On well-quasi-ordering lower sets of finite trees, a new proof

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Abstract

In 1964, Nash-Williams proved in [6] that the iterated lower sets (ideals) of finite trees are well-quasi-ordered by the subset relation. However, acknowledges that his proof is very complicated. Robertson in 1997, [13], conjectured a "Lifting lemma" which claims that every ideal can be described by a finite tree in such way that an embedding relation between two description trees of two ideals lifts back to the subset relation between the corresponding two ideals.We prove the lifting conjecture in the affirmative. As a corollary, a simple proof of the result of [6] follows.

1 Introduction

We start with an easy analogous example. Let $\mathbb{N}_0$ denote the set of non-negative integers. Let $L\mathbb{N}_0$ be the set of all finite subsets of $\mathbb{N}_0$ (ordered by the subset relation "$\subseteq$") such that each $S \in \mathbb{N}_0$ has the property that if $k \in S$ and $j \leq k$ then $j \in S$. Then clearly, $S = \{0, 1, 2, \ldots, m\} = [m]$ for some $m \in \mathbb{N}_0$. If $S = [m]$ and $S' = [m']$, then $S \subseteq S'$ if and only if $m \leq m'$. What is nice here is that the $\subseteq$ relation is replaced by the basic

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relation $\leq$ of $\mathbb{N}_0$. We deduce that $L\mathbb{N}_0$ has an obvious order preserving isomorphism to $\mathbb{N}_0$, which sends $[m]$ to m. Define $L^n\mathbb{N}_0 = L(L^{(n-1)}\mathbb{N}_0)$, and we observe that inductively $L^n\mathbb{N}_0$ has an order preserving isomorphism to $\mathbb{N}_0$, for all $n \geq 1$. Hence $\mathbb{N}_0$ is totally ordered implies so is $L^n\mathbb{N}_0$, for any n. This example is trivial of course, since we have sets that are finite and totally ordered. However, it shows the key idea of our proof. Trees can have arbitrarily long anti-chains. A set of trees we study can be infinite, and need not have a maximal element. Nevertheless, we ask if we have a similar reduction of the iterated sets to the set of trees in a such a way that the relation between these reduced trees governs the subset relation between the iterated sets? The conjecture of Robertson is positive to this question. We start with basic definitions and state the conjecture formally.

A *quasi-order* $\leq$ on a set Q is a reflexive and transitive relation. Let Q be a set quasi-ordered by $\leq$. A *lower set* of a set Q (or an *ideal*, for short) is a subset I of Q which is closed down under $\leq$. That is, for all $y \in I$ and all $x \in Q$, if $x \leq y$ then $x \in I$. We say Q is *well-quasi-ordered*, (*wqo*), if every infinite sequence in Q is "good", (A sequence $q_1, q_2, \ldots$ of Q is *good* if there exist indexes i, j such that $i < j$ and $q_i \leq q_j$).

Trees in this paper are finite, rooted and directed away from the root. We assume the edge set $E(T)$ of a tree T is labeled from a finite ordinal, say $[m] = \{0, 1, \ldots, m\}$, and the vertex set $V(T)$ is labeled from a set Q, that is quasi-ordered by $\leq_Q$. The root of T, denoted $root(T)$, is a special vertex that has a *root-edge label* from $[m]$, in addition to its vertex label from Q. Hence, we assume every vertex $v \in V(T)$ has a pair of labels, $l(v) = (p, q)$, where $p \in [m], q \in Q$, and refer to p as $l_1(v)$ and to q as $l_2(v)$. If v is not a root, then we can treat $l_1(v)$ as labeling the unique edge $e = uv$ and will sometimes refer to it as the *edge-label of e*. Where no misunderstanding can occur we will write $l(e)$ instead of $l_1(v)$ for emphasis and clarity. Given trees T, T', we say that T is a *topological minor of T'* (and write $T \leq_t T'$) if a subdivision of T is isomorphic (without the labels) to a subtree T'' of T', such that $l_2(v) \leq_Q l_2(v'')$ for each $v \in V(T)$, where v'' is the isomorphism corresponding vertex in $V(T'')$. Note that the relation $T \leq_t T'$ is an injective mapping $f : V(T) \to V(T')$ and also that it ignores the edge label $l_1(v)$.

However, if $T \leq_t T'$, then f maps every edge e of T to a unique path $f(e)$ in T'. The map f is said to satisfy the *Kruskal-Friedman gap-condition* (written $T \leq_{KF} T'$) if in addition $l(e) \leq l(e')$ for every edge e' in $f(e)$. A stronger version of $\leq_{KF}$ has one more condition that $l_1(root(T)) \leq l_1(v')$ for every $v' \in V(T')$ that is on the path from $root(T')$ to $f(root(T))$. This stronger version is what we use throughout this paper. The following is Robertson's Lifting Conjecture.

Conjecture 1 (Lifting Lemma) *For every $\leq_{KF}$-ideal $\mathcal{I}$, there exists a finite structure-tree $\psi(\mathcal{I})$ that is labeled from a wqo set such that for any two ideals $\mathcal{I}, \mathcal{I}'$ if $\psi(\mathcal{I}) \leq_{KF} \psi(\mathcal{I}')$ then $\mathcal{I} \subseteq \mathcal{I}'$.*

Let $\mathcal{F}(Q)$ denote the set of all finite trees with vertices labeled from Q, and let $\mathcal{F}(m, Q)$ denote the set of all finite trees we get from $\mathcal{F}(Q)$ by labeling the edges from $[m], m \geq 0$. Note that when $m = 0$, the relation $\leq_{KF}$ reduces to $\leq_t$ and so is a natural generalization of Kruskal's relation $\leq_t$. Kruskal [3] proved in 1960 that $\mathcal{F}(Q)$ is wqo by $\leq_t$. Nash-Williams gave in [2] a short and elegant proof of [3].

Theorem 2 *(**J. Kruskal [3]**) If Q is wqo, then $\mathcal{F}(Q)$ is wqo by $\leq_t$.*

Friedman [1] strengthened Kruskal's Theorem by the following:

Theorem 3 *(**H. Friedman, [1]**) If Q wqo, then $\mathcal{F}(m, Q)$ is wqo by $\leq_{KF}$.*

The class LQ of ideals of Q is quasi-ordered by the subset relation " $\subseteq$ ". Inductively, $L^nQ = L(L^{n-1}Q)$, is defined to be n-th iterated ideal of ideals. Note that due to Rado's counterexample, a set Q is wqo by $\leq$ relation does not imply LQ is wqo by the subset relation $\subseteq$. In fact, Kruskal has shown by an unpublished example that for each n there exists a set $Q(n)$ such that $L^nQ(n)$ is wqo and $L^{(n+1)}Q(n)$ is not. Then, Nash-Williams in [6] proved this does not occur for finite trees:

Theorem 4 *(**Nash-Williams**) For all $n \in \mathbb{N}$, if L^nQ is wqo, then $L^n\mathcal{F}(Q)$ is wqo by $\leq_t$.*

We assume the following result from [10]: (see next section for the definition of ψ)

Theorem 5 *For every ideal $\mathcal{I} \subseteq \mathcal{F}(m, Q)$, $m \geq 0$ an integer and Q wqo, there is a finite tree $\psi(\mathcal{I})$ in $\mathcal{F}(2m + 4, Q')$ that constructs precisely the elements of $\mathcal{I}$, where Q' is wqo.*

By applying Theorem5 and the Lifting Lemma, we shall show a short proof of:

Theorem 6 *For all $n \in \mathbb{N}$, if L^nQ is wqo, then $L^n\mathcal{F}(m, Q)$ is wqo.*

In the next section we prove the lifting conjecture and Theorem 6. We conclude with discussion on future directions of this research.

2 The Lifting Lemma and proof of Theorem 6

First, we define a basic operation of constructing a new tree by "tree-summing" a finite number of trees in $\mathcal{F}(m, Q)$. The *null-tree*, Γ, is defined by $V(\Gamma) = E(\Gamma) = \emptyset$. Clearly, Γ is not a rooted tree. Let $T_1, T_2, \ldots, T_n, n \geq 0$, be pairwise vertex disjoint trees or Γ. We call each $T_i, 1 \leq i \leq n$ a *summand*. For $p \in [m]$ and $q \in Q$ the *tree-sum* of $T_1, T_2, \ldots, T_n$ is given by $T = Tree(T_1, T_2, \ldots, T_n)_{(p,q)}$ where for all i the labels of T_i is preserved, t_0 is a new vertex (the root of T) labeled by (p, q), $V(T) = V(T_1) \cup V(T_2) \cup \ldots \cup V(T_n) \cup \{t_0\}$, and its set of edges is $E(T_1) \cup E(T_2) \cup \ldots \cup E(T_n) \cup \{e_i = (t_0, root(T_i)) : 1 \leq i \leq n, T_i \neq \Gamma\}$. (note that $Tree(T_1, T_2, \Gamma)_{(p,q)}$ is isomorphic to $Tree(T_1, T_2)_{(p,q)}$ but allowing the null-tree to appear in a tree-sum will be convenient later). Note also that for each new edge e_i we have $l(e_i) = l_1(root(T_i))$.

Next we define a tool called "rst-cell", that is used to construct trees by using tree-sum operation:

$$H = (k; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; \mathcal{I}_\infty)_{\mathcal{I}_0} \tag{1}$$

is an *rst-cell* where the parameters in H are defined as follows: $k \in \mathbb{N} \cup \{0, \infty\}$ is the *width* of H and is denoted by $k(H)$; we allow $k(H) = \infty$ only if $n = 0$ and $\mathcal{I}_\infty = \emptyset$; $\mathcal{I}_i$ for $i = \infty$ or $i = 0, 1, \ldots, n$ is an ideal of $\mathcal{F}(m, Q)$; If $1 \leq i \leq n$, then $\mathcal{I}_i \neq \emptyset$. Every tree in $\mathcal{I}_0$ has height zero. If $0 < i < \infty$, we say $\mathcal{I}_i$ is a *middle-component* of H. We say $\mathcal{I}_\infty$ is the *right-component* of H, and $\mathcal{I}_0$ is the *root-component* of H. Two rst-cells H and H' are assumed to be *equal* if they differ only by a permutation of their middle-components.

Let $\mathcal{H}$ be a set of rst-cells and $H \in \mathcal{H}$ be as in Equation (1). An ideal $\mathcal{I}$ is said to satisfy condition

(*) if $Tree(T_1, \ldots, T_k, T_{k+1}, \ldots, T_{k+n}, T_{k+n+1}, \ldots, T_{k+n+s})_{(p,q)}$ belongs to $\mathcal{I}$, where $s \geq 0$, $(p, q) \in \mathcal{I}_0$ is called the **root-part** ; the **left-part** for $a = 1, \ldots, k, T_a \in \mathcal{I} \cup \{\Gamma\}$; the **middle-part** for $a = k+1, \ldots, k+n$, $T_a \in \mathcal{I}_a \cup \{\Gamma\}$; **right-part** for $a = k+n+1, \ldots, s$, $T_a \in \mathcal{I}_\infty$. We define the ideal $I(\mathcal{H})$ to be the intersection of all ideals satisfying (*).

Let $\alpha(\mathcal{I}) = \max\{l_1(root(T)) : T \in \mathcal{I}\}$. Let T be a tree, $\mathcal{H}$ be a set of rst-cells and $H \in \mathcal{H}$ be as in Equation (1). Then, we say that T *conforms to H in $I(\mathcal{H})$* if either T is in a component ideal of H such that $l_1(root(T)) \leq \alpha(\mathcal{I}_0)$ or T is a tree-sum as in (*) where the left-part summands are from $I(\mathcal{H}) \cup \{\Gamma\}$. The following is obvious by definition and by the fact that $I(\mathcal{H})$

is an ideal:

Theorem 7 *Let $\mathcal{H}$ be a set of rst-cells. Let T be a tree. Then, $T \in I(\mathcal{H})$ if and only if T conforms to H in $I(\mathcal{H})$.*

Let H be a width zero rst-cell. If $\mathcal{I}'$ is a component of H and if $I(\{H\}) \subseteq \mathcal{I}'$, then we assume H is of the form $(0;\mathcal{I}';\emptyset)_{\mathcal{I}_0}$. Further, if $\alpha(\mathcal{I}_0) < \alpha(\mathcal{I}')$, then we call H a *trimmer-cell.* and denote it by a unique symbol H^*. Note that if $H^* = (0;\mathcal{I}';\emptyset)_{\mathcal{I}_0}$ is a trimmer-cell then $\mathcal{I}' \not\subseteq I(\{H^*\}) \subset \mathcal{I}'$. A set of rst-cells $\mathcal{H}$ need not have a trimmer-cell. So if $\mathcal{H}$ has no trimmer-cell we assume H^* is null. For convenience, if H^* is null then we assume $\{H^*, H_1, H_2 \ldots, H_t\} = \{H_1, H_2 \ldots, H_t\}$.

Definition 8 *Let $\mathcal{I}$ be an ideal such that $\alpha(\mathcal{I}) = m$. Then, a set of rst-cells $\mathcal{H} = \{H^*, H_1, H_2 \ldots, H_t\}$ with at most one trimmer-cell H^* is a proper spanning set of $\mathcal{I}$ if the following three properties hold:*

(P1) *$I(\{H^*\}) \subset \mathcal{I}$ and if $\mathcal{I}'$ is a component of H in $\mathcal{H}$-$\{H^*\}$, then $\mathcal{I}' \subset \mathcal{I}$. (induction axiom)*
(P2) *$I(\mathcal{H}) = \mathcal{I}$. (spanning axiom)*
(P3) *for any component $\mathcal{I}'$ that is not a root-component of H^*, $\alpha(\mathcal{I}') = m$. (chain axiom)*

Let $\mathcal{I}$ be an ideal. A middle or a right-component $\mathcal{I}'$ in a proper span $\mathcal{H}$ of $\mathcal{I}$ is a *descendant of* $\mathcal{I}$. Recursively, any descendant $\mathcal{I}''$ of $\mathcal{I}'$ is also a descendant of $\mathcal{I}$. Call a sequence $X = (\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \ldots)$ a *descendant-sequence*, if for each $i \geq 1$, $\mathcal{I}_i$ is a descendant of $\mathcal{I}_{i-1}$. Let $X = (\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \ldots)$ be a descendant-sequence. For some $i \geq 1$, if $\mathcal{I}_{i+1}$ is a component of a trimmer-cell and $\mathcal{I}_{i+1} = \mathcal{I}_j$, for some $j < i$, then we say $\mathcal{I}_i$ is an *ancestor-dependent* or more specific $\mathcal{I}_j$-dependent. The difference $\delta = |i - j|$ is the *distance* of $\mathcal{I}_i$ from its ancestor $\mathcal{I}_j$. If $\mathcal{I}_j \neq \mathcal{I}_{i+1}$ for all $j, 0 \leq j < i$, then we say $\mathcal{I}_i$ is *ancestor-free.* A recursive definition of "Structure-tree" follows:

Definition 9 (Structure-tree) *Let $\mathcal{I}$ be an ideal in $\mathcal{F}(m, Q)$, where $m \geq 0$ is an integer and Q is a wqo set. Let $\mathcal{H}(\mathcal{I}) = \{H^*, H_1, H_2, \ldots, H_t\}$ be a proper span of $\mathcal{I}$. If $\mathcal{I} = \emptyset$, then set $\psi(\mathcal{I}) = (0, q_0)$. Otherwise,*

$$\psi(\mathcal{I}) = Tree(R^*, R_1, R_2, \ldots, R_t)_{(\alpha(\mathcal{I}), \hat{q})}, \tag{2}$$

where $R^*(\mathcal{I})$ corresponds to $H^* \in \mathcal{H}(\mathcal{I})$ and for $i = 1, 2, \ldots t$, $R_i(\mathcal{I})$ corresponds to $H_i = (k_i; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_{n_i}; \mathcal{I}_\infty)_{\mathcal{I}_0} \in \mathcal{H}$, such that :

(i) If $H^*(\mathcal{I})$ is null, then $R^*(\mathcal{I}) = \Gamma$. Otherwise

$$R^*(\mathcal{I}) = Tree(I(\{H^*\}))_{(2m+4,\mathcal{I}_0)}.$$

(ii) For each $i, 1 \leq i \leq t$, construct the summand $R_i(\mathcal{I})$ on the path P_{n_i+3} with vertices $v_r, v_\infty, v_0, v_1, v_2, \ldots, v_{n_i}$, rooted at v_r, set $l(v_r(R_i)) = (2m+3, (k_i, \mathcal{I}_0))$, set $l(v_\infty(R_i)) = (2m+2, q_0)$ and for $0 \leq j \leq n_i$, set $l(v_j(R_i)) = (2m+1, q_0)$.

(iii)Join $(0, q_0)$ to $v_0(R_i)$ by an edge. For each $j, j = \infty$ or $1 \leq j \leq n_i$, using a proper span $\mathcal{H}(\mathcal{I}_j)$ of $\mathcal{I}_j$, join the root of $\hat{\psi}(\mathcal{I}_j)$ to v_j by an edge, where $\hat{\psi}(\mathcal{I}_j) = \psi(\mathcal{I}_j)$, if $\mathcal{I}_j$ is ancestor-free. Otherwise change $l_1(\psi(\mathcal{I}_j))$ to $(1 + \alpha(I(\{H^*(\mathcal{I}_j)\})) + \alpha(\mathcal{I}_j), [\hat{q}, \delta])$, set $H^*(\mathcal{I}_j)$ to null and proceed to (i).

Proof. (**of the Lifting Lemma1**) Let $\mathcal{I}$ and $\mathcal{I}' \in L\mathcal{F}(m, Q)$ with their structure-trees $\psi(\mathcal{I}) = Tree(R^*, R_1, \ldots, R_t)_{(\alpha(\mathcal{I}),\hat{q})}$ and $\psi(\mathcal{I}') = Tree(R'^*, R'_1, \ldots, R'_{t'})_{(\alpha(\mathcal{I}'),\hat{q})}$ be given. Their summands correspond to the rst-cells obtained from proper spans $\mathcal{H}(\mathcal{I})$ and $\mathcal{H}'(\mathcal{I}')$. For a contradiction, assume $\psi(\mathcal{I}) \leq_{KF} \psi(\mathcal{I}')$ by a gap embedding f and that $\mathcal{I} \not\subseteq \mathcal{I}'$, where the ideal $\mathcal{I}$ is chosen to be as small as possible. For the chosen $\mathcal{I}$ we may also choose $\psi(\mathcal{I}')$ so that $f(root(\psi(\mathcal{I}))) = root(\psi(\mathcal{I}'))$. By the gap-condition, we see that for any edge e of $\psi(\mathcal{I})$, if $l(e) \geq 2m+2$, then $f(e) = e'$ is an edge of $\psi(\mathcal{I}')$. If $R^* \neq \Gamma$, then its root has unique largest label $2m+4$, and so we have $R^* \leq_{KF} R'^*$ and $f(root(R^*)) = root(R'^*)$. Hence, we have $\psi(I(\{H^*\})) \leq_{KF} \psi(I(\{H'^*\}))$ and so by induction $I(\{H^*\}) \subseteq I(\{H'^*\})$. Similarly and by rearranging if necessary, we have $R_i \leq_{KF} R'_i$ and $f(root(R_i)) = v_r(R_i) = v_r(R'_i) = root(R'_i)$, for $i = 1, 2, \ldots, t$. Let the corresponding rst-cells are $H_i = (k_i; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_{n_i}; \mathcal{I}_\infty)_{\mathcal{I}_0} \in \mathcal{H}$, and $H'_i = (k'_i; \mathcal{I}'_1, \mathcal{I}'_2, \ldots, \mathcal{I}'_{n'_i}; \mathcal{I}'_\infty)_{\mathcal{I}'_0} \in \mathcal{H}'$.

We show that every counterexample tree T of minimal height $h, h \geq 1$, that conforms to H_i in $I(\mathcal{H})$ also conforms to H'_i in $I(\mathcal{H}')$, which is absurd. From $f(root(R_i)) = root(R'_i)$, we have $k_i \leq k'_i$ and $\mathcal{I}_0 \subseteq \mathcal{I}'_0$. Hence, (using induction on h for $k_i \leq k'_i$ summands of T for the left-part) it suffices to show that for every component $\mathcal{I}_a$ of H_i there is a unique component $\mathcal{I}'_{a'}$ of H'_i such that $\mathcal{I}_a \subseteq \mathcal{I}'_{a'}$. Also, if $\mathcal{I}_a = \mathcal{I}_\infty$, then we must show that $\mathcal{I}'_{a'} = \mathcal{I}'_\infty$. Let P be the path induced by $v_r(R_i), v_\infty(R_i), v_0(R_i), v_1(R_i), \ldots, v_{n_i}(R_i)$, and also P' be the path induced by $v_r(R'_i), v_\infty(R'_i), v_0(R'_i), v_1(R'_i), \ldots, v_{n'_i}(R'_i)$. By the gap-condition, we have $f(P) \subseteq P'$, since any other edge incident to P' has label at most $2m$. In particular, $f(v_\infty(R_i)) = v_\infty(R'_i)$. We have $\hat{\psi}(\mathcal{I}_a) \leq_{KF} \hat{\psi}(\mathcal{I}'_{f(a)})$ for some descendant $\mathcal{I}'_{f(a)}$ of $\mathcal{I}'$ such that

$f(root(\hat{\psi}(\mathcal{I}_a)) = root(\hat{\psi}(\mathcal{I}'_{f(a)}))$. Further, if $a = \infty$ then either $f(a) = \infty$ or $\mathcal{I}'_{f(a)}$ is a descendant of $\mathcal{I}'_\infty$. We claim that $\psi(\mathcal{I}_a) \leq_{KF} \psi(\mathcal{I}'_{f(a)})$. The case $\mathcal{I}_a$ ancestor-free is trivial. If it is $\mathcal{I}$-dependent, then $l_1(root(\hat{\psi}(\mathcal{I}_a))) \geq m+1$. So, by the gap-condition, $\mathcal{I}'_{f(a)}$ and all of its ancestors (except $\mathcal{I}'$) are also ancestor-dependent. Let $(\mathcal{I}' = \mathcal{I}'_{a_r}, \ldots, \mathcal{I}'_{a_{(r-1)}}, \ldots, \mathcal{I}'_{a_1}, \ldots, \mathcal{I}'_{a_0} = \mathcal{I}'_{f(a)})$ be the subsequence of the descendant-sequence such that $\mathcal{I}'_{a_i}$ is $\mathcal{I}'_{a_{i+1}}$-dependent, $0 \leq i \leq r-1$. For $0 \leq i \leq r$, we have $l_1(root(\hat{\psi}(\mathcal{I}'_{a'_i}))) \geq l_1(root(\hat{\psi}(\mathcal{I}_a)))$. By subtracting $m+1$ from each side of these inequalities (see Definition 9 (iii)), and since we also have $\psi(\mathcal{I}) \leq \psi(\mathcal{I}')$, we deduce that $\psi(I(\{H^*(\mathcal{I}_a)\})) \leq_{KF} \psi(I(\{H^*(\mathcal{I}'_{f(a)}\}))$.

The claim follows. By the choice of $\mathcal{I}$, we have $\mathcal{I}_a \subseteq \mathcal{I}'_{f(a)}$. Since every ancestor $\mathcal{I}''$ of $\mathcal{I}'_{f(a)}$, by (P3), has $\alpha(\mathcal{I}'') = m$, we have $(\mathcal{I}', \mathcal{I}'_{a'}, \ldots, \mathcal{I}'', \ldots, \mathcal{I}'_{f(a)})$, a strictly descending descendant-sequence, where $\mathcal{I}'_{a'}$ in H'_i. Hence, $\mathcal{I}_a \subseteq \mathcal{I}'_{a'}$ as needed. □

Proof. (of Theorem 4) Let $X = \mathcal{I}_1, \mathcal{I}_2, \ldots$ be any infinite sequence of $L^n\mathcal{F}(m, Q)$. The case $n = 1$ is obvious. Inductively, let $\bar{\mathcal{I}}_i = \{T \in \mathcal{F}(2^n(m+4)-4, L^nQ) : T \leq_{KF} \psi(\mathcal{I}), \mathcal{I} \in \mathcal{I}_i\}, i \geq 1$. Let $\psi(\bar{\mathcal{I}}_1), \psi(\bar{\mathcal{I}}_2), \cdots$ be the structure-tree sequence in $\mathcal{F}(2^{(n+1)}(m+4)-4, L^{n+1}Q))$. By Friedman' Theorem, we have $i, j, i < j$ such that $\psi(\bar{\mathcal{I}}_i) \leq_{KF} \psi(\bar{\mathcal{I}}_j)$ and by the Lifting Lemma, $\bar{\mathcal{I}}_i \subseteq \bar{\mathcal{I}}_j$. Now let $\mathcal{I} \in \mathcal{I}_i$. Then $\psi(\mathcal{I})$ and $\psi(\mathcal{I}')$ for some $\mathcal{I}' \in \mathcal{I}_j$, are in $\bar{\mathcal{I}}_j$ such that $\psi(\mathcal{I}) \leq_{KF} \psi(\mathcal{I}')$. This lifts to $\mathcal{I} \subseteq \mathcal{I}'$. Since $\mathcal{I}_j$ is closed under $\subseteq$, we have $\mathcal{I} \in \mathcal{I}_j$. Hence $\mathcal{I}_i \subseteq \mathcal{I}_j$. So, X is good. □

One may observe that we can continue iteration using transfinite induction. Assuming the result of Křiž, [2], such an iteration does yield wqo property of iterated ideals beyond the finite case. This in turn is directly related, [4], to some of the deeper results of Nash-Williams in [7], [8]. Our next paper will study this particular subject [11].

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