q^{l\lambda_i}) = \prod_{i=1}^{k} \frac{q^{(l+1)\lambda_i} - 1}{q^{\lambda_i} - 1}$$

with degree ln. Let

$$V = V(n) := \big(v_{\lambda,l}(q)\big) \in \mathbb{Z}[q]^{P(n)\times\mathbb{N}}$$

be the $p(n) \times \infty$ matrix of these polynomials, where λ ranges over the set $P(n)$ of $p(n)$ partitions of n and l ranges over $\mathbb{N}$. Are the $p(n)$ rows of V linearly independent over the field $\mathbb{Q}$?

I am really impressed by a beautiful negative solution of Martin Klazar to this problem, presented in the next section. I was sure the conjecture was true!! The conjecture is included in my paper on the q-chromatic function, Advances in Mathematics 2007.

Let me include here my motivation for posting the problem, and write down a weaker conjecture which would still serve my purpose; as far as I know, the method of Martin of the next section is in principle not applicable for its rejection.

Let $G = (V, E)$ be a graph and n a positive integer. Let $V = \{1, \dots, n\}$ and for $k \in \mathbb{N} = \{1, 2, \dots\}$ let $V(G, k)$ denote the set of all vertex colourings $s : V \to \{0, \dots, k-1\}$ such that $s(u) \neq s(v)$ whenever $uv \in E$. We define the *q-chromatic function* as follows.

$$M_q(G, k) = \sum_{s \in V(G,k)} q^{\sum_{v \in V} s(v)}.$$

Note that $M_q(G, k)|_{q=1}$ is the classical chromatic polynomial of G. We recall some notation:

For $k \in \mathbb{N}$, let $(k)_q = q^{k-1} + \dots + q + 1$ denote a *quantum integer*, with the convention that $(0)_q = 0$, and let $(k)!_q = \prod_{1 \leq n \leq k}(n)_q$, with the convention that $(0)!_q = 1$. For $0 \leq n \leq k$ the *quantum binomial coefficients* are defined by

$$\binom{k}{n}_q = \frac{(k)!_q}{(n)!_q (k-n)!_q}.$$

Example: a simple quantum binomial formula

$$(a-z)(a-qz)\dots(a-q^{k-1}z) = \sum_{i=0}^{k} (-1)^i \binom{k}{i}_q q^{i(i-1)/2} a^{k-i} z^i$$

leads to a well-known formula for the summation of the products of distinct powers. This gives the q-chromatic function for the complete graph.

Observation 5.3.1. For $k \in \mathbb{N}$, the q-chromatic function of the complete graph on $n \leq k$ vertices is given by

$$M_q(K_n, k) = n! \binom{k}{n}_q q^{n(n-1)/2}$$

and $M_q(K_n, k) = 0$ for $n > k$.

The next theorem provides a natural way to extend the q-chromatic function from positive integers to real numbers, by extending quantum integers $(k)_q$ to *quantum numbers* $(y)_q = \frac{q^y - 1}{q-1}$ for real variables y and $q \neq 1$ (and $(y)_1 = y$ by continuity, $\lim_{q \to 1}(y)_q = y$.)

A graph $G = (V, E)$ is *connected* if it has a path between any pair of vertices. If a graph is not connected then its maximum connected subgraphs are called *connected components*. If $G = (V, E)$ is a graph and $A \subset E$ then let $C(A)$ denote the set of the connected components of graph (V, A) and $c(A) = |C(A)|$. If $W \in C(A)$ then let $|W|$ denote the number of vertices of W.

Theorem 5.3.2. For $k \in \mathbb{N}$,

$$M_q(G, k) = \sum_{A \subset E} (-1)^{|A|} \prod_{W \in C(A)} (k)_{q^{|W|}}.$$

Proof. We use the principle of inclusion and exclusion (PIE): If $\{I_e : e \in E\}$ are finite sets indexed by a finite set E and $\cap_{e \in A} I_e =: I_A$ then

$$|\cup_{e \in E} I_e| = \sum_{1 \leq a \leq |E|} (-1)^{a-1} \sum_{A \subset E, |A| = a} |I_A|.$$

The next considerations connect the PIE with the geometric series formula.

$$M_q(G,k) = \sum_{s:V\to\{0,\dots,k-1\}} q^{\sum_{v\in V} s(v)} - \sum_{s\in\cup_{e\in E} I_e} q^{\sum_{v\in V} s(v)},$$

where $I_e, e = uv \in E$, denotes the set of functions $s : V \to \{0,\dots,k-1\}$ for which $s(u) = s(v)$.

By PIE this equals

$$\sum_{A\subset E} (-1)^{|A|} \sum_{s\in I_A} q^{\sum_v s(v)} =$$

$$\sum_{A\subset E} (-1)^{|A|} \prod_{W\in C(A)} \sum_{0\le i\le k-1} q^{i|W|} = \sum_{A\subset E} (-1)^{|A|} \prod_{W\in C(A)} (k)_{q^{|W|}}.$$

□

The following function called *dichromate* is extensively studied in combinatorics. It is equivalent to the Tutte polynomial.

$$B(G,x,y) = \sum_{A\subset E} x^{|A|} y^{c(A)}.$$

The formula of Theorem 5.3.2 leads naturally to a definition of *q-dichromate*.

Definition 5.3.3. For variables x, y with $y \in \mathbb{R}$,

$$B_q(G,x,y) = \sum_{A\subset E} x^{|A|} \prod_{W\in C(A)} (y)_{q^{|W|}}.$$

Note that $B_{q=1}(G,x,y) = B(G,x,y)$ and $M_q(G,k) = B_q(G,-1,k)$ by Theorem 5.3.2.

Let $x_1, x_2, \dots$ be commuting indeterminates and let $G = (V,E)$ be a graph. The q-chromatic function $M_q(G,y)$ restricted to non-negative integer y is the principal specialization of X_G, the *symmetric function generalisation of the chromatic polynomial* defined by Stanley:

Definition 5.3.4.

$$X_G(x_0,x_1,\dots) = \sum_{s\in\cup_{k\in\mathbb{N}_0} V(G,k)} \prod_{v\in V} x_{s(v)},$$

the sum over all proper colourings of G by $\{0,1,\dots\}$.

Therefore $M_q(G,k) = X_G(x_i = q^i(0 \leq i \leq k-1), x_i = 0(i \geq k))$.

Stanley further defined *symmetric function generalisation of the bad colouring polynomial*:

Definition 5.3.5.

$$XB_G(t, x_0, x_1, \dots) = \sum_{s:V \to \{0,1,\dots\}} (1+t)^{b(s)} \prod_{v \in V} x_{s(v)},$$

where the sum ranges over ALL colourings of G by $\{0,1,\dots\}$ and $b(s) := |\{uv \in E : s(u) = s(v)\}|$ denotes the number of monochromatic edges of f.

Noble and Welsh defined the *U-polynomial* and showed that it is equivalent to XB_G. I. Sarmiento proved that the polychromate defined by Brylawski is also equivalent to the U-polynomial.

Definition 5.3.6.

$$U_G(z, x_1, x_2 \dots) = \sum_{A \subset E(G)} x(\tau_A)(z-1)^{|A|-|V|+c(A)},$$

where $\tau_A = (n_1 \geq n_2 \geq \dots \geq n_l)$ is the partition of $|V|$ determined by the connected components of A, $x(\tau_A) = x_{n_1} \dots x_{n_l}$.

The following observation is straightforward.

Observation 5.3.7. For $k \in \mathbb{N}$,

$$B_q(G, z-1, y) = (z-1)^{|V|} U_G(z, x_1, x_2, \dots)|_{x_i = (z-1)(y)_{q^i}}.$$

On the other hand, it seems plausible that the q-dichromate determines the U-polynomial. If true, then the q-dichromate would provide a compact representation of the multivariate generalisations of the Tutte polynomial mentioned above.

It is not difficult to observe that the conjecture I posted in the workshop would imply that the q-dichromate determines the U-polynomial. In fact, the following weaker conjecture still implies the same statement.

Let $\tau = (n_1 \geq n_2 \geq \dots n_k)$ be a partition of n. We let $w(\tau)$ be the following function of two variables q, y:

$$w(\tau, q, y) = \prod_{i=1}^{k} (y)_{q^{n_i}}.$$

Conjecture 5.3.8. Only trivial rational linear combination of $w(\tau, q, y)$'s is identically zero.

5.4 Martin Klazar: negative solution of the problem of Martin Loebl for large n

I will prove two results. 1. For $n \geq 34$ the answer is negative, the rows of $V(n)$ are linearly dependent over $\mathbb{Q}$. 2. On the other hand, for $n \leq 6$ the rows are linearly independent.

1. Linear dependence for $n \geq 34$. We consider the generating function

$$R_\lambda(q,x) := \sum_{l\geq 0} v_{\lambda,l}(q)x^l \in \mathbb{Z}[q][[x]]$$

with the entries of the λ-th row as coefficients of x^l. We have

$$\begin{aligned}
R_\lambda(q,x) &= \sum_{l\geq 0} x^l \prod_{i=1}^{k} \frac{1-q^{\lambda_i(l+1)}}{1-q^{\lambda_i}} = \prod_{i=1}^{k} \frac{1}{1-q^{\lambda_i}} \sum_{l\geq 0} x^l \prod_{j=1}^{k}(1-q^{\lambda_j(l+1)}) \\
&= \prod_{i=1}^{k} \frac{1}{1-q^{\lambda_i}} \sum_{l\geq 0} x^l \sum_{I\subset[k]} (-1)^{|I|} q^{(l+1)\sum_{j\in I}\lambda_j} \\
&= \prod_{i=1}^{k} \frac{1}{1-q^{\lambda_i}} \sum_{I\subset[k]} (-1)^{|I|} q^{\sum_{j\in I}\lambda_j} \sum_{l\geq 0} \left(xq^{\sum_{j\in I}\lambda_j}\right)^l \\
&= \prod_{i=1}^{k} \frac{1}{1-q^{\lambda_i}} \sum_{I\subset[k]} (-1)^{|I|} \frac{q^{\sum_{j\in I}\lambda_j}}{1-xq^{\sum_{j\in I}\lambda_j}}.
\end{aligned}$$

For further reference we denote

$$r_\lambda(q) := \prod_{i=1}^{k} \frac{1}{1-q^{\lambda_i}}.$$

The roots of the denominator of $r_\lambda(q)$ are the primitive m-th roots of unity $\alpha = \exp(2r\pi i/m)$, $0 \leq r < m$, $(r,m) = 1$, $m \leq n$, and have multiplicity

$$s = \sum_{i,\ m|\lambda_i} 1 \leq \left\lfloor \frac{n}{m} \right\rfloor.$$

Decompositions into partial fractions show that every $r_\lambda(q)$ is a $\mathbb{C}$-linear combination of the rational functions from the set S defined as

$$\{(1-\alpha q)^{-s} \mid \alpha = \exp(\frac{2r\pi i}{m}), 0 \leq r < m, (r,m) = 1, 1 \leq s \leq \frac{n}{m}, 1 \leq m \leq n\}$$

(constant term does not occur in any decomposition, so $s = 0$ may be omitted). The cardinality of this set is

$$|S| = \sum_{m=1}^{n} \varphi(m) \left\lfloor \frac{n}{m} \right\rfloor = \sum_{m=1}^{n} \varphi(m) \sum_{e,\ em \le n} 1 =$$

$$= \sum_{j=1}^{n} \sum_{m|j} \varphi(m) = \sum_{j=1}^{n} j = \frac{(n+1)n}{2}$$

(we used the identity $\sum_{m|j} \varphi(m) = j$). So every $R_\lambda(q,x)$ is a $\mathbb{C}$-linear combination of the rational functions from the $(n+1)^2 n/2$-element set

$$T := \{(1-\alpha q)^{-s} q^t (1-xq^t)^{-1} \mid \alpha, s \text{ are as in } S \text{ and } 0 \le t \le n\}.$$

If $p(n) > (n+1)^2 n/2$, the number of these linear combinations exceeds the dimension and there exist numbers $\alpha_\lambda \in \mathbb{C}$, $\lambda \in P(n)$, which are not all zero and such that

$$\sum_{\lambda \in P(n)} \alpha_\lambda \cdot R_\lambda(q,x) \equiv 0.$$

This means that

$$\sum_{\lambda \in P(n)} \alpha_\lambda \cdot v_{\lambda,l}(q) \equiv 0 \quad \text{for every } l \in \mathbb{N}.$$

Because $v_{\lambda,l}(q)$ have integral coefficients, we can in fact take α_λ in $\mathbb{Q}$ and so in $\mathbb{Z}$.

As $p(n) > (n+1)^2 n/2$ for $n \ge 40$, for these n the rows of $V(n)$ are linearly dependent over $\mathbb{Q}$. We can do a little better by noting that only half of the partial fractions are needed to express the rational functions $r_\lambda(q)$. Since $r_\lambda(q)$ have real coefficients, the partial fractions $(1-\alpha q)^{-s}$ and $(1-\overline{\alpha} q)^{-s}$ have the same coefficient in the linear combination expressing $r_\lambda(q)$. We can replace the set S by the real generating set

$$S' := \{(1-\alpha q)^{-s} + (1-\overline{\alpha} q)^{-s} \mid \alpha, s \text{ are as in } S\}.$$

Its cardinality is

$$|S'| = n + \left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{2} \sum_{m=3}^{n} \varphi(m) \left\lfloor \frac{n}{m} \right\rfloor = \frac{n + \lfloor n/2 \rfloor + |S|}{2} = \frac{n^2 + 3n + 2\lfloor n/2 \rfloor}{4}$$

because for $m = 1, 2$ the m-th primitive roots of unity are real but for $m > 2$ they come in complex conjugate pairs. Accordingly we get a smaller real generating set T' for $R_\lambda(q,x)$'s, with size

$$|T'| = \frac{(n+1)(n^2+3n+2\lfloor n/2\rfloor)}{4}.$$

If $p(n) > |T'|$, which happens for $n \geq 34$, then the rows of $V(n)$ are linearly dependent over $\mathbb{Q}$.

2. Linear independence for $n \leq 6$. For $m \geq 0$, the coefficient $[q^m]v_{\lambda,l}(q)$ of q^m in $v_{\lambda,l}(q)$ is the number of partitions of m into distinguishable parts $\lambda_1, \ldots, \lambda_k$ such that each part λ_i is used at most l times. If $m \leq l$, the restriction on the multiplicity of parts may be dropped with no effect and the coefficient equals to the number of all unrestricted partitions of m into distinguishable parts $\lambda_1, \ldots, \lambda_k$. Therefore

$$[q^m]v_{\lambda,l}(q) = [q^m]\frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})\ldots(1-q^{\lambda_k})} = [q^m]r_\lambda(q) \;\text{ for }\; 0 \leq m \leq l.$$

It follows that if the rational functions $\{r_\lambda(q) \mid \lambda \in P(n)\}$ are linearly independent over $\mathbb{Q}$, then so are the rows of $V(n)$. Considering poles, it is not hard to show that it is so for $n \leq 6$. We present the case $n = 6$, for $n \leq 5$ the arguments are similar and easier.

The partitions of $n = 6$ are $\lambda = 6, 51, 42, 41^2, 3^2, 321, 31^3, 2^3, 2^21^2, 21^4, 1^6$; $p(6) = 11$. The corresponding rational functions are

$$\begin{aligned}
r_6(q) &= (1-q^6)^{-1}\\
r_{51}(q) &= (1-q^5)^{-1}(1-q)^{-1}\\
r_{42}(q) &= (1-q^4)^{-1}(1-q^2)^{-1}\\
r_{41^2}(q) &= (1-q^4)^{-1}(1-q)^{-2}\\
r_{3^2}(q) &= (1-q^3)^{-2}\\
r_{321}(q) &= (1-q^3)^{-1}(1-q^2)^{-1}(1-q)^{-1}\\
r_{31^3}(q) &= (1-q^3)^{-1}(1-q)^{-3}\\
r_{2^3}(q) &= (1-q^2)^{-3}\\
r_{2^21^2}(q) &= (1-q^2)^{-2}(1-q)^{-2}\\
r_{21^4}(q) &= (1-q^2)^{-1}(1-q)^{-4}\\
r_{1^6}(q) &= (1-q)^{-6}.
\end{aligned}$$

Suppose now that

$$\sum_{\lambda \in P(6)} \alpha_\lambda \cdot r_\lambda(q) \equiv 0$$

for some $\alpha_\lambda \in \mathbb{C}$. We see that quite a few α_λ are zero because $r_\lambda(q)$ has a pole α with multiplicity s and α has pole multiplicity $< s$ in every other $r_\kappa(q)$, $\kappa \neq \lambda$, with yet undetermined coefficient α_κ. We record this by writing "unique pole m^s" if α is a primitive m-th root of unity. Thus $\alpha_6 = 0$ (unique pole 6^1), $\alpha_{51} = 0$ (unique pole 5^1), $\alpha_{1^6} = 0$ (unique pole 1^6), $\alpha_{21^4} = 0$ (unique pole 1^5), $\alpha_{2^3} = 0$ (unique pole 2^3), and $\alpha_{3^2} = 0$ (unique pole 3^2). We are left with five partitions $42, 41^2, 321, 31^3, 2^21^2$ and there is no longer unique pole with maximum multiplicity. To break the impasse we calculate α_{42} and α_{41^2} by sending q to the two primitive 4-th roots of unity. We have

$$r_{42}(q) = (1-q^4)^{-1}(1-q^2)^{-1} \sim \frac{1/2}{1-q^4} \quad \text{if} \quad q \to \pm i$$

but

$$r_{41^2}(q) = (1-q^4)^{-1}(1-q)^{-2} \sim \frac{\pm i/2}{1-q^4} \quad \text{if} \quad q \to \pm i.$$

Thus

$$\alpha_{42} + i\alpha_{41^2} = \alpha_{42} - i\alpha_{41^2} = 0$$

and $\alpha_{42} = \alpha_{41^2} = 0$. Now $\alpha_{2^21^2} = 0$ (unique pole 2^2), $\alpha_{31^3} = 0$ (unique pole 1^4) and $\alpha_{321} = 0$ is forced. Thus all α_λ, $\lambda \in P(6)$, are zero. We conclude that the eleven rows of $V(6)$ are linearly independent over $\mathbb{Q}$. We have proven stronger result: they remain linearly independent even if the $v_{\lambda,l}(q)$ in $V(6)$ are truncated to the powers of q with exponents $\leq l$.

5.5 Anna de Mier: Ping-pong ball problem

Suppose we are given three bins A, B, and C and infinitely many (ping-pong) balls numbered $1, 2, 3, \ldots$ In the first turn we put the balls $1, 2, 3$ in bin A, then we move two of them to bin B and finally one of them to bin C. In subsequent turns, we put the next three balls in bin A; among all the balls that are now in this bin, we move two to bin B; finally, we move one of the balls of bin B to bin C. The ping-pong ball problem *asks for the number of different tuples (A_n, B_n, C_n) such that A_n, B_n, C_n are the n-subsets of $[3n]$ that correspond to the balls present in each bin after n turns (irrespective of* *the order in which the balls got to the bins). A simpler question might be to determine if the corresponding generating function is algebraic or not.*

This problem is a particular case of the *tennis ball problem with parameters* introduced by the author (*SIAM J. Disc. Math.* **21** (2007) 130–140). In turn, the version with parameters generalizes a previous statement with only two bins, whose associated generating function is algebraic (de Mier and Noy, *Theoret. Comput. Sci.* **346** (2005), 254–264). We only know of trivial bounds for the general case. The first terms of the sequence of the ping-pong ball problem are $6, 63, 856, 13479, 233496, 4324102$.

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