Planar graphs without 3-, 7-, and 8-cycles are 3-choosable *

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Abstract

A graph is k-choosable if it can be colored whenever every vertex has a list of available colors of size at least k. Grötzsch's theorem states that every planar triangle-free graph is 3-colorable. However, Voigt [13] gave an example of such a graph that is not 3-choosable, thus Grötzsch's theorem does not generalize naturally to choosability. We prove that every planar triangle-free graph without 7- and 8-cycles is 3-choosable.

1 Introduction

All graphs considered in this paper are simple and finite. The concept of list colorings and choosability was introduced by Vizing [11] and independently

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by Erdős et al. [3]. A *list assignment* of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. An *L-coloring* is a function $\lambda : V(G) \to \bigcup_v L(v)$ such that $\lambda(v) \in L(v)$ for every $v \in V(G)$ and $\lambda(u) \neq \lambda(v)$ whenever u and v are adjacent vertices of G. If G admits an L-coloring then it is *L-colorable*. A graph G is *k-choosable* if, for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, there exists an L-coloring of G.

Thomassen [9] proved that every planar graph is 5-choosable. Voigt [12] showed that not all planar graphs are 4-choosable. By 3-degenericity, every planar triangle-free graph is 4-choosable, and Voigt [13] exhibited an example of a non-3-choosable triangle-free planar graph.

Sufficient conditions for 3-choosability of planar graphs are studied intensively. We present a table of known results for triangle-free graphs, where the additional assumptions are given by other forbidden cycle lengths. Many other criteria, some of them applicable even to graphs with triangles, were studied, see e.g. [7, 8] for more results in this direction.

3	4	5	6	7	8	9	authors	year
×	×						Thomassen [10]	1995
×			×	×		×	Zhang and Xu [15]	2004
×		×			×	×	Zhang [14]	2005
×		×	×				Lam, Shiu and Song [5]	2005
×					×	×	Zhang, Xu and Sun [16]	2006
×					×	×	Zhu, Lianying and Wang [17]	2007
×			×	×	×		Lidický [6]	submitted
×			×	×			Dvořák, Lidický, Škrekovski [2]	submitted
×				×	×		This paper	

There are many possible combinations of cycles one may try to forbid. We would like to explicitly mention one:

Problem 1.1. *Is there k such that forbidding all odd cycles of length $\leq k$ is a sufficient condition for 3-choosability of planar graphs?*

Such a condition makes the graph locally bipartite and would strengthen the result of Alon and Tarsi [1] that every bipartite planar graph is 3-choosable.

We use the following notation. Let G be a plane graph. We denote the set of its vertices by $V(G)$, the set of its edges by $E(G)$, and the set of its faces by $F(G)$. A vertex of G is *inner*, if it does not belong to the outer face of G. We denote the degree of a vertex v by $\deg(v)$. In a plane graph G, we denote the size of a face f (the length of its facial walk) by $\ell(f)$. A vertex of degree d (respectively at least d, respectively at most d) is said to be a *d-vertex* (respectively a *$(\geq d)$-vertex*, respectively a *$(\leq d)$-vertex*). The notion of an *l-face* (respectively an *$(\geq l)$-face*, respectively an *$(\leq l)$-face*) is defined analogously regarding the size of a face. Given a graph G and $S \subseteq V(G)$, let $G - S$ be the graph obtained from G by removing vertices in S and the edges incident with them. A vertex v and a face f are incident if $v \in V(f)$. Similarly, an edge uv and a face f are incident if $uv \in E(f)$. Faces f_1 and f_2 are adjacent if they share an edge.

2 Colorings planar graphs without 3-, 7-, 8-cycles

Our goal is to prove the following theorem.

Theorem 2.1. *Every plane graph G without 3-, 7- and 8-cycles is 3-choosable. Moreover, any precoloring of a 4- or 5-face h can be extended to a list coloring of G provided that each vertex not in $V(h)$ has at least three available colors.*

Proof. Suppose that Theorem 2.1 is false, and let G be a minimal counterexample. In case that h is precolored, we assume that h is the outer face of G. We shall get a contradiction by using the Discharging Method. Here is an overview of the proof: First we study some reducible configurations. Next, each vertex and face is assigned an initial charge such that the total charge is negative. Afterwards, the charge of faces and vertices is redistributed according to prescribed rules in such a way that the total charge stays unchanged. Under the assumption that the reducible configuration are not present in G, we show that the final charge of each vertex and each face is non-negative, which contradicts the fact that the total charge is negative.

Reducible configurations. We use the term *configuration* for a graph H, possibly with degree constraints on its vertices when considering H as a potential subgraph of G. We say that a configuration H is *reducible* if it cannot appear in the minimal counterexample G.

Lemma 2.2. *The following configurations are reducible:*

(1) *a non-precolored (≤ 2)-vertex v;*

(2) *an even cycle C_{2k} whose vertices are not precolored and have degree 3;*

(3) *two 4-cycles* $v_1v_2v_3v_4$ *and* $v_1v_5v_6v_7$ *consisting of non-precolored mutually distinct vertices* $v_1, \ldots, v_7$*, such that* v_1 *is a* 4*-vertex and* v_i *has degree* 3 *for* $2 \le i \le 7$*, see Figure 1.*

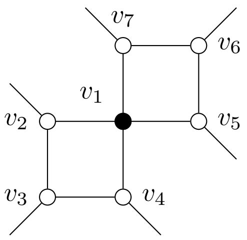

Figure 1: A reducible configuration.

Proof. Let L be an arbitrary list assignment of G such that each vertex is assigned precisely 3 colors. We show that G is L-colorable provided that it contains one of the three configurations.

If G has a non-precolored 2-vertex v, then by the minimality of G, the graph $G - v$ is L-colorable. This coloring can be extended to v, since it has three available colors and at most two neighbors.

Suppose now that G contains an even cycle C, whose vertices are non-precolored and of degree 3. Observe that every even cycle is 2-choosable (see also [3]). By the minimality of G, there exists an L-coloring of $G - C$. Since each vertex of C has only one neighbor in $G - C$ and C is 2-choosable, we can extend this coloring to C.

Finally suppose that G contains the third configuration K. Remove this configuration, and by the minimality of G, L-color $G - K$. Let $L'(v_1) = L(v_1)$ and let $L'(v_i)$ be obtained from $L(v_i)$ by removing the color of the neighbor of v_i in $G - K$, for $2 \le i \le 7$. We show that there exists a proper L'-coloring φ of v_2, v_3 and v_4 such that $|L'(v_1) \setminus \{\varphi(v_2), \varphi(v_4)\}| \ge 2$. Consider the following cases:

- $L'(v_2) \cap L'(v_4) \neq \emptyset$: Let a be a common color of v_2 and v_4. We color v_2 and v_4 by a, and extend this coloring to v_3.
- $L'(v_2) \cap L'(v_4) = \emptyset$: Then $|L'(v_2) \cup L'(v_4)| \ge 4$. Hence, there exists a color $a \in (L'(v_2) \cup L'(v_4)) \setminus L'(v_1)$. Without loss of generality assume that $a \in L'(v_2)$. We assign a to v_2, and afterwards L'-color v_3 and v_4.

Since the 4-cycle $v_1v_5v_6v_7$ is 2-choosable, we can extend this coloring to an L'-coloring of K, giving an L-coloring of G. $\square$

We can assume that *the outer face h of G is a precolored 4- or 5-cycle*: if G has no precolored 4- or 5-face, then every vertex has degree ≥ 3 according Lemma 2.2(1). Euler's formula implies that G has a 4- or 5-face, so we can color it.

We can also assume that *G contains no separating 4- or 5-cycle*. Otherwise, if C is such a cycle, then again by the minimality of G, color first the part of G outside of C, and then extend the coloring of C to the part of G inside C.

Lemma 2.3. *A 4-face $f \neq h$ cannot be adjacent to 5- or 6-face and it cannot share three edges with other 4-faces. If a 4-face shares edges with two other* 4*-faces, then they surround a vertex of degree three.*

Proof. Let $f = v_1v_2v_3v_4$ be a 4-face sharing at least one edge with a face $f' = v_1v_2u_3 \ldots u_t$, where $t \in \{4, 5, 6\}$. As G has no triangles, $u_3 \neq v_4$ and $u_t \neq v_3$. If $u_3 = v_3$, then $\deg(v_2) = 2$, thus $v_1v_2v_3$ is a part of the boundary of the outer face h. In this case, we remove v_2 and color v_4 instead. Therefore, $u_3 \neq v_3$, and by symmetry, $u_t \neq v_4$.

Suppose that $t = 5$. If $u_4 \notin \{v_3, v_4\}$, then $v_1u_5u_4u_3v_2v_3v_4$ would be a 7-cycle, and if $u_4 \in \{v_3, v_4\}$, then G contains a triangle, which is a contradiction. Therefore, G does not contain a 4-face adjacent to a 5-face.

Consider the case that $t = 6$. If $\{u_4, u_5\} \cap \{v_3, v_4\} = \emptyset$, then $v_1u_6u_5u_4u_3 v_2v_3v_4$ would be an 8-cycle, thus assume that say $u_4 \in \{v_3, v_4\}$. As G does not contain triangles, $u_4 \neq v_3$, and hence $u_4 = v_4$. But, the 4-cycle $v_4v_1v_2u_3$ separates v_3 from u_5, which is a contradiction. It follows that G does not contain a 4-face adjacent to a 6-face.

Suppose now that $t = 4$ and that f shares an edge with one more 4-face f''. Assume first that $f'' = v_3v_4u_5u_6$. Observe that $\{u_5, u_6\} \cap \{v_1, v_2\} = \emptyset$. If $\{u_5, u_6\} \cap \{u_3, u_4\} = \emptyset$, then $v_1u_4u_3v_2v_3u_6u_5v_4$ is an 8-cycle, thus assume that say $u_5 \in \{u_3, u_4\}$. As G does not contain triangles, $u_5 \neq u_4$, thus $u_5 = u_3$. However, G then contains a separating 4-cycle $u_3v_2v_1v_4$.

It follows that $f'' = v_1v_4u_5u_6$. By symmetry, f does not share the edge v_2v_3 with a 4-face, thus f does not share edges with three 4-faces. Also, as G does not contain 8-cycles, $\{u_5, u_6\} \cap \{u_3, u_4\} \neq \emptyset$. Note that $u_5 \neq u_3$ because of the separating 4-cycle $u_3v_2v_1v_4$, and $u_3 \neq u_4$ and $u_6 \neq u_3$, as G does not contain triangles. It follows that $u_4 = u_6$, thus v_1 has degree three and it is surrounded by 4-faces f, f' and f''. $\square$

Lemma 2.4. *No two* 5*-faces* f *and* f' *distinct from* h *are adjacent.*

Proof. Let $f = v_1v_2v_3v_4v_5$ and $f' = v_1v_2u_3u_4u_5$. As $f, f' \neq h$, v_1 and v_2 have degree at least three, thus $v_3 \neq u_3$ and $v_5 \neq u_5$. As G does not contain triangles, $v_3 \neq u_5$ and $v_5 \neq u_3$. As $v_2v_3v_4v_5v_1u_5u_4u_3$ is not an 8-cycle, $\{v_3, v_4, v_5\} \cap \{u_3, u_4, u_5\} \neq \emptyset$. By symmetry, we may assume that $v_4 \in \{u_3, u_4\}$. As G does not contain triangles, $v_4 \neq u_3$, thus $v_4 = u_4$. However, at least one of 4-cycles $u_4u_3v_2v_3$ or $u_4u_5v_1v_5$ is distinct from h, contradicting Lemma 2.3. $\square$

Initial charges. We assign the initial charge to each inner vertex v and the initial charge to each inner face f, respectively, by

$$\mathrm{ch}(v) := 2\deg(v) - 6 \quad \text{and} \quad \mathrm{ch}(f) := \ell(f) - 6.$$

A vertex v of h has initial charge $\mathrm{ch}(v) := 2\deg(v) - 4$ and the outer face h has initial charge $\mathrm{ch}(h) := 0$.

It is easy to see that every vertex has non-negative initial charge, and that only the inner (≤ 5)-faces have negative charge. We are interested in the total amount of charge of G. By Euler's formula, the total amount of charge is

$$\begin{aligned} &\textstyle\sum_{v\in V(G)} \mathrm{ch}(v) + \sum_{f\in F(G)} \mathrm{ch}(f) = \\ &= \textstyle\sum_{v\in V(G)}(2\deg(v) - 6) + 2\ell(h) + \sum_{f\in F(G)}(\ell(f) - 6) + 6 - \ell(h) \\ &= (4|E(G)| - 6|V(G)|) + (2|E(G)| - 6|F(G)|) + 6 + \ell(h) \\ &= 6(|E(G)| - |V(G)| - |F(G)|) + 6 + \ell(h) \\ &= -6 + \ell(h). \end{aligned}$$

As $\ell(h) \leq 5$, the total charge is negative.

Rules. We use the following discharging rules to redistribute the initial charge, see Figure 2. A vertex v is *big* if $\deg(v) \geq 4$ or it is precolored and $\deg(v) = 3$.

Rule 1. *Let a* (≥ 9)*-face* f *share an edge* e *with a* 4*-face* $g \neq h$*. If* g *contains only one big vertex, then* f *sends charge* $1/3$ *to* g *through the edge* e*.*

Rule 2. *Let two* (≥ 9)*-faces* f_1 *and* f_2 *share a 3-vertex* v *with a 4-face* $g \neq h$ *which contains only one big vertex. Let* e *be the common edge of* f_1 *and* f_2 *that is incident with* v*. Then each of* f_1 *and* f_2 *sends charge* $1/6$ *to* g *through the edge* e*.*

Rule 3. *Let a* (≥ 9)*-face* f *share a common edge* uv *with a 4-face* g*, which has no precolored vertex, and* $\deg(v) = 4$*. Let* uvw *be a part of the facial walk of* f*. If* v *is the only big vertex of* g*, then* f *sends charge* $1/6$ *to* g *through the edge* vw*.*

Rule 4. *A* (≥ 9)*-face sends charge* $1/3$ *to an adjacent 5-face (distinct from* h*) through their common edge* $e = uv$*, if* u *and* v *are of degree three.*

Rule 5. *A 6-face sends charge* $1/4$ *to an adjacent 5-face (distinct from* h*) through their common edge* $e = uv$*, if* u *and* v *are of degree three.*

Rule 6. *A big vertex* v *sends charge to an incident 4-face* g *distinct from* h*. If* $\deg(v) = 4$ *and* v *is not precolored, or* $\deg(v) = 3$ *(and* v *is precolored), then* v *sends charge* 1*. Otherwise* v *sends charge* $4/3$ *to* g*.*

Rule 7. *A big vertex sends charge* $1/2$ *to every adjacent 5- or 6-face distinct from* h*.*

Note that rules apply simultaneously. Hence, for example Rule 1 and Rule 2 can both send charge from one face to some other. Also multiplicity is considered, for example, a face can send charge to another face through several edges.

Final charges. We use $\mathrm{ch}^*(x)$ to denote the final charge of a vertex or face x. Next we show that the final charge of every vertex and face is non-negative, thus establishing the theorem.

Let v be a vertex of degree d of G. If v is not big, then its initial charge is zero, and no charge is sent or received by it, hence its final charge is zero as well. Therefore, assume that v is big. If $d = 3$, then v is incident with h, hence its initial charge is 2, As v sends charge of at most 1 to each of the two incident faces distinct from h, its final charge is nonnegative. Therefore, assume that $d \geq 4$.

The vertex v sends charge by Rules 6 and 7 to 4-, 5-, and 6-faces. Let a be the number of 4-faces distinct from h incident with v. Let b be the

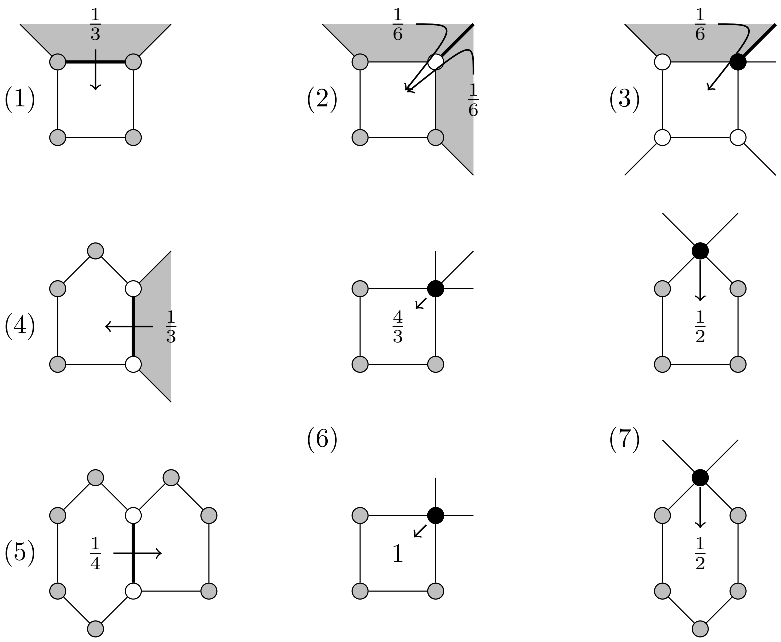

Figure 2: The discharging rules R1–R7. A black vertex denotes a big vertex, a white vertex denotes a non-precolored 3-vertex, and a gray vertex can be of any degree in G. A thick edge is used for transferring charge and a gray face is a (≥ 9)-face.

number of 5-faces and 6-faces (other than h) incident with v. The final charge of v is

$$\mathrm{ch}^*(v) \;\geq\; 2d - 6 - \frac{4}{3}a - \frac{1}{2}b.$$

If $a = 0$, then the final charge of v is at least $2d - 6 - \frac{1}{2}b \geq \frac{3d}{2} - 6 \geq 0$. Suppose now that $a > 0$. A 4-face distinct from h cannot be adjacent to a 5- or 6-face by Lemma 2.3. Hence if v is not incident with h, there must be at least two (≥ 7)-faces incident with v, and if v is incident with h, then there must be at least one (≥ 7)-face incident with v. In both cases, $a + b \leq d - 2$. The final charge of v is at least $2d - 6 - \frac{4}{3}(a + b) \geq \frac{2d-10}{3}$, which is nonnegative if $d \geq 5$.

Finally, consider the case that $d = 4$. Since $a > 0$, we have $a + b \leq 2$. If v is incident with h, then its initial charge is 4, and the final charge is at least $4 - \frac{4}{3}(a + b) \geq \frac{4}{3}$. If v is not incident with h, then its initial charge is 2, and it sends at most one to each incident face of length at most 6, thus its final charge is at least $2 - (a + b) \geq 0$. We conclude that the final charge of each vertex is nonnegative.

Let f be an arbitrary face of G. If f is the outer face h, then $\mathrm{ch}^*(h) = \mathrm{ch}(h) = 0$. Therefore, we assume that $f \neq h$. If $\ell(f) \leq 6$, then the boundary of f is a cycle, thus if f contains a precolored vertex of degree two, then it contains at least two precolored vertices of degree at least three, and these two vertices are big. Similarly, if $\ell(f) \leq 6$ and f is incident with a precolored vertex of degree three, then f contains at least two big vertices. We consider the following cases regarding $\ell(f)$:

$\ell(f) \geq 9$: We show that f *sends charge of at most* $1/3$ *through each of its edges*. Then,

$$\mathrm{ch}^*(v) \geq \ell(f) - 6 - \frac{\ell(f)}{3} \geq \frac{2\ell(f)}{3} - 6 \geq 0.$$

Let $e = uv$ be an edge of f and let g be the face incident with e distinct from f. If $g = h$, then no charge is sent through e, hence assume that $g \neq h$. Note that if f sends charge through e only once, then this charge is at most $1/3$. We consider the following subcases regarding the size of g:

- $\ell(g) = 4$ *and g is incident with only one big vertex:* f sends charge $1/3$ to g through e by R1. The face f can send more charge through e only by rule R3. Then, we may assume that v is a 4-vertex, vw is an edge of f and it is incident with some 4-face g' for which v is also the only big incident vertex, and no vertex of g' is precolored. As v is the only big vertex of g, no vertex of g is precolored as well. But then g and g' form a reducible configuration, by Lemma 2.2(3).
- $\ell(g) = 4$ *and g is incident with more than one big vertex:* then the charge is sent through e only by R3, for the total of at most $1/6 + 1/6 = 1/3$.
- $\ell(g) = 5$: In this case, f sends either at most $1/3$ through e by rule R4 (if both u and v have degree three) or at most twice $1/6$ by R3 (if u or v have degree four).

- $\ell(g) = 6$: The face f sends at most twice $1/6$ through e by rule R3.
- $\ell(g) \geq 9$: The charge of $1/6$ is sent at most twice through e by rules R2 or R3.

This case analysis establishes the claim.

$\ell(f) = 6$: By Lemma 2.2(2), f cannot consist of non-precolored vertices of degree three, thus f contains a big vertex v. The face f receives $1/2$ from v by Rule 7, and at most twice sends $1/4$ by Rule 5 (as two 5-faces distinct from h cannot share an edge and f contains a big vertex). Therefore, $\mathrm{ch}^*(f) \geq 0 + 1/2 - 2/4 = 0$.

$\ell(f) = 5$: The face f has initial charge -1 and it sends no charge. By Lemmas 2.3 and 2.4, f is not adjacent to any face of length at most 5 distinct from h. We consider several possibilities regarding the number of big vertices incident with f.

If f contains at least two big vertices, then Rule 7 applies twice, and thus $\mathrm{ch}^*(f) \geq -1 + 2/2 = 0$.

If f contains one big vertex v, then no vertex of f except possibly for v is precolored. Note that Rule 7 applies once. Moreover, f contains three edges whose endvertices are non-precolored vertices of degree 3. The charge is received by f through these three edges by Rules 4 and 5. Thus, $\mathrm{ch}^*(f) \geq -1 + 1/2 + 3/4 > 0$.

If f is incident with no big vertex, then all its vertices are of degree 3 and are not precolored. Then, f receives charge by Rules 4 and 5 through each incident edge, and $\mathrm{ch}^*(f) \geq -1 + 5/4 > 0$.

$\ell(f) = 4$: By Lemma 2.2(2), the face f must contain a big vertex. If f contains at least two big vertices, then Rule 6 applies twice, and $\mathrm{ch}^*(f) \geq -2 + 2 = 0$. Therefore, we may assume that f is incident with exactly one big vertex v. In particular, no vertex of f other than v is precolored, and if v is precolored, then $\deg(v) \geq 4$.

If at most one edge of f is shared with another 4-face, then at least three edges of f are incident with faces of degree at least 9. After applying Rule 6 and three times Rule 1, we obtain $\mathrm{ch}^*(f) \geq -2 + 1 + 3/3 = 0$. By Lemma 2.3, the 4-face f cannot share three edges with other 4-faces. Therefore, we may assume that f shares exactly two edges with other 4-faces f_1 and f_2, and the three 4-faces surround a

3-vertex y. Note that $v \neq y$, otherwise v is precolored and hence f contains at least two big vertices.

If v is incident with f_1 or f_2, then Rule 6, twice Rule 1 and twice Rule 2 apply and $\mathrm{ch}^*(f) \geq -2+1+2/3+2/6 = 0$. Now assume that v is not adjacent to any of the other two 4-faces. If v is precolored or $\deg(v) \geq 5$, then Rule 6 and twice Rule 1 apply and $\mathrm{ch}^*(f) \geq -2+4/3+2/3 = 0$. Finally, if v is a non-precolored 4-vertex, then Rule 6, twice Rule 1, and twice Rule 3 apply, and we infer that $\mathrm{ch}^*(f) \geq -2+1+2/3+2/6 = 0$.

□

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