Backbone Colorings and Generalized Mycielski's Graphs*

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Abstract

For a graph G and its spanning tree T the *backbone chromatic number*, $\mathrm{BBC}(G,T)$, is defined as the minimum k such that there exists a coloring $c\colon V(G) \to \{1,2,\dots,k\}$ satisfying $|c(u)-c(v)| \geq 1$ if $uv \in E(G)$ and $|c(u)-c(v)| \geq 2$ if $uv \in E(T)$.

Broersma et al. [1] asked whether there exists a constant c such that for every triangle-free graph G with an arbitrary spanning tree T the inequality $\mathrm{BBC}(G,T) \leq \chi(G)+c$ holds. We answer this question negatively by showing the existence of triangle-free graphs R_n and their spanning trees T_n such that $\mathrm{BBC}(R_n,T_n) = 2\chi(R_n)-1 = 2n-1$.

In order to answer the question we obtain a result of independent interest. We modify the well known Mycielski's construction and construct triangle-free graphs J_n, for every integer n, with chromatic number n and 2-tuple chromatic number $2n$ (here 2 can be replaced by any integer t).

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1 Introduction

1.1 Backbone colorings

The backbone coloring problem is related to frequency assignment problems in the following way, the transmitters are represented by the vertices of a graph and they are adjacent in the graph if the corresponding transmitters are close enough or transmitters are strong enough. The problem is to assign frequency channels to the transmitters in such a way that the interference is kept at an "acceptable" level. One way of putting these requirements together is following: Given graphs G_1, G_2 such that G_1 is a spanning subgraph of G_2. Determine a coloring of G_2 that satisfies certain restriction of one type in G_1 and of the other type in G_2.

In this concept, backbone colorings were introduced and motivated and put into a general framework of related coloring problems in [1]. Let us recall some basic definitions. In the sequel we deal with undirected simple graphs, i.e. without loops and/or multiedges. By the symbol $[n]$ we understand the set $\{1, 2, \ldots, n\}$, by the symbol $\chi(G)$ the chromatic number of G, and by the symbol $G[W]$ the subgraph induced by the vertex set $W \subseteq V(G)$. For a graph G, we define a coloring $\nu : V \to \{1, 2, \ldots, k\}$ to be a *backbone k-coloring* of a graph G with a backbone graph $H \subseteq G$, if for every two different vertices u and v of G, it holds

- $|\nu(u) - \nu(v)| \geq 1$, if $uv \in E(G) \setminus E(H)$;
- $|\nu(u) - \nu(v)| \geq 2$, if $uv \in E(H)$.

The minimum k for which G with backbone H admits a backbone k-coloring is called the *backbone chromatic number* of G with backbone H. It is denoted by $\mathrm{BBC}(G, H)$. In this paper we consider only the case when a backbone graph H is acyclic.

We refer to several results concerning backbone colorings of graphs. The connection between the backbone chromatic number and the chromatic number is studied in [1]. The authors showed that the backbone chromatic number of a graph G is at most $2\chi(G) - 1$, while they provided examples where this bound is attained. To show this inequality it is sufficient to color the graph G with colors 1, 3,..., $2\chi(G) - 1$. The decision problem if there exists a backbone coloring of a graph G with backbone tree T with l colors is NP-complete for $l \geq 5$. Broersma et al. in [2] showed that the backbone chromatic number of planar graphs with backbone matchings is at most six. Other results on backbone colorings appear in [3, 5].

We deal with the intriguing question posed by Broersma et al. [1].

Question 1.1. *Does there exists a constant c such that* $\mathrm{BBC}(G,T) \leq \chi(G)+c$ *holds for every triangle-free graphs G with T being a tree?*

We will present infinite class of triangle-free graphs answering the question negatively. More precisely, for every integer n, we will show the existence of a triangle-free graph G with a backbone tree T such that $\mathrm{BBC}(G,T) = 2\chi(G)-1 = 2n-1$.

1.2 Triangle-free graphs and their colorings

For integers k and t we define a *t-tuple k-coloring* of a graph G to be a function $c\colon V(G) \to \binom{[k]}{t}$ such that $c(u) \cap c(v) = \emptyset$ whenever $uv \in E(G)$. The minimum possible k that G has a t-tuple k-coloring is called the *t-tuple chromatic number*, denoted by $\chi_t(G)$.

The procedure of giving a negative answer to Question 1.1 will proceed in the following steps:

I. For a given triangle-free graph G, we will construct an infinite triangle-free graph R_G with a backbone tree T_G such that $\mathrm{BBC}(R_G,T_G) \geq \chi_2(G)-1$ and $\chi(R_G)=\chi(G)$.

II. For a given triangle-free graph G, we will present a Mycielski-type construction of a triangle-free graph $J(G)$ such that $\chi_2(J(G)) \geq \chi_2(G)+2$ and $\chi(J(G)) \leq \chi(G)+1$. In particular, it follows that $\chi(J_n)=n$ and $\chi_2(J_n)=2n$, where $J_n = J^{n-2}(K_2)$ and K_2 is the complete graph on 2 vertices.

III. From the previous two steps, $\mathrm{BBC}(R_{J_n},T_{J_n}) \geq 2n-1 = 2\chi(R_{J_n})-1$. The graph R_{J_n} is infinite; however, by the principle of compactness there exists a finite (connected) subgraph $R_n \subseteq R_{J_n}$ such that $\mathrm{BBC}(R_n, T_{J_n}[V(R_n)]) \geq 2n-1$. $T_{J_n}[V(R_n)]$ is a subforest of R_n thus it can be extended to a spanning tree T_n of R_n. We know that $\mathrm{BBC}(R_n,T_n) \geq 2n-1$ and $\chi(R_n) \leq \chi(R_{J_n}) = n$. Actually, equalities hold since $\mathrm{BBC}(G,T) \leq 2\chi(G)-1$ for any graph G with backbone T.

The construction from Step I follows an idea of Broersma et al. [1], however it requires additional work. It will be described in Section 2.

Now we discuss Step II in a detail. Its task is to construct a triangle-free graph J_n, for every integer n, such that $\chi(J_n) = n$ and $\chi_2(J_n) = 2n$. The well known *fractional chromatic number* of a graph G is defined as

$$\chi_{\mathrm{f}} = \inf_{t \in \mathbb{N}} \frac{\chi_t(G)}{t}.$$

Since $\chi(G) \geq \frac{\chi_2(G)}{2} \geq \chi_{\mathrm{f}}(G)$, it is natural to look for a class, say F_n, of triangle-free graphs such that $\chi(F_n) = \chi_{\mathrm{f}}(F_n) = n$. The existence of such graphs was proved by Erdős [4] using the probabilistic method. Indeed, in Section 2 Erdős proves an existence of graphs with chromatic number n on kn vertices (for an integer k) such that its independent sets have size at most k; moreover these graphs are assumed to have girth at least l for a given integer l.[1] Thus the fractional chromatic number of such a graph is at least $\frac{kn}{k} = n$.

Instead of the probabilistic proof, we offer a construction suitable for our situation (not considering fractional chromatic number). It is a generalization of Mycielski's construction. It will be precisely described in Section 3.

2 Relation between backbone colorings and 2-tuple colorings

In this section, for a given graph G we construct a pair of graphs (R_G, T_G) as mentioned in Step I in the introduction.

Definition 2.1. Let G be a graph. We define a pair of infinite graphs (R_G, T_G) in the following way:

1. The graph R_G is the OR-product of G and $\mathbb{N}$ (as independent set), i.e.
   - $V(R_G) = V(G) \times \mathbb{N}$;
   - $E(R_G) = \{\{(v_1, n_1), (v_2, n_2)\} \mid v_1 v_2 \in E(G)\}$.
2. Now we define the spanning tree T_G of R_G, see Figure 1 for the construction. We will gradually construct trees T_G^i for $i \in \mathbb{N}$ satisfying $T_G^{i+1} \supset T_G^i$. The tree T_G is then defined as $\cup_{i=1}^{\infty} T_G^i$. In particular we

[1] Note that k and n are interchanged in [4].

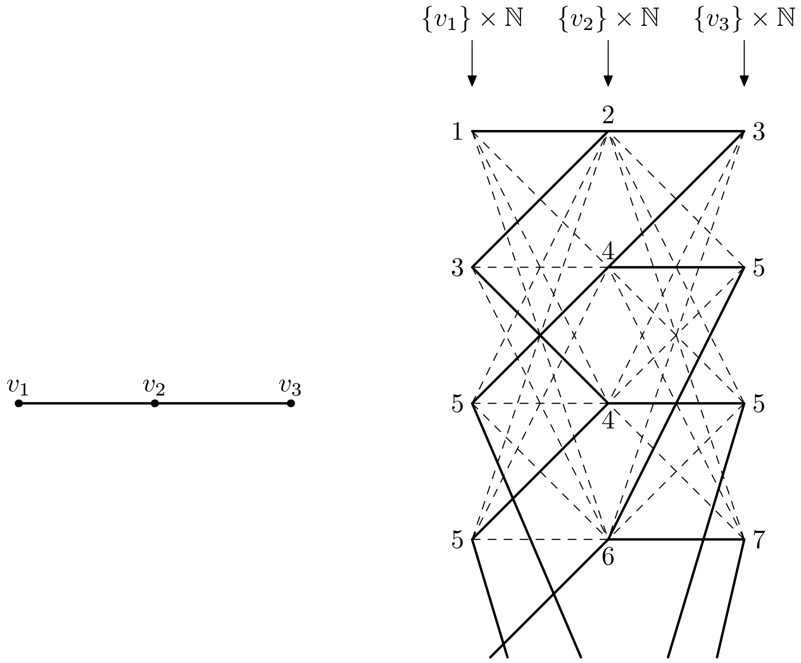

Figure 1: The path P_3 and the pair (R_{P_3}, T_{P_3}). Labels at vertices are the smallest such i that the corresponding vertex belongs to $T^i_{P_3}$. For example $T^3_{P_3}$ consist of vertices with labels 1, 2 and 3, and thick edges connecting them.

let T_G^1 to be a single vertex $(v_0, 1) \in R_G$ where $v_0 \in V(G)$ is a fixly chosen vertex.

Now we define trees T_G^i precisely. We define them recursively—suppose that $i \geq 2$ and T_G^{i-1} is already defined. To every leaf $l = (u, n)$ of T_G^{i-1} we attach $\deg_G(u)$ new vertices. More precisely, for every neighbor v of u in G we attach to l a new vertex (v, n') in $\{v\} \times \mathbb{N}$ (we assume that n' is the smallest possible in order to use all vertices of R_G). Thus we get T_G^i.

An example of the construction is depicted on Figure 1. Notice, that the spanning tree T_G is not defined uniquely; however, for our purposes it is not important to have a unique definition. The following lemma easily follows from the construction.

Lemma 2.2. *For any graph G, the pair (R_G, T_G) has the following properties:*

1. *T_G is a spanning tree of R_G.*
2. *If G is triangle-free then R_G is triangle-free.*
3. *For every vertex (v, j) of R_G and for every edge uv of G, there exits an integer j' such that $\{(v, j), (u, j')\}$ is an edge of T_G.*

□

The following proposition relates the 2-tuple chromatic number of a graph G and the backbone chromatic number of the pair (R_G, T_G).

Proposition 2.3. *Let G be a graph. Then*

1. $\chi(R_G) = \chi(G)$;
2. $\mathrm{BBC}(R_G, T_G) \geq \chi_2(G) - 1$.

Proof. We prove each of the claims separately:

1. The graph $R_G[V(G) \times \{1\}]$ is isomorphic to G, hence $\chi(G) \leq \chi(R_G)$.

   On the other hand, any coloring of G induces a coloring of R_G: For every $v \in V(G)$ and $n \in \mathbb{N}$ the vertices $(v, n) \in V(R_G)$ are assigned with the color of v. Hence $\chi(R_G) \leq \chi(G)$.

2. First, observe that $\mathrm{BBC}(R_G, T_G)$ is finite since

$$\mathrm{BBC}(R_G, T_G) \leq 2\chi(R_G) - 1 = 2\chi(G) - 1.$$

Let $k = \mathrm{BBC}(R_G, T_G)$ and let ν be a backbone k-coloring of (R_G, T_G). Our goal will be to construct a 2-tuple $(k+1)$-coloring c of G. First, we define a function $c' \colon V(G) \to 2^{[k]} \setminus \{\emptyset\}$:

$$c'(v) = \{n \in [k] \mid \text{ exists } j \in \mathbb{N} : \nu(v, j) = n\}.$$

Now, we define a function $c \colon V(G) \to \binom{[k+1]}{2}$ in the following way:

- $c(v) = \{i, i+1\}$ if $c'(v) = \{i\}$.
- $c(v)$ is any 2-element subset of $c'(v)$ if $|c'(v)| \geq 2$.

It remains to show that c is a 2-tuple coloring of G. First, observe that $c'(u) \cap c'(v) = \emptyset$ for every $uv \in E(G)$, since $\{(u, j_1), (v, j_2)\} \in E(R_G)$ for every $j_1, j_2 \in \mathbb{N}$. For any $uv \in E(G)$, we will show that $c(u) \cap c(v) = \emptyset$ by considering three cases:

$|c'(u)| \geq 2$ and $|c'(v)| \geq 2$: Since $c'(u) \cap c'(v) = \emptyset$, we infer $c(u) \cap c(v) = \emptyset$.

$c'(u) = \{i\}$ and $|c'(v)| \geq 2$ (or vice versa): Since $c'(u) \cap c'(v) = \emptyset$, we infer $i \notin c(v)$. It remains to show that $i + 1 \notin c(v) \subseteq c'(v)$. For a contradiction, suppose that $i + 1 \in c'(v)$. Let $(v, j) \in R_G$ be a vertex such that $\nu(v, j) = i + 1$ and (u, j') be its neighbor in T_G due to Lemma 2.2(3). Then $\nu(u, j') = i$ since $c'(u) = \{i\}$. It contradicts the fact that ν is a backbone coloring of (R_G, T_G).

$c'(u) = \{i_1\}$ and $c'(v) = \{i_2\}$: Since $c'(u) \cap c'(v) = \emptyset$, we have $i_1 \neq i_2$. Thus without loss of generality, we can assume that $i_1 < i_2$. Moreover, $i_1 + 1 \notin \{i_2\}$ from a similar reason as in the previous case. Thus, $i_1 + 1 < i_2$ implies that $c(u) \cap c(v) = \emptyset$.

□

3 Mycielski-type construction

Mycielski [6] was among the first authors who showed the existence of triangle-free graphs with arbitrarily large chromatic number. We wish, in addition, to relate the chromatic number and the 2-tuple chromatic number. More precisely, we will show that for every $n \in \mathbb{N}$ there exists a triangle-free graph which chromatic number is n and 2-tuple chromatic is $2n$. We will present a construction that increases the chromatic number by 1, the 2-tuple chromatic number by 2, and preserves the property being triangle-free. The construction has several steps.

Construction D. Suppose that we are given a graph G; a set $W \subseteq V(G)$ such that $G[W]$ is bipartite with parts A_1 and A_2 (possibly empty); and two vertices $z_1, z_2 \notin V(G)$.

We construct a graph[2] $D = D(G, W, z_1, z_2)$. Let[3] $W' = W \times \{W\}$ be a copy of W and let w' be an abbreviation for $(w, W) \in W'$ where $w \in W$. We define

$$\begin{aligned}
V(D) &= V(G) \cup W' \cup \{z_1, z_2\}, \text{ and} \\
E(D) &= E(G) \\
&\cup \{w_1' w_2' \mid w_1, w_2 \in W, \text{ and } w_1 w_2 \in E(G)\} \\
&\cup \{v w' \mid v \in V(G) \setminus W, w \in W, \text{ and } vw \in E(G)\} \\
&\cup \{w' z_i \mid w \in A_i, i \in \{1, 2\}\}.
\end{aligned}$$

An example of the construction is depicted on Figure 2. It is easy to check that the following lemma holds:

Lemma 3.1. *The graph $D = D(G, W, z_1, z_2)$ from Construction D has the following properties:*

1. *If G is triangle-free then D is triangle-free.*
2. *The graph $D[(V(G) \setminus W) \cup W']$ is isomorphic to G.*

□

We define another auxiliary graph:

Construction H. Suppose that we are given a graph G, and two vertices $z_1, z_2 \notin V(G)$.

[2] Formally, the graph D also depends on a partition of W to A_1 and A_2. For our purposes, it will be convenient to suppose that W is always already given with such a partition.

[3] For the most of the purposes $W \times \{W\}$ could be replaced by $W \times \{1\}$. However it will be convenient later on to get different copies for different W.

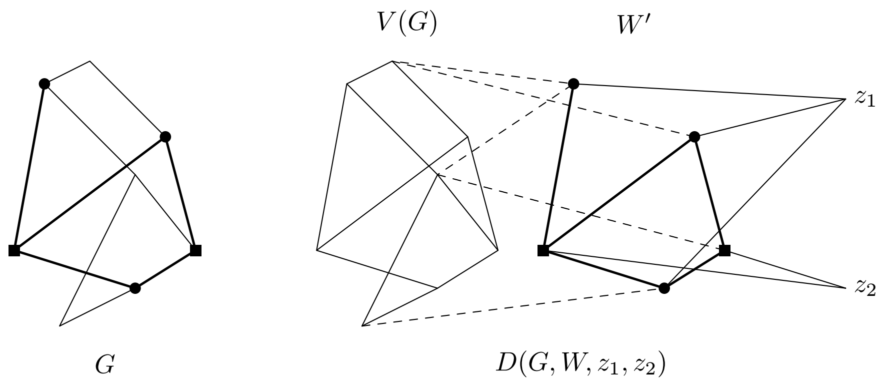

Figure 2: An example of Construction D. In the graph G, the subgraph $G[W]$ is indicated by a thick line; circular vertices belong to A_1 and square vertices belong to A_2.

We define the graph $H(G, z_1, z_2)$ in the following way: Order all $W \subseteq V(G)$ such that $G[W]$ is bipartite in a sequence W_1, W_2,..., W_t (and choose parts A_1, A_2 for each of them). Construct graphs $D_i = D(G, W_i, z_1, z_2)$. Finally, define

$$H = H(G, z_1, z_2) = \bigcup_{i=1}^{t} D_i,$$

i.e. H consists of union of sets $D(G, W_i, z_1, z_2)$ where G, z_1 and z_2 are identified in all the copies, however the sets W'_i are not identified (see Figure 3).

The following lemma is the key lemma for our construction.

Lemma 3.2. *Let $H = H(G, z_1, z_2)$ be a graph from Construction H.*

1. *If G is triangle-free, then H is also triangle-free.*
2. *Let k be the 2-tuple chromatic number of G. Then, there is no 2-tuple $(k+1)$-coloring c of H such that $c(z_1) = c(z_2)$.*

Proof. The first claim easily follows from Lemma 3.1(1).

For the second claim, assume to the contrary that c is a 2-tuple $(k+1)$-coloring of H such that $c(z_1) = c(z_2) = \{k, k+1\}$. Recall that H contains G. Let $W = \{v \in V(G) \mid c(v) \cap \{k, k+1\} \neq \emptyset\}$. It is easy to see that $G[W]$ is bipartite, thus there exists $i \in [t]$ (where t is defined as in Construction H)

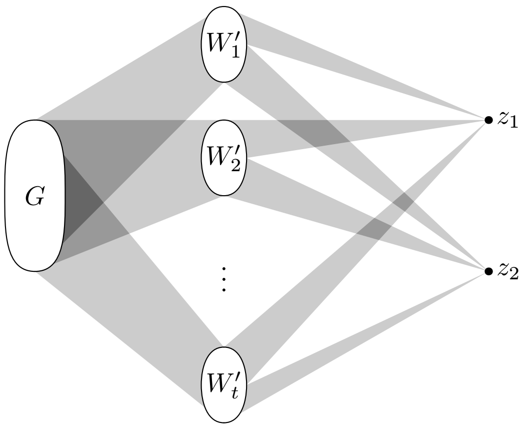

Figure 3: A scheme of Construction H.

such that $W = W_i$. The graph $G' = H[(V(G) \setminus W_i) \cup W_i']$ is isomorphic to G according Lemma 3.1(2) (where $W_i' = W_i \times \{W_i\}$ is defined as in Construction D).

We claim that $c(v) \cap \{k, k+1\} = \emptyset$ for every $v \in V(G')$: If $v \in V(G) \setminus W_i$ then it follows from the definition of $W = W_i$, if $v \in W_i'$ then either z_1 or z_2 is a neighbor of v. Thus c restricted to G' is a 2-tuple $(k-1)$-coloring of a graph isomorphic to G contradicting the assumptions of the lemma. $\square$

Finally, for a graph G we define the graph $J(G)$ that will satisfy our requirements.

Construction J. Let G be a graph, $k = \chi_2(G)$, $r = \binom{k+1}{2} + 1$, and let Z be the graph with $V(Z) = \{z_1, z_2, \ldots, z_r\}$ and $E(Z) = \emptyset$. For every $i, j \in [r]$, $i \neq j$, let G_{ij} be an isomorphic copy of G, formally, $V(G_{ij}) = V(G) \times \{\{i, j\}\}$ and $E(G_{ij}) = \{\{(u, \{i, j\}), v, \{i, j\}\} \mid uv \in E(G)\}$. Then, we define

$$J(G) = \bigcup_{\{i,j\} \subset [r]} H(G_{ij}, z_i, z_j),$$

i.e., $J(G)$ consists of an independent set Z where between each two vertices z_i, z_j of Z there is inserted a copy of $H(G, z_i, z_j)$.

Theorem 3.3. *Let G be a graph. The graph $J(G)$ satisfies the following properties:*

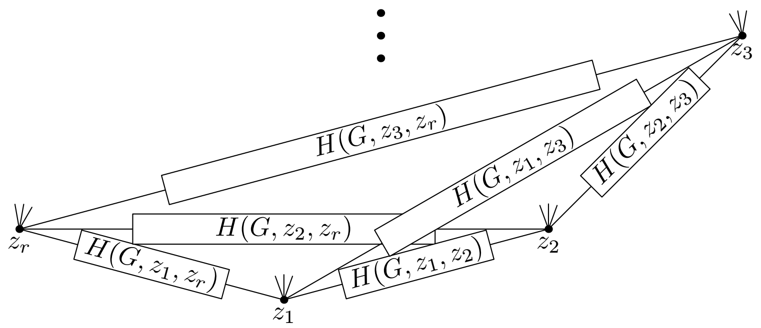

Figure 4: A scheme of Construction J.

1. *If G is triangle-free then $J(G)$ is also triangle-free.*
2. $\chi_2(J(G)) \geq \chi_2(G) + 2$.
3. $\chi(J(G)) \leq \chi(G) + 1$.

In fact it is not difficult to derive that $\chi_2(J(G)) = \chi_2(G) + 2$ and $\chi(J(G)) = \chi(G) + 1$, but we will not need it for our purposes.

Proof. We prove each of the claims separately:

1. This claim follows from Lemma 3.2(1) and from the fact that no two z_i and z_j are adjacent in $H(G_{ij}, z_i, z_j)$.
2. We use the notation from Construction J. Let $k = \chi_2(G)$. We will show that there is no 2-tuple $(k+1)$-coloring c of $J(G)$. For a contradiction, suppose that such c exists. From the pigeonhole principle, there are $i, j \in [r]$ such that $c(z_i) = c(z_j)$. But it contradicts Lemma 3.2(2) for $H = H(G_{ij}, z_i, z_j)$.
3. Again, we use the notation from Construction J. Let $l = \chi(G)$, we will show that there is an $(l+1)$-coloring γ of $J(G)$. First, we color all the vertices of Z with color $l+1$, i.e. $\gamma(Z) = \{l+1\}$. Then it is sufficient to color every $H(G_{ij}, z_i, z_j)$ separately. For notational convenience, we will color $H(G, z_1, z_2)$ following Construction H so that $\gamma(z_1) = \gamma(z_2) = l + 1$. The graph G is l-colorable, hence the coloring γ can be extended to G so that γ is a coloring of G using only colors 1, 2,..., l. Finally, for $i \in [t]$ and for $(w, W_i) \in W_i \times \{W_i\}$

we define $\gamma(w, W_i) = \gamma(w)$. It is easy to check that γ is an $(l+1)$-coloring of $H(G, z_1, z_2)$.

□

Corollary 3.4. *For every $n \in \mathbb{N}$ there exists a (connected) triangle-free graph J_n such that $\chi(J_n) = n$ and $\chi_2(J_n) = 2n$.*

Proof. Let J_1 be the graph consisting of a single vertex. For $n \geq 2$, let $J_n = J^{n-2}(K_2)$, where $J^0(K_2) = K_2$. By mathematical induction, Theorem 3.3 implies that $\chi(J_n) \leq n$ and $\chi_2(J_n) \geq 2n$. On the other hand, it is easy to see that $\chi_2(G) \leq 2\chi(G)$ for any graph G. □

The proof of the following corollary, answering negatively Question 1.1, is explicitly written in the introduction (Step III):

Corollary 3.5. *For every $n \in \mathbb{N}$ there exists a (finite) triangle-free graph R_n and its spanning tree T_n such that* $\mathrm{BBC}(R_n, T_n) = 2\chi(R_n) - 1 = 2n - 1$.

□

4 Conclusion

We showed the existence of triangle-free graphs R_n such that their backbone colorings (with suitable spanning tree) need $2\chi(R_n) - 1 = 2n - 1$ colors. However, these graphs contain 4-cycles. For further research, it could be interesting to describe the behavior of maximum possible backbone number for graphs with given chromatic number χ and given girth g.

The construction of a graph J_n can be generalized for every $t \geq 2$ in an obvious way to get triangle-free graphs J_n^t such that $\chi(J_n^t) = n$ and $\chi_t(J_n^t) = tn$ (compare with Corollary 3.4). In a bit more detail, to construct graphs J_n^t, consider t-colorable subgraphs W instead of bipartite subgraphs in Constructions D and H and put $r = \binom{k+t-1}{t} + 1$ in Construction J.

If we want to avoid a use of the principle of compactness, one can (with additional work) find a concrete pair (R'_G, T'_G) of finite graphs such that Proposition 2.3 is valid even if we replace (R_G, T_G) by (R'_G, T'_G) (considering a suitable finite iteration of the construction in Definition 2.1). Thus one get a purely constructive proof. On the other hand, we are aware only of a technical proof for a concrete pair (R'_G, T'_G), thus we decided not to include this proof.

5 Acknowledgement

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