The packing chromatic number of infinite product graphs*

Jiří Fiala [a] Sandi Klavžar [b†] Bernard Lidický [a]

[a] Department of Applied Mathematics and Inst. for Theoretical Computer Science (ITI)[1],
Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic
Email: {fiala,bernard}@kam.mff.cuni.cz

[b] Department of Mathematics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia
Email: sandi.klavzar@uni-mb.si

Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that the vertex set $V(G)$ can be partitioned into disjoint classes $X_1, \ldots, X_k$, where vertices in X_i have pairwise distance greater than i. For the Cartesian product of a path and the 2-dimensional square lattice it is proved that $\chi_\rho(P_m \,\square\, \mathbb{Z}^2) = \infty$ for any $m \geq 2$, thus extending the result $\chi_\rho(\mathbb{Z}^3) = \infty$ of Finbow and Rall [4]. It is also proved that $\chi_\rho(\mathbb{Z}^2) \geq 10$ which improves the bound $\chi_\rho(\mathbb{Z}^2) \geq 9$ of Goddard et al. [5]. Moreover, it is shown that $\chi_\rho(G \,\square\, \mathbb{Z}) < \infty$ for any finite graph G. The infinite hexagonal lattice $\mathcal{H}$ is also considered and it is proved that that $\chi_\rho(\mathcal{H}) \leq 7$ and $\chi_\rho(P_m \,\square\, \mathcal{H}) = \infty$ for $m \geq 6$.

*The research was initiated at WFAP'07 — Second Workshop on Frequency Assignment Problems in Wireless Networks organized in September 23–27, 2007 at Sádek u Třebíče by DIMATIA and ITI, whose generous support of all authors is gratefully acknowledged.

†Supported by the Ministry of Science of Slovenia under the grant P1-0297. The author is also with the University of Maribor, Slovenia and with the Institute of Mathematics, Physics and Mechanics, Ljubljana.

[1]Supported by the Ministry of Education of the Czech Republic as project 1M0021620808.

Keywords: Packing chromatic number; Cartesian product of graphs; Cubic and hexagonal lattices;

1 Introduction

The concept of packing coloring comes from the area of frequency planning in wireless networks. This model emphasizes the fact that some frequencies are used more sparely than the others.

In graph terms, we ask for a partition of the vertex set of a graph G into disjoint classes $X_1, \ldots, X_k$ (representing frequency usage) according to the following constraints. Each color class X_i should be an *i-packing*, that is, a set of vertices with the property that any distinct pair $u, v \in X_i$ satisfies $\mathrm{dist}(u, v) > i$. Here $\mathrm{dist}(u, v)$ denotes the usual shortest path distance between u and v. Such partition is called a *packing k-coloring*, even though it is allowed that some sets X_i may be empty. The smallest integer k for which there exists a packing k-coloring of G is called the *packing chromatic number* of G and it is denoted by $\chi_\rho(G)$. This concept was introduced by Goddard et al. [5] under the name *broadcast chromatic number*. The term packing chromatic number was later (even if the corresponding paper was published earlier) proposed by Brešar et al. [1].

Sloper [7] followed with a closely related concept, the eccentric coloring. An *eccentric coloring* of a graph is a packing coloring in which a vertex v is colored with a color not larger than the eccentricity of v. His results among others imply that the infinite 3-regular tree has packing chromatic number 7.

The determination of the packing chromatic number is quite difficult. In particular, it is NP-complete for general graphs [5]. In addition, in the same paper it was also proved that it is NP-complete to decide whether $\chi_\rho(G) \leq 4$. But things are much worse: Fiala and Golovach showed that determining $\chi_\rho(G)$ is one of few inherent problems that are NP-complete on trees [2].

The following interesting phenomena was the starting point for our investigations. The packing chromatic number of the infinite square lattice $\mathbb{Z}^2$ is finite, more precisely, Goddard et al. [5] showed that it lies between 9 and 23. In Theorem 3.11 we improve the lower bound to 10. On the other hand, Finbow and Rall [4] proved that the packing chromatic number of the infinite cubic lattice $\mathbb{Z}^3$ is unbounded. So where does a step from a finite number to the infinity occur? In Section 3 we prove that the packing chromatic number is unbounded already on two layers of the square lattice, that is, $\chi_\rho(P_2 \,\square\, \mathbb{Z}^2) = \infty$. On the other hand, in the next section we prove that $\chi_\rho(G \,\square\, \mathbb{Z}) < \infty$ for any finite graph G, hence in particular $\chi_\rho(P_n \,\square\, P_m \,\square\, \mathbb{Z}) < \infty$ for arbitrary m and n. In fact, we prove a slightly more general theorem: for the strong product of the complete graph on $n \geq 1$ vertices K_n with the two-way infinite path we have $\chi_\rho(K_n \boxtimes \mathbb{Z}) < 4^n$.

Just like square and cubic lattices, the hexagonal lattice $\mathcal{H}$ is important in different applications, for instance in the field of frequency assignment. Brešar et al. [1] showed that $6 \leq \chi_\rho(\mathcal{H}) \leq 8$ and asserted (without a proof) that the actual lower bound is 7. This was later indeed verified, using a computer, by Vesel [8]. In Section 4 we exhibit a tiling of the hexagonal lattice using 7 colors; see Theorem 4.1. As a consequence $\chi_\rho(\mathcal{H}) = 7$. We also investigate the situation of the hexagonal lattice with more hexagonal layers and we prove that $\chi_\rho(P_m \,\square\, \mathcal{H}) = \infty$ for every $m \geq 6$.

2 Cartesian products with a single infinite path

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ where vertices (g, h) and (g', h') being adjacent whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. The Cartesian product operation is associative and commutative [6]. The subgraph of $G \,\square\, H$ induced by $\{g\} \times V(H)$ is isomorphic to H and it is called an *H-layer*. Similarly one defines the *G-layer* for a vertex h of H. The *strong product* $G \boxtimes H$ of graphs G and H can be described as the graph obtained from $G \,\square\, H$ by adding edges between (g, h) and (g', h') provided that $gg' \in E(G)$ and $hh' \in E(H)$. Layers of the strong product are defined analogously as the layers of the Cartesian product.

It will be convenient to present our results by the Cartesian product with the 2-way infinite path $\mathbb{Z}$. In this notation the square lattice $\mathbb{Z}^2$ can be viewed as the product $\mathbb{Z} \,\square\, \mathbb{Z}$ and the cubic lattice $\mathbb{Z}^3$ as $\mathbb{Z} \,\square\, \mathbb{Z} \,\square\, \mathbb{Z}$.

We first prove that the packing chromatic number of the strong product of the complete graph K_n with the infinite path is asymptotically exponential in n.

Theorem 2.1. *For any $n \geq 1$ it holds that $\chi_\rho(K_n \boxtimes \mathbb{Z}) < 4^n$ and $\chi_\rho(K_n \boxtimes \mathbb{Z}) = \Omega(e^n)$.*

Proof. We first observe that every single infinite path $\mathbb{Z}$ allows a packing coloring using colors from k up to $4k - 1$, for any $k \geq 1$: we use the coloring pattern $(k, k{+}1, k{+}2, \ldots, 2k{-}1)$ repeatedly on even vertices and the pattern $(2k, 2k+1, 2k+2, \ldots, 4k-1)$ on odd vertices. The resulting pattern is

$$(k, 2k, k+1, 2k+1, ..., 2k-1, 3k-1, k, 3k, ..., 2k-2, 4k-2, 2k-1, 4k-1)\,.$$

To prove the theorem consider $\mathbb{Z}$-layers of $K_n \boxtimes \mathbb{Z}$; see Figure 1. We color the i-th layer $\mathbb{Z}$ with the above pattern by using colors from the interval $[4^{i-1}, 4^i - 1]$. This particular packing coloring of $K_n \boxtimes \mathbb{Z}$ needs $4^n - 1$ colors in total.

Figure 1: Decomposition of $K_n \boxtimes \mathbb{Z}$ into $\mathbb{Z}$-layers from Theorem 2.1 is on the left. The infinite strip of width two of the triangular lattice and its decomposition into $\mathbb{Z}$-layers is on the right.

To show the lower bound we proceed as follows. Let N be a (large) positive integer and consider the subgraph $G_N = K_n \boxtimes P_N$ of $K_n \boxtimes \mathbb{Z}$. Suppose f is a packing coloring of $K_n \boxtimes \mathbb{Z}$ using at most c colors. Then for any $i \geq 1$, at most $\left\lceil \frac{N}{i+1} \right\rceil$ vertices of G_N can have color i. Since G_N has $n\,N$ vertices we infer that

$$\left\lceil \frac{N}{2} \right\rceil + \left\lceil \frac{N}{3} \right\rceil + \left\lceil \frac{N}{4} \right\rceil + \cdots + \left\lceil \frac{N}{c+1} \right\rceil \geq n\,N\,.$$

Since for any $k \geq 1$, $\frac{\lceil N/k \rceil}{N} \leq \frac{1}{k} + \frac{1}{N}$, it follows that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{c+1} \geq n + 1 - \frac{c}{N}\,. \qquad (1)$$

The sum from inequality (1) is the $(c+1)$'th harmonic number H_{c+1}. It is well-known that H_c grows about as fast as the natural logarithm of c. Therefore, since c is fixed and N can be arbitrarily large, we obtain that $\ln c$ must be of order n and so c must be of order at least e^n. □

Corollary 2.2. *For any finite graph G, $\chi_\rho(G \,\square\, \mathbb{Z}) < \infty$.*

Proof. Let G be of order n, then G is a (spanning) subgraph of K_n. Therefore $G \,\square\, \mathbb{Z}$ is a (spanning) subgraph of $K_n \boxtimes \mathbb{Z}$ and by Theorem 2.1 the assertion follows. □

Returning to the Cartesian product of paths we observe that Corollary 2.2 immediately implies that for any $m, n \geq 1$, $\chi_\rho(P_m \,\square\, P_n \,\square\, \mathbb{Z}) < \infty$.

Finbow and Rall [4] proved that the infinite triangular lattice has infinite packing chromatic number. On the other hand, we can apply Theorem 2.1 to show that the packing chromatic number is finite for every finite strip of the triangular lattice; see the right-hand side of Figure 1.

3 Square lattices

In this section we focus on the case when two factors of the Cartesian product are 2-way infinite paths. In particular we prove that $\chi_\rho(P_m \,\square\, \mathbb{Z}^2) = \infty$ for $m \geq 2$ and that $\chi_\rho(\mathbb{Z}^2) \geq 10$.

Our approach on proving that some lattice L cannot be covered by a finite number of packings is based on arguments using the notion of the density of a packing. The idea is, roughly speaking, to assign first a unit area to every vertex of L. Then we redistribute the area to vertices covered by the packing such that areas at vertices from the packing are equal and as large as possible. In this way we can define a density for every vertex from the packing as the reciprocal of the area.

Formally we proceed as follows. Let X_k be a k-packing in L. For every x from L and a positive integer l we denote by $N_l(x)$ the set vertices at distance at most l from x, i.e. $N_l(x) := \{y : y \in L, \mathrm{dist}(x, y) \leq l\}$. Observe that for arbitrary vertices u and v of X_k the sets $N_{\lfloor k/2 \rfloor}(u)$ and $N_{\lfloor k/2 \rfloor}(v)$ are disjoint, since the vertices u and v are at distance greater than k.

Let k be an odd number, x be a vertex from X_k, and y be a vertex at distance $\left\lceil \frac{k}{2} \right\rceil$ from x. Then there is no vertex from X_k in $N_{\lfloor k/2 \rfloor}(y)$. Hence y is not in $N_{\lfloor k/2 \rfloor}(z)$ of any vertex z from X_k. We redistribute the unit area assigned to y to vertices of X_k by sending the reciprocal of its degree to every of its neighboring sets $N_{\lfloor k/2 \rfloor}(x)$ as follows:

Definition 3.1. The *k-area* $A(x, k)$ assigned to a vertex $x \in V(L)$ is defined by

$$A(x,k) := \begin{cases} |N_{\lfloor k/2 \rfloor}(x)| & \text{for } k \text{ even}, \\ |N_{\lfloor k/2 \rfloor}(x)| + \displaystyle\sum_{\substack{y \in V(G) \\ \mathrm{dist}(x,y) = \lceil k/2 \rceil}} \frac{|N_1(y) \cap N_{\lfloor k/2 \rfloor}(x)|}{\deg(y)} & \text{for } k \text{ odd}. \end{cases}$$

If the k-area is the same for all vertices of the lattice L we define $A(k) := A(x,k)$, where x is chosen arbitrarily.

By abuse of language we only speak of area instead of k-area if k is clear from the context. See Figure 2 for an example of distribution of the area in $\mathbb{Z}^2$. Note that the area $A(k)$ is in particular well-defined for lattices that are vertex transitive.

Figure 2: Coverage of $\mathbb{Z}^2$ by X_2 on the left and by X_3 on the right. Vertices from the packings are black. The dotted cross shapes correspond to $N_1(x)$. The white vertices on the right are not covered by any set $N_1(x), x \in X_3$. For every white vertex, each adjoining set $N_1(x)$ receives $\frac{1}{4}$ or $\frac{2}{4}$ of its area, depending on the mutual position.

The definition of the area is justified in the following fundamental observation.

Proposition 3.2. *If a finite graph G has a packing k-coloring and all areas $A(i)$, $1 \le i \le k$, are well-defined, then*

$$\sum_{i=1}^{k} \frac{1}{A(i)} \ge 1\,.$$

Proof. If G has n vertices then any color class X_i can contain at most $\frac{n}{A(i)}$ vertices. Therefore, $n = |V_G| = |X_1| + \cdots + |X_k| \le \frac{n}{A(1)} + \cdots + \frac{n}{A(k)}$, and the assertion follows. □

Definition 3.3. Let G be a graph. Then the *density* of a set of vertices $X \subset V(G)$ is

$$d(X) := \limsup_{l\to\infty} \max_{x\in V} \left\{ \frac{|X \cap N_l(x)|}{|N_l(x)|} \right\}.$$

The following claim goes immediately:

Observation 3.4. *Let G be a graph and $X \subsetneq V(G)$. Then for every $\varepsilon > 0$ there exists l_0 such that for every vertex $x \in V(G)$ and $l > l_0$,*

$$\frac{|X \cap N_l(x)|}{|N_l(x)|} < d(X) + \varepsilon\,.$$

We now get an analogue of Proposition 3.2.

Lemma 3.5. *For every finite packing coloring with k classes $X_1, X_2, \ldots, X_k$ of a graph G holds that*

$$\sum_{i=1}^{k} d(X_i) \geq d(X_1 \cup X_2) + \sum_{i=3}^{k} d(X_i) \geq d\Big(\bigcup_{i=1}^{k} X_i\Big) = 1\,.$$

Proof. We apply iteratively the following argument that for any vertex x and arbitrarily positive small ε, every sufficiently large l satisfies that

$$\frac{|N_l(x) \cap (X \cup Y)|}{|N_l(x)|} \leq \frac{|N_l(x) \cap X|}{|N_l(x)|} + \frac{|N_l(x) \cap Y|}{|N_l(x)|} \leq d(X) + d(Y) + \varepsilon.$$

□

Let x be a vertex of a graph G. We denote the boundary of $N_l(x)$ by $\Delta N_l(x) := \{y : \mathrm{dist}(y, x) = l\}$.

Lemma 3.6. *If for a graph G the area $A(k)$ is well-defined, and if*

$$\lim_{l\to\infty} \frac{|\Delta N_l(x)|}{|N_l(x)|} = 0\,,$$

then for any k-packing X_k it holds that $d(X_k) \leq \frac{1}{A(k)}$.

Proof. We choose a vertex x arbitrarily and use the following estimate: $|X_k \cap N_l(x)| \leq \frac{|N_l(x)|}{A_k} + |\{y : l - k \leq \mathrm{dist}(y, x) \leq l\}|$. Here the first summand estimates the maximum number of vertices z of X_k such that $N_{\lfloor k/2 \rfloor}(z) \subset N_l(x)$. The second summand simply roughly estimates all the remaining vertices of $N_l(x)$. According to our assumptions the right summand is negligible in comparison with $N_l(x)$ if l is large enough and the claim follows. □

We now focus our attention to the lattice $P_2 \,\square\, \mathbb{Z}^2$.

Lemma 3.7. *For every k and the lattice $P_2 \,\square\, \mathbb{Z}^2$,*

$$A(k) = \begin{cases} k^2 + 2 & \text{for } k \text{ even}\,, \\ k^2 + 1 & \text{for } k \text{ odd}\,. \end{cases}$$

Proof. Observe that in a single layer of $\mathbb{Z}^2$ for any vertex $x \in \mathbb{Z}^2$ and integer i it holds that $|\{y : \mathrm{dist}(x, y) = i\}| = 4i$. Then the number of vertices at distance at most l in $\mathbb{Z}^2$ from any fixed vertex is $1 + \sum_{i=1}^{l} 4i$.

In the lattice $P_2 \,\square\, \mathbb{Z}^2$ we first consider the case of an even $k = 2l$. We count the size of N_l in both layers separately by using the previous observation we get that:

$$A(k) = |N_l(x)| = 1 + \sum_{i=1}^{l} 4i + 1 + \sum_{i=1}^{l-1} 4i = 4l^2 + 2 = k^2 + 2\,.$$

If $k = 2l + 1$ is odd then we first discuss the case of $k = 1$. In this case $A(1) = 1 + \frac{5}{5} = 2$ since $N_0(x)$ is just a single vertex and it has 5 neighbors.

For the case of $l \geq 1$ we have to distinguish four kinds of vertices that are at distance $l + 1$ from some vertex x:

- four such vertices have one neighbor in $N_l(x)$ — those from the same $\mathbb{Z}^2$-layer as x that share a coordinate with x,
- $4l$ vertices have two neighbors in $N_l(x)$ — those remaining from the same layer,
- another four vertices have also two neighbors in $N_l(x)$ — those from the other layer but which share a coordinate with x,
- $4l - 4$ vertices have three neighbors in $N_l(x)$ — all the remaining vertices from the other layer.

In total we have:

$$A(k) = |N_l(x)| + 4\frac{1}{5} + 4l\frac{2}{5} + 4\frac{2}{5} + (4l - 4)\frac{3}{5} = 4l^2 + 2 + 4l = k^2 + 1\,.$$

□

We now are ready to prove the main result of this section, i.e. that the packing chromatic number of two layers of the square lattice is infinite.

Theorem 3.8. *For any $m \geq 2$, it holds that $\chi_\rho(P_m \,\square\, \mathbb{Z}^2) = \infty$.*

Proof. To get the result it is enough to prove the case $m = 2$. Let V be the vertex set of $P_2 \,\square\, \mathbb{Z}^2$.

We show that the sum of densities of all optimal k-packings is strictly less than one and get a contradiction with Lemma 3.5.

Since the lattice $P_2 \,\Box\, \mathbb{Z}^2$ satisfies assumptions of Lemma 3.6 (cf. also Lemma 3.7), we can bound densities in terms of area, and for areas use an explicit expression given by Lemma 3.7.

However, this approach does not work such straightforwardly — the case of optimal 1- and 2-packings need to be treated separately: Observe that the box $P_2 \,\Box\, P_2 \,\Box\, P_2$ (the cube) cannot contain more than five vertices from $X_1 \cup X_2$. Hence we can bound the density of $d(X_1 \cup X_2)$ by $\frac{5}{8}$ since the whole lattice $P_2 \,\Box\, \mathbb{Z}^2$ can be partitioned into such boxes.

We get a contradiction by the following estimate that holds for any packing coloring $X_1, \ldots, X_k$:

$$\begin{aligned} d\Big(\bigcup_{i=1}^{k} X_i\Big) \;&\leq\; d(X_1 \cup X_2) + \sum_{i=3}^{k} d(X_i) \;\leq\; \frac{5}{8} + \sum_{i=3}^{\infty} \frac{1}{A(i)} \;\leq \\ &\leq\; \frac{5}{8} + \sum_{i=3}^{15} \frac{1}{A(i)} + \int_{i=15}^{\infty} \frac{di}{i^2} \;\leq\; 0.9329 + \frac{1}{15} \;<\; 1\,. \end{aligned}$$

Here the exact value of the sum of the first 15 summands was obtained by a computer program. □

In the rest of the section we focus our attention on the square lattice $\mathbb{Z}^2$ and improve the lower bound of its packing chromatic number from 9 to 10. We base the argument on an observation that the best packing patterns for X_1 and other for X_k with even k significantly overlap.

Lemma 3.9. *For the lattice $\mathbb{Z}^2$ and every k it holds that $A(k) = \left\lfloor \frac{k^2}{2} \right\rfloor + k + 1$.*

Proof. In the proof of Lemma 3.7 we have already observed that $|\{y : \mathrm{dist}(x, y) = i\}| = 4i$ for every vertex $x \in \mathbb{Z}^2$ and every i.

In the case of an even $k = 2l$ we have

$$A(k) = |N_l(x)| = 1 + \sum_{i=1}^{l} 4i = 2l^2 + 2l + 1 = \frac{k^2}{2} + k + 1\,.$$

In the case of an odd $k = 2l + 1$ we have four vertices at distance $l + 1$ from x that have a single neighbor in $N_l(x)$ and the remaining $4l$ vertices at distance $l + 1$ have two neighbors in $N_l(x)$. We get that

$$A(k) = |N_l(x)| + 4\frac{1}{4} + 4l\frac{2}{4} = 2l^2 + 4l + 2 = \left\lfloor \frac{k^2}{2} \right\rfloor + k + 1\,.$$

□

We now show that the best possible coverage of $\mathbb{Z}^2$ by $X_1 \cup X_2$ covers $\frac{5}{8}$ of the lattice which improves the bound $\frac{1}{2} + \frac{1}{6}$ corresponding to the case where X_1 and X_2 are treated separately.

Lemma 3.10. *The density $d(X_1 \cup X_2)$ on $\mathbb{Z}^2$ is at most $\frac{5}{8}$.*

Proof.

Figure 3: The graph O.

We first define a graph O on eight vertices consisting of a cycle $v_1, \dots v_6, v_1$, a chord v_1v_4 and two vertices v_7 and v_8 of degree one adjacent to v_1 and v_4 respectively.

In Figure 3 is depicted an embedding of the graph O in $\mathbb{Z}^2$. We say that the position of O is $[x, y]$ if in such an embedding of O the vertex v_1 is placed at $[x, y]$.

The square lattice $\mathbb{Z}^2$ can be partitioned into copies of O, e.g. those at copies of O placed at positions $[4i + 2j, 2j]$ where $i, j \in \mathbb{Z}$. This partition is depicted in Figure 4 and through the proof we assume that it is fixed.

Figure 4: A partition of $\mathbb{Z}^2$ into isomorphic copies of O.

Assume that $X_1, \dots, X_k$ is a packing k-coloring of $\mathbb{Z}^2$. Let X be the union of X_1 and X_2. We bound the density of X according to Definition 3.3, but first we present some properties of X and O. For this purpose, a copy of O is called a *z-copy* if it contains exactly z vertices of X.

The goal is to show that on average every copy of O contains at most 5 vertices of X.

We assume that the partition contains a 6-copy $O[x, y]$ and without lost of generality assume, that $v_3, v_6, v_7, v_8 \in X_1$ and $v_2, v_5 \in X_2$.

Figure 5: A 6-copy $O[x, y]$ is the bottom left copy of O. The others are possibilities for a 5-copy $O[x + 2, y + 2]$.

We claim that if the partition contains another 6-copy $O[x+2i, y+2i]$ for some $i > 0$ then there exists $j \in [0, i]$ such that $O[x+2j, y+2j]$ contains strictly less than 5 vertices of X.

Observe that v_6 and v_7 of $O[x+2, y+2]$ do not belong to X. There are four possibilities of extending X such that $O[x+2, y+2]$ contains five vertices of X. They are depicted in Figure 5. All four possibilities force that v_6 and v_7 from $O[x+4, y+4]$ do not belong to X. Hence it becomes an invariant which propagates through the diagonal up to $O[x+2i, y+2i]$. This contradiction proves the claim.

Note that the choice of the diagonal depends on the configuration of X_1 and X_2 on $O[x, y]$. It is essential for the next argument that the diagonal can be always traversed such that the x coordinate grows. In the sequel we refer to such a configuration as to an *O-strip*.

It may happen that two O-strips have different orientations and hence they cross. Assume that the partition contains appropriate 6-copies $O[x-2i, y-2i]$ and $O[x-2j, y+2j]$ for positive i, j such that $O[x, y]$ is in the intersection of the two corresponding O-strips.

Assume also that between $O[x, y]$ and $O[x-2i, y-2i]$ are only 5-copies as well as for the other O-strip. We reuse the invariant from the previous paragraph and get that X contains no v_5, v_6, v_7 or v_8 of $O[x, y]$. Moreover, at most three vertices of $v_1, \dots, v_4$ may belong to X. Hence $O[x, y]$ contains at most three vertices of X. See Figure 6.

Figure 6: Intersection of two O-strips. In every possible intersection some vertices are forced to be in X_1, X_2, or they are not covered at all. The square vertices are not forced.

Now we are ready to prove the limit on the density of X. For every 6-copy C we traverse the diagonal while increasing the first coordinate. We either encounter a z-copy D where $z < 5$ or the diagonal consist only of 5-copies. The z-copy D is *a pairing copy* for C. Note that D can be in two pairs but then $z < 4$.

Let x be an arbitrary vertex. We use the fact that $\lim_{l\to\infty} \frac{|\Delta N_l(x)|}{|N_l(x)|} = 0$ on $\mathbb{Z}^2$. We denote by $O_l(x)$ the set of copies of O which are included in $N_l(x)$.

Now we show that $|X \cap N_l(x)| \leq 5|O_l| + c|\Delta N_l(x)|$. If a 6-copy and its pair copy are both in $O_l(x)$ then they contribute to $X \cap N_l(x)$ at most 10 vertices. Indeed, if the two copies are paired with a single copy of O then these three contain at most 15 vertices of X.

Observe that the number of 6-copies which has no pair copy in O_l is linear in $|\Delta N_l(x)|$ since traversing a diagonal of a copy of O without its pair in $O_l(x)$ ends on the boundary. Note that $O_l(x)$ does not have to cover whole $N_l(x)$ but it can miss linearly many vertices of the boundary. See Figure 7.

Figure 7: Bounding density of X in $N_l(x)$

Finally, the density of X is:

$$d(X) \leq \limsup_{l\to\infty} \left(\frac{5}{8} + \frac{c|\Delta N_l(x)|}{|N_l(x)|} \right) = \frac{5}{8}.$$

□

Theorem 3.11. *For the infinite square lattice $\mathbb{Z}^2$ it holds that $10 \leq \chi_\rho(\mathbb{Z}^2)$.*

Proof. We compute an upper bound on the density of the union of packings $X_1, X_2, \ldots, X_9$. The bound for the union of X_1 and X_2 is given in Lemma 3.10. The other packings are bounded separately by using Lemma 3.9.

$$d\Big(\bigcup_{i=1}^{9} X_i\Big) \quad \leq \quad \frac{5}{8} + \sum_{i=3}^{9} \frac{1}{A(i)} \quad = \quad \frac{3830381}{3837600} \; < \; 1.$$

Finally Lemma 3.5 implies that the packing chromatic number of $\mathbb{Z}^2$ is at least 10. □

4 Hexagonal lattices

We now turn our attention to the infinite hexagonal lattice $\mathcal{H}$. We first exhibit its packing coloring of $\mathcal{H}$ that uses only 7 colors. This result was already presented during the workshop Cycles and Colourings 2007 [3], but has not been published so far.

Theorem 4.1. *For the hexagonal lattice $\mathcal{H}$, $\chi_\rho(\mathcal{H}) \leq 7$.*

Proof. We exhibit a tiling of $\mathcal{H}$; refer to Figure 8. One class of the bipartition of the lattice $\mathcal{H}$ is the first color class X_1. The other class of bipartition can be covered by packings $X_2, \dots, X_7$. The pattern for filling the hexagonal lattice consists of 12 vertices. It is bordered by a bold line in the figure. $\square$

Figure 8: The pattern for partitioning hexagonal lattice using 7 packings of pairwise different width.

Our next goal is to show that six layers of the hexagonal lattice cannot be covered by a finite number of packings of pairwise different width. We follow the same approach as we have used for proving Theorem 3.8. We number the hexagonal layers of $P_6 \,\square\, H$ by $1,2,3,4,5,6$ where layer 1 and layer 6 are on the boundary. Every vertex is in one layer.

Lemma 4.2. *For every $l \geq 6$, the density of X_{2l} on $P_6 \,\square\, \mathcal{H}$ is at most $\frac{1}{9l^2-36l+66}$. The upper bounds on $d(X_2), d(X_4), \dots, (X_{10})$ are given in the next table.*

l	1	2	3	4	5
$d(X_{2l}) \leq$	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{1}{34}$	$\frac{1}{65}$	$\frac{1}{111}$

Proof. We count the size of $N_l(x)$ and obtain an upper bound on the density due to Lemma 3.6. The size of $N_l(x)$ depends on the choice of x. More precisely it depends on the layer of x. The smallest size of $N_l(x)$ is for x in one of the boundary layers. On the other hand it is the largest for layers 3 and 4. Hence we bound the size $N_l(x)$ from below by the size of N_l of vertices in layer 1.

Let y be a vertex of $\mathcal{H}$. Then the number of vertices at distance l is $3l$. Hence the number of vertices at distance at most l including y is

$$|N_{\mathcal{H}l}| := 1 + \sum_{i=1}^{l} 3i = 1 + 3\frac{(l+1)l}{2}.$$

For a vertex x in the layer 1 we compute the size of $N_l(x)$ in the following way:

$$|N_l(x)| = \sum_{i=l-5}^{l} |N_{\mathcal{H}i}| = 9l^2 - 36l + 66.$$

Note that the last equality holds only for $l \geq 6$. The values of $N_l(x)$ for smaler values of l were computed explicitly.

□

Lemma 4.3. *Any packings X_1, X_2, X_3, and X_4 on $P_3 \,\square\, \mathcal{H}$ staisfy that:*

- $d(X_3) \leq \frac{2}{18}$.
- $d(X_1 \cup X_2 \cup X_4) \leq \frac{12}{18}$.

Proof. We partition $P_3 \,\square\, \mathcal{H}$ into copies of $P_3 \,\square\, C_6$. The graph $P_3 \,\square\, C_6$ and partitioning of $\mathcal{H}$ into disjoint copies of C_6 are depicted in Figure 9.

The graph $P_3 \,\square\, C_6$ consists of three copies of C_6. We call them layer 1, layer 2, and layer 3 where layer 2 is the middle one.

The first claim of the lemma follows from the simple fact that $|X_3 \cap (P_3 \,\square\, C_6)| \leq 2$.

In the rest of the proof we abbreviate $X := X_1 \cup X_2 \cup X_4$.

Assume that it is possible to cover 13 vertices of $P_3 \,\square\, C_6$ by X. Then there is a copy C of C_6 such that $|X \cap C| = 5$. There are two possibilities of such a covering: either $|X_2 \cap C| = 1$ or $|X_2 \cap C| = 2$.

First we discuss the case that there are two layers with five vertices of X. The only possibility is that they are not neighbors because of vertices from X_4. Hence these layers are 1 and 3. Two cases of possible layer 1 are depicted in Figure 10. These two cases are compatible four cases for layer 3. We determined them by the position of a vertex from X_4 which is unique. It is not possible to cover more than one vertex in layer 2, therefore we get at most 11 covered vertices.

Now we know that one layer contains five vertices and the other two contain four vertices. We introduce two observations about X_2 and $X_1 \cup X_2$ which give us more information about possible structure of the layers.

The first observation is that if one of the layers contains two vertices of X_2 then the neighboring layer(s) does not contain any vertices of X_2. This holds since all vertices in the neighboring layers are at distance at most two from the vertices of X_2.

The second observation is that $P_3 \,\square\, C_6$ contains at most 11 vertices of $X_1 \cup X_2$. So let there be 12 such vertices. One layer may contain at most four vertices of $X_1 \cup X_2$. Hence every layer contains four of them. Moreover, every layer contains exactly one vertex of X_2 since every layer must contain at least one. Take the middle layer and let v be the vertex from X_2. Since we want to cover four vertices of the middle layer, the vertices of X_1 are determined by the position of v. Then vertices of X_1 are also determined in the other two layers since there must be three of them in each; refer Figure 9. Now the only two vertices left for X_2 in layers one and three are too close to each other hence it is not possible to cover 12 vertices by $X_1 \cup X_2$.

Figure 9: On the left-hand side is a possible tiling of the hexagonal lattice using C_6's. On the right-hand side is a coverage of $C_6 \square P_3$ by X_1 and X_2 which contains 9 vertices of X_1 and a vertex of X_2 in the middle layer. There are only two other candidate vertices for X_2, which are square vertices. But they are too close to be both in X_2.

Figure 10: Layer 1 contains five vertices of X. There are two possibilities. The first one is on the left and the second one is on the right. Layer 3 contains also five vertices of X. Vertices from the middle layer are assigned lists of available packings.

Since $X_1 \cup X_2$ covers at most 11 vertices and we want to cover 13 vertices, we derived that two vertices must be from X_4. These two vertices must be in layer 1 and layer 3. Hence the layer containing five vertices of X must be layer 1 or layer 3. Assume assume without lost of generality that it is layer 1. Note that it cannot be the middle one since it does not contain a vertex from X_4. The other two layers must each contain four vertices of X.

Hence the middle layer must contain one vertex from X_2 and three vertices of X_1. This implies that the first layer contains only one vertex from X_2. Hence we know the configurations for layer 1 and layer 2. See Figure 11. We observe that there are only three vertices in layer 3 which can be in X. Hence we failed to include 13 vertices of $P_3 \square C_6$ to X.

$\square$

Note that in the following lemma we get a better estimate while counting

Figure 11: Let layer 1 contain five vertices of X and layer 2 contain four vertices of X. They must look as depicted. Vertices of the third layer have assigned lists of possible colors. But there are only three with nonempty list.

on $P_3 \,\square\, \mathcal{H}$ instead of $P_6 \,\square\, \mathcal{H}$. We are able to obtain at most $\frac{1}{21.8}$ for $P_6 \,\square\, \mathcal{H}$.

Lemma 4.4. *The density of any packing X_5 on $P_3 \,\square\, \mathcal{H}$ is at most $\frac{1}{21.9}$.*

Proof. We bound the density using Lemma 3.6. We compute $A(x, 5)$ in $P_3 \,\square\, \mathcal{H}$ for a vertex x in one of two outer layers. Assume layer 1 for x. Then the area consists of vertices in $N_2(x)$ together with the part obtained from vertices at distance three from x. We distinguish several types of these vertices.

- six vertices from the layer 1 have one neighbor in $N_2(x)$,
- three vertices from the layer 1 have two neighbors in $N_2(x)$,
- six vertices from the layer 2 have two neighbors in $N_2(x)$,
- three vertices from the layer 3 have two neighbors in $N_2(x)$.

In total we have:

$$A(x, 5) = 15 + \frac{6}{4} + \frac{6}{4} + \frac{12}{5} + \frac{6}{4} = 21.9\,.$$

For a vertex x from the middle layer the area $A(x, 5)$ is 25.4 hence we can estimate the area by 21.9 for any vertex of $P_3 \,\square\, \mathcal{H}$. Refer to Figure 12 for three hexagonal layers of $P_3 \,\square\, \mathcal{H}$ and $N_2(x)$. □

Figure 12: Three layers of hexagonal lattice. Black square corresponds to x. Black vertices correspond to vertices from $N_2(x)$ and white vertices are at distance 3 from x.

Theorem 4.5. *For any $m \geq 6$ it holds that $\chi_\rho(P_m \,\square\, \mathcal{H}) = \infty$.*

Proof. Assume $m = 6$. We show that the sum of densities of all k-packing is strictly less than 1 and we get a contradiction with Lemma 3.5.

The lattice $P_6 \,\square\, \mathcal{H}$ can be partitioned into two copies of $P_3 \,\square\, \mathcal{H}$. Hence we can use bound on $X_1 \cup X_2 \cup X_3 \cup X_4$ from Lemma 4.3. Also X_5 can be bounded using Lemma 4.4. Since a $(2l+1)$-packing is also a $2l$-packing we bound the density of X_{2l+1} by the density of X_{2l}. Note that the density of X_{2l} may be bounded by $\frac{1}{2l^2}$.

We get the contradiction by the following estimate that holds for any packing coloring $X_1, \ldots, X_k$:

$$
\begin{aligned}
d\Big(\bigcup_{i=1}^{k} X_i\Big) &\leq \frac{14}{18} + \frac{1}{21.9} + \sum_{i=6}^{\infty} d(X_i) \\
&\leq \frac{541}{657} + \sum_{i=6}^{59} d(X_i) + \sum_{i=30}^{\infty} \frac{2}{(2i)^2} \\
&\leq 0.982 + \frac{1}{2}\int_{i=29}^{\infty} \frac{di}{i^2} \quad \leq \quad 0.982 + \frac{1}{58} \quad < \quad 1\,.
\end{aligned}
$$

Again, the exact value of the sum of the first 59 summands was enumerated by a computer program. □

5 Conclusion

In our opinion the following related problems deserve further exploration:

- When the step to infinity occurs for $\chi_\rho(P_m \,\square\, \mathcal{H})$?
- Determine the exact value of $\chi_\rho(\mathbb{Z}^2)$ as suggested by Goddard et al. [5].
- Do planar cubic graphs have bounded packing chromatic number? (Brought to us by R. Škrekovski.)

References

[1] B. Brešar, S. Klavžar and D. F. Rall, On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, Discrete Appl. Math. 155 (2007) 2303–2311.

[2] J. Fiala and P. A. Golovach, Complexity of the packing chromatic problem on trees, manuscript, 2008.

[3] J. Fiala and B. Lidický, Packing chromatic number for lattices. Abstract in: Workshop Cycles and Colourings 2007 (I. Fabrici, M. Horňák, S. Jendrol', eds.), IM Preprint, series A, No. 8/2007.

[4] A. Finbow and D. F. Rall, On the packing chromatic number of some lattices, submitted to Discrete Appl. Math., special issue LAGOS'07.

[5] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris and D. F. Rall, Broadcast chromatic numbers of graphs, Ars Combin. 86 (2008) 33–49.

[6] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, Wiley-Interscience, New York, 2000.

[7] C. Sloper, An eccentric coloring of trees, Australas. J. Combin. 29 (2004) 309–321.

[8] A. Vesel, private communication, 2007.