Backbone colorings of graphs with bounded degree

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Abstract

We study backbone colorings, a variation on classical vertex colorings: Given a graph G and a spanning subgraph H of G (the backbone of G), a backbone coloring for G and H is a proper vertex k-coloring of G in which the colors assigned to adjacent vertices in H differ by at least 2. The minimal $k \in \mathbb{N}$ for which such a coloring exists is called the backbone chromatic number of G. We show that for a graph G of maximum degree Δ with the backbone graph being a d-degenerated subgraph of G, the backbone chromatic number is at most $\Delta + d + 1$ and moreover, in the case when the backbone graph being a matching we prove that backbone chromatic number is at most $\Delta + 1$. We also present examples where these bounds are attained.

Finally, the asymptotic behavior of the backbone chromatic number is studied regarding the degrees of G and H. We prove for any sparse graph G that if the maximum degree of a backbone graph is small compared to the maximum degree of G, then the backbone chromatic number is at most $\Delta(G) - \sqrt{\Delta(G)}$.

1 Introduction

The backbone coloring problem is related to frequency assignment problems in the following way: the transmitters are represented by the vertices of a

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graph and they are adjacent in the graph if the corresponding transmitters are close enough or transmitters are strong enough. The problem is to assign frequency channels to the transmitters in such a way that interference is kept at an "acceptable" level. One way of putting these requirements together is following: Given graphs G_1, G_2 such that G_1 is a spanning subgraph of G_2. Determine a coloring of G_2 that satisfies certain restriction of one type in G_1 and of the other type in G_2. In this way, backbone colorings were introduced and motivated and put into a general framework of related coloring problems in [1].

In further we deal with undirected simple graphs, i.e. without loops and/or multiedges, although we recall some basic definitions. For a graph G we define a coloring $\nu : V \to \{1, 2, \ldots, k\}$ to be a *vertex λ-backbone k-coloring* of a graph G with backbone graph $H \subseteq G$ if for every two different vertices u and v of G, one has

- $|\nu(u) - \nu(v)| \geq 1$, if $uv \in E(G) \setminus E(H)$;
- $|\nu(u) - \nu(v)| \geq \lambda$, if $uv \in E(H)$.

The minimum k for which G admits a vertex backbone k-coloring is called the *λ–backbone chromatic number* of G with *backbone* H, $\mathrm{BBC}_\lambda(G, H)$. When we speak about the 2-backbone chromatic number of G with backbone T, we write $\mathrm{BBC}(G, T)$ instead of $\mathrm{BBC}_2(G, T)$.

We refer to several results concerning backbone colorings. At the beginning some special classes of graphs were studied in view of backbone colorings, see [3, 2]. The connection between the backbone chromatic number and the chromatic number was studied in [1, 4]. It was shown that the 2-backbone chromatic number of G with backbone $H \subseteq G$ is at most twice its chromatic number, since if we use only odd numbers for coloring of G that the conditions of the backbone coloring of G are satisfied. The authors in [1] also provided examples where this bound is attained. The backbone coloring of planar graphs was also studied with respect to their chromatic number. Using the Four Color Theorem one can prove that the 2-backbone chromatic number of planar graphs with backbone matchings is at most six, and moreover if a backbone graph is a tree then the 2-backbone chromatic number is at most seven, see Broersma et al. [3].

The complexity of the decision problem: "Is there a backbone k-coloring of a graph G with backbone tree T?" was shown to be NP-complete even for $k \geq 5$, see [3]. For recent results on λ-backbone colorings see also [2, 8].

As mentioned above the backbone chromatic number has been mostly investigated in view of the chromatic number. The main goal of this paper is to study the behavior of backbone colorings according to maximum degree of a graph G and the maximum degree of a backbone graph H. Since the chromatic number is connected to the maximum degree of a graph, many results were derived in this direction but no general results concerning the maximum degree of graphs and the backbone colorings are known.

In the first section we deal with the backbone coloring of a graph G of maximum degree Δ with the backbone graph R. We show that there is always backbone coloring of a graph G with backbone graph H with at most $\Delta + \mathrm{col}(H)$ colors, where $\mathrm{col}(G) = \max\{\delta(R) + 1 \,|\, R \subseteq G\}$ represents the *coloring number* of a graph G. Similarly, a graph G is said to be d-*degenerate* if every its subgraph has a vertex of degree at most d. Further, we deal with graphs with backbone forests, and especially matchings. We show that $\mathrm{BBC}(G,T) \leq \Delta + 2$ if the backbone graph is a tree T and $\mathrm{BBC}(G,M) \leq \Delta + 1$ in the case that the backbone graph is a matching M. We also show that there are non-trivial classes of graphs where these bounds are sharp. We conclude by investigating the asymptotic behavior of the λ-backbone colorings of graphs. Surprisingly, if a sparse graph G is large enough and H is a backbone graph with $\Delta(H) << \Delta$, then the λ-backbone chromatic number of G with $\lambda << \Delta$, is at most $\Delta - \sqrt{\Delta}$.

2 Degenerated graphs

As mentioned above in this section we present several results concerning λ-backbone colorings of graphs with backbones being d-degenerated graphs. For the sake of a clear and simple exposition, we deal only with 2-backbone colorings of graphs, but with small technically involved modifications one can prove similar bounds for λ-backbone colorings.

Theorem 2.1. *Let G be a graph of maximum degree Δ and let T be a d-degenerated subgraph of G. Then,* $\mathrm{BBC}(G,T) \leq \Delta + d + 1$.

Proof. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G such that each v_i is preceded by at most d neighbors from T. Such an ordering is possible since T is d-degenerated.

Now, we apply the following procedure for coloring the vertices:

for each color $c \in \{1, \ldots, \Delta + d + 1\}$; **do**
    for each $i \in \{1, \ldots, n\}$; **do**

if v_i is not colored **and**
  neither c appears on the neighborhood of v_i in G
  nor $c-1$ appears on the neighborhood of v_i in T **then**
 $\lambda(v_i) := c$.

We claim that after applying the above procedure we obtain a proper backbone coloring λ of G. Obviously, if v_iv_j is an edge of G then $\lambda(v_i) \neq \lambda(v_j)$. Moreover, if v_iv_j is an edge of T then $|\lambda(v_i) - \lambda(v_j)| \geq 2$. Thus, it is enough to show that each vertex has assigned a color.

Suppose that v_i is an uncolored vertex of G after the procedure is completed. Notice that the following holds:

- each preceding neighbor v_j of v_i in T forbids only the colors $\lambda(v_j)$ and $\lambda(v_j)+1$ to be assigned to v_i;
- each succeeding neighbor v_j of v_i in T forbids only the color $\lambda(v_j)+1$ to be assigned to v_i;
- each preceding neighbor v_j of v_i in $G-T$ forbids only the color $\lambda(v_j)$ to be assigned to v_i;
- each succeeding neighbor v_j of v_i in $G-T$ forbids no color to be assigned to v_i.

Now, easily follows that at most $2d + \Delta - d = \Delta + d$ colors are forbidden to v_i. Since we have available $\Delta + d + 1$ colors, it follows that in the procedure v_i is colored, a contradiction. □

The above bound is sharp for $d = 1$ and G being an odd cycle or a complete graph. In the former case T is a spanning path and in the later case it is a spanning star of G.

3 Matching backbones

In this section we study graphs with backbone being a matching. In this case we show that an upper bound on the 2-backbone chromatic number presented above can be decreased.

Proposition 3.1. *Let M be a matching in the cycle C_n. Then*

$$\mathrm{BBC}(C_n, M) \leq 3.$$

Proof. Let $C_n = v_1 v_2 \cdots v_n$. If n is even and M is perfect, then color the vertices one by one alternatively by colors 1 and 3 as they appear on C_n.

So, assume now that n is odd or M is not perfect. Then, C_n has a vertex, say v_n, non-incident with any edge of M. Now, color the vertices $v_1, v_2, \ldots, v_{n-1}$ alternatively by 1 and 3. Finally, choose a color for v_n from $\{1,2,3\}$ which does not appear at v_1 and v_{n-1}. $\square$

Notice, that the above result can be easily extended to $\mathrm{BBC}_\lambda(G, M) \leq \lambda + 1$ for any $\lambda \geq 2$.

Proposition 3.2. *Let M be a matching in the complete graph K_n, $n \geq 3$. Then $\mathrm{BBC}(K_n, M) \leq n$.*

Proof. Let $e_1, e_2, \ldots, e_s$ with $s \leq \lfloor \frac{n}{2} \rfloor$ be the edges of the matching M. We may assume that $s \geq 2$; otherwise color K_n as usually with colors 1 and n assigned on the possible single edge of M.

For each $i \in \{1, \ldots, s\}$, color the end-vertices of e_i by the colors i and $s+i$. The remaining vertices color one by one with the colors $2s+1, \ldots, n$. The procedure gives a proper backbone coloring of K_n, since $s \geq 2$ and $s \leq \lfloor \frac{n}{2} \rfloor$. $\square$

The backbone coloring is influenced by certain structures in the graphs and their backbones. We refer to a result concerning an existence of special structure.

Theorem 3.3 (Bryant [5]). *For a 2-connected graph G the following three statements are equivalent:*

(i) G is a complete graph or a cycle;

(ii) the removal from G of any two non-adjacent vertices disconnects it;

(iii) the removal from G of any two vertices at distance 2 apart disconnects it.

Let x, y be two non-adjacent neighbors of a vertex v in a graph G such $G - x - y$ is connected. Then we say that $(v; x, y)$ is a *fork*. We do not distinguish between $(v; x, y)$ and $(v; y, x)$. Notice that above theorem claims that each 2-connected graph distinct from a cycle and a complete graph contains a fork. Now, we show the existence of a fork that "avoids" a given vertex in a particular class of 2-connected graphs.

Proposition 3.4. *Let G be a 2-connected graph whose all vertices are of degree $d \geq 3$ except a particular vertex v which is of degree $< d$. Then, G has a fork $(w; x, y)$ such that $v \neq x$ and $v \neq y$.*

Proof. Suppose that the claim is false and G is a counterexample. By 2-connectivity of G, the graph $G - v$ is connected. Notice that $G - v$ has at least three vertices, so it is distinct from K_2.

We claim that $G - v$ *is not 2-connected*. Otherwise, if $G - v$ is 2-connected, then by the degree assumptions of G, it follows that G is neither a complete graph nor a cycle. So by Theorem 3.3, $G - v$ contains some fork $(w; x, y)$. Observe that $(w; x, y)$ is a fork in G as well, a contradiction.

Since $G - v \neq K_2$ is connected but not 2-connected, it follows that $G - v$ has at least two end-blocks in its tree-block representation. Let B be an end-block. Note that B has at least three vertices, and so each vertex of B has degree at least 2 in B. Let w be the unique cut-vertex of $G - v$ that belongs to B. By 2-connectivity of G, v has a neighbor in every end-block of $G - v$, which is distinct from the unique cut-vertex of that end-block. Thus, v has neighbor in $B - w$, say z. Let x be a neighbor of w in B distinct from z; we can choose x since $d_B(w) \geq 2$. Define $H = G - v - (B - w)$ and choose a neighbor y of w in H.

We claim that $(w; x, y)$ *is a fork of* G, which will give us a contradiction. Notice that x and y are non-adjacent neighbors of w; otherwise they belong to a same block of $G - v$. In order to establish the claim, it is enough to show that $G - x - y$ is connected. Observe that $B - x$ is connected and z from $B - x$ is adjacent to v. Thus, $B - x + v$ is connected. From other side $H - y$ may not be a connected; this may happen only if y is a cut-vertex in $G - v$. As we observed before, v has a neighbor in every end-block of $G - v$ distinct from the unique cut-vertex of that end-block. This implies that $H - y + v$ is connected. Finally, from connectivity of $B - x + v$ and $H - y + v$, the connectivity of $G - x - y$ easily follows. ☐

We proceed with the theorem.

Theorem 3.5. *Let M be a matching in a graph G of maximum degree Δ. Then* $\mathrm{BBC}(G, M) \leq \Delta + 1$.

Proof. Obviously, we may assume that G is connected. By Propositions 3.1 and 3.2, we may also assume that G is neither an odd cycle nor a complete graph.

Consider an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G such that each v_i $(i < n)$ has a succeeding neighbor. Since G is connected, such an ordering exists, and it can be constructed by a depth-first search starting at the vertex v_n. Actually, we may choose for v_n any vertex of G.

Consider the procedure from the proof of Theorem 2.1 on G regarding the above ordering. A vertex v_i with $i \neq n$, has at most Δ forbidden colors, so the procedure color it. But the last vertex v_n may have forbidden all $\Delta + 1$ colors. Suppose, that this is the case.

We claim that G *is a regular graph* and M *is a perfect matching*. Otherwise, we may choose for v_n a vertex that is of degree $< \Delta$ or that it is not incident with an edge of M. In both cases, the procedure will color also the vertex v_n.

We now claim that G *is 2-connected*. Otherwise, let B be an end-block in G incident with a cut-vertex v. By Proposition 3.4, B has a fork $(w; x, y)$ with $w, x, y \in V(B)$ and $v \notin \{x, y\}$. Since $x \neq v$ and $y \neq v$, it follows that $G - x - y$ is connected, so $(w; x, y)$ is a fork in G as well. By the definition of the fork, we can order vertices of G in order $v_1, v_2, \ldots, v_n$ such that $v_1 = x$, $v_2 = y$ and $v_n = v$. According to this ordering v_1 and v_2 receive the same color by the procedure, which assures that v_n is also colored.

Finally, we may assume that G is 2-connected. Since G is neither an odd cycle nor a complete graph, Theorem 3.3 assures existence of a fork. Now, we can apply a similar argument as above in order to color G. This establish the proposition. □

4 Graphs with large backbone chromatic number

In this section we show that there exist classes of graphs with backbone trees, for which the upper bound presented above is sharp. As mentioned in the previous section, such classes are complete graphs and odd cycles. One may wonder if all such graphs are only of these two types as it is the case of the usual coloring.

In what follows, we show that there exist graphs distinct from complete graphs and odd cycles with backbone trees T such that $\mathrm{BBC}(G, T) = \Delta(G) + 2$.

Proposition 4.1. *Let* $\Delta \in \mathbb{N}$. *There exists a graph* G *with maximum degree* Δ *and a backbone tree* T *such that*

$$\mathrm{BBC}(G, T) = \Delta + 2.$$

Proof. We construct the desired graph. At first observe that the backbone chromatic number of the complete graph K_n with the backbone been a spanning star $S_{1,n-1}$ is $n+1$. Moreover, in any optimal coloring the central vertex of the star $S_{1,n-1}$ must be colored by 1 or $n+1$. Notice that $n = \Delta(K_n) + 1$.

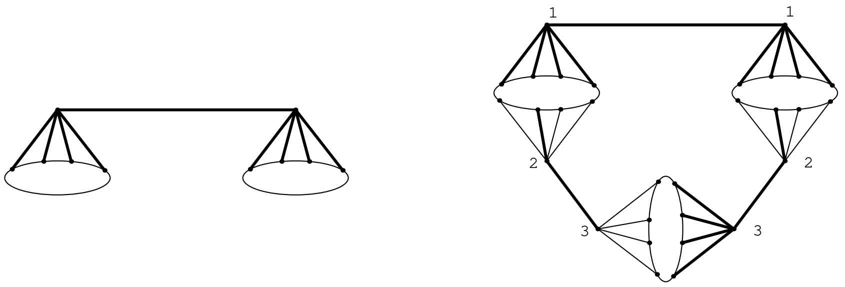

Figure 1: Graphs R and G

Next, let R be the graph on $2n+2$ vertices $v_1, \ldots, v_{2n+2}$ with the edge set

$$\begin{aligned} E(R) &= \{v_i v_j \mid i,j \in \{1,\ldots,n+1\}\} \\ &\cup \{v_i v_j \mid i,j \in \{n+2,\ldots,2n+2\}\} \\ &\cup \{v_{n+1} v_{n+2}\}, \end{aligned}$$

and let its backbone tree be a double spanning star $S_{n,n}$, with centers $x = v_{n+1}$ and $y = v_{n+2}$. See the graph on the left side of Fig. 1 for an illustration. Observe that any backbone $(\Delta(R)+1)$-coloring of R has a property that the colors of x and y comprise the set $\{1, \Delta(R)+1\}$.

Finally, we define G to be the graph on $3n+6$ vertices $v_1, \ldots, v_{3n+6}$ with the edge set

$$\begin{aligned} E(G) &= \{v_i v_j \mid i,j \in \{1,\ldots,n+1\}\} \\ &\cup \{v_i v_j \mid i,j \in \{n+2,\ldots,2n+2\}\} \\ &\cup \{v_i v_j \mid i,j \in \{n+3,\ldots,2n+3\}\} \\ &\cup \{v_i v_j \mid i,j \in \{2n+4,\ldots,3n+4\}\} \\ &\cup \{v_i v_j \mid i,j \in \{2n+5,\ldots,3n+5\}\} \\ &\cup \{v_i v_j \mid i,j \in \{3n+6,1,\ldots,n\}\} \\ &\cup \{v_{n+1} v_{n+2}, v_{2n+3} v_{2n+4}, v_{3n+5} v_{3n+6}\}. \end{aligned}$$

For an illustration see the graph on the right side of Fig. 1, where $v_{n+1} = x_1$, $v_{n+2} = y_1$, $v_{2n+3} = y_2$, $v_{2n+4} = y_3$, $v_{3n+5} = x_3$, $v_{3n+6} = x_2$. Notice that the graph G can be constructed from three copies of R by identification of some cliques. The backbone tree T is the one with the thick edges in the graph on the right side of Fig. 1, i.e.

$$\begin{aligned} E(T) &= \{v_i v_{n+1} \mid i \in \{1,\dots,n\}\} \\ &\cup \{v_i v_{n+2} \mid i \in \{n+3,\dots,2n+2\}\} \\ &\cup \{v_i v_{2n+4} \mid i \in \{2n+5,\dots,3n+4\}\} \\ &\cup \{v_{n+1}v_{n+2}, v_{2n+3}v_{2n+4}, v_{3n+5}v_{3n+6}, v_{3n+6}v_1, v_{2n+2}v_{2n+3}\}. \end{aligned}$$

We show that the graph G has backbone chromatic number at least $\Delta+2$, and hence equal to $\Delta+2$ by Theorem 2.1. Let us suppose for a contradiction that the backbone chromatic number of the graph G is at most $\Delta+1$. The graph G contains a copy of K_{n+1}, hence $\mathrm{BBC}(G,T) \geq n+2 = \Delta+1$. The vertices $x_1 = v_{n+1}$ and $y_1 = v_{n+2}$ are colored by 1 and $\Delta+1$, respectively, since they correspond to the vertices x and y on Fig. 1. Without loss of generality, let x_1 be colored by 1. Then all colors from 3 to $n+2$ are used among the vertices $v_1,\dots,v_n$. Hence, the vertex $x_2 = v_{3n+6}$ must be colored by 1 or 2. Similarly, the vertex $y_2 = v_{2n+3}$ is colored by $n+1$ or $n+2$. The vertex $y_3 = v_{2n+4}$ is colored by 1, since the edge $y_2y_3 \in E(T)$ is a backbone edge and $\{v_{2n+4},\dots,v_{3n+4}\}$ induces a copy of K_{n+1}. This implies that the vertex $x_3 = v_{3n+5}$ can be colored only by the color 1 or 2. Again x_2x_3 is a backbone edge, and hence it cannot be colored with the colors from $\{1,2\}$, a contradiction. $\square$

In the previous claim the maximum degree of the backbone graph is the same as the maximum degree of the graph G. We show that there exist graphs with backbone graphs (forests) for which $\Delta(T) < \Delta(G)$ and still the bound from Theorem 2.1 is sharp.

Proposition 4.2. *Let $\Delta \in \mathbb{N}$. There exist a graph G of maximum degree Δ and a backbone tree T with $\Delta > \Delta(T)$ such that*

$$\mathrm{BBC}(G,T) = \Delta+2.$$

Proof. Let J be the graph on $2n+5$ vertices $v_1,\dots,v_{2n+5}$ with the edge set

$$\begin{aligned} E(J) &= \{v_i v_j \mid i,j \in \{1,\dots,n+1\}\} \\ &\cup \{v_i v_j \mid i,j \in \{n+2,\dots,2n+2\}\} \\ &\cup \{v_i v_j \mid i,j \in \{n+3,\dots,2n+3\}\} \\ &\cup \{v_i v_j \mid i,j \in \{2n+5,1,\dots,n\}\} \\ &\cup \{v_{n+1}v_{n+2}, v_{2n+3}v_{2n+4}, v_{2n+4}v_{2n+5}\}, \end{aligned}$$

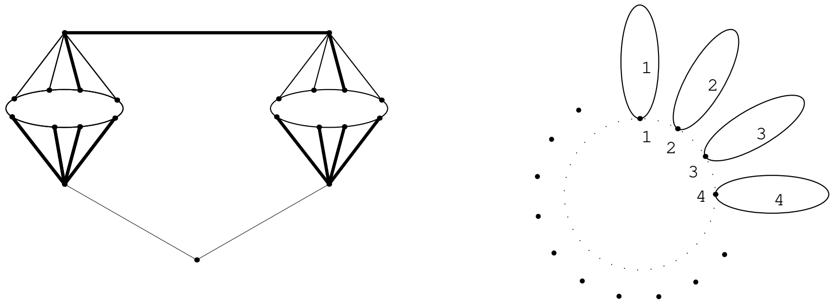

Figure 2: Graphs J and G

and let the backbone tree be the one with the thick edges in the graph on the left side of Fig. 2, i.e.

$$\begin{aligned} E(T) &= \{v_i v_{2n+5} \,|\, i \in \{1, \ldots, n\}\} \\ &\cup \{v_i v_{2n+3} \,|\, i \in \{n+3, \ldots, 2n+2\}\} \\ &\cup \{v_1 v_{n+1}, v_{n+1} v_{n+2}, v_{n+2} v_{n+3}\}. \end{aligned}$$

At first observe that in any backbone coloring with $n+2$ colors the vertex $x = v_{2n+4}$ of the graph J cannot obtain neither color 1 nor $n+2$ by using a similar argument as in the proof of the previous proposition.

Finally, we construct G consisting of n disjoint copies J_i of J with vertices x_i corresponding to x, and all the edges between vertices x_i, $i \in \{1, \ldots, n\}$. Notice that x_i's induce a copy of K_n. The backbone graph of G is union of backbone graphs of J_i and a star $S_{1,n-1}$ on vertices x_i's with center x_1.

Suppose now that the central vertex x_1 of $S_{1,n-1}$ is colored with the color c. The vertices $x_2, \ldots, x_n$ cannot be colored with the colors 1, $n+2$, c. Moreover, at least one color $c+1$ or $c-1$ is also not used by these vertices. Hence, we have at most $\Delta + 1 - 4 = n + 2 - 4 = n - 2$ available colors for coloring $\{x_2, \ldots, x_n\}$, which is impossible. This contradiction establish the claim. $\square$

5 Asymptotic behavior of the backbone chromatic number

In this section we present an upper bound on the backbone chromatic number of a sparse graph G with a backbone graph H such that $\Delta(H) << \Delta(G)$. We also show that the bound is asymptotically almost best possible. Let us recall standard well known Talagrand's inequality [10], see also [9, p. 81] and its consequence.

Theorem 1 (Talagrand's Inequality). *Let X be a non–negative random variable, not identically 0, which is determined by n independent trials $T_1, T_2, \ldots, T_n$, and for some c, $r > 0$ satisfying the following:*

(a) changing the outcome of any trial can affect X by at most c; and

(b) for any s, if $X \geq s$ then there is a set of at most rs trials whose outcomes certify that $X \geq s$.

Then, for any $0 \leq t \leq \mathbf{E}(X)$ it holds

$$P(|X - E(X)| > t + 60c\sqrt{r\mathbf{E}(X)}) \leq 4e^{-\frac{t^2}{8c^2 r\mathbf{E}(X)}}.$$

Theorem 2 (Simple Concentration Bound [9, p. 79]). *If X is a random variable determined by n independent trials $T_1, T_2, \ldots, T_n$ such that changing the outcome of any trial changes the values of X by at most by c, then*

$$P(|X - E(X)| > t) \leq 2e^{-\frac{t^2}{2c^2 n}}$$

for $t > 0$.

For a purposes of the following theorem we also recall the Lovász Local Lemma [6], see also [9, p. 40].

Theorem 3 (Lovász Local Lemma). *Consider a set $\mathcal{E}$ of events such that for each $A \in \mathcal{E}$*

(a) $P(A) \leq p < 1$; and

(b) A is mutually independent of a set of all but at most d other events.

If $4pd \leq 1$ then with positive probability, none of the events in $\mathcal{E}$ occur.

Now, we show that a sparse graph G of large enough maximum degree $\Delta >> \lambda$ and the backbone H having maximum degree $d << \Delta$ satisfy $\mathrm{BBC}(G,H) \le \Delta - \sqrt{\Delta}$. This is an interesting fact since for backbone graphs with high maximum degree, there exist graphs with 2-backbone chromatic number equals to $\Delta + 2$, due to Proposition 4.1. Let us recall that the *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$, is the set of all vertices adjacent to it, and so $v \notin N_G(v)$.

Theorem 5.1. *Let G be a graph of large enough maximum degree Δ and H its subgraph of maximum degree $d \le \frac{\Delta}{6}$. If the neighborhood $N_G(v)$ of any vertex $v \in V(G)$ contains at most $\binom{\Delta}{2} - B$ edges, for $B \ge \Delta^{\frac{3}{2}} \sqrt{3000\, e^{16} \log \Delta}$, then*

$$\mathrm{BBC}_\lambda(G,H) \le \Delta + 1 + (2\lambda - 1)d - \frac{B}{e^8 \Delta}.$$

Proof. Let $\mathcal{C} = \{1, \dots, \lfloor \frac{\Delta}{2} \rfloor\}$ and $c = |\mathcal{C}|$. For every vertex x of G we assign to x a uniformly random color from $\mathcal{C}$ with probability $\frac{1}{c}$. Next, we uncolor all vertices that are present in some conflict, i.e. there is an edge between two vertices sharing the same color or there is a backbone edge between two vertices with colors at distance 1. In such a situation, we uncolor both end-vertices. We are interested in the random variable X_v, that counts the number of colors assigned to at least two non-adjacent neighbors of a given vertex $v \in V(G)$ and retained by all of them.

For each $v \in V(G)$, let A_v be the event that $X_v < \frac{B}{e^8 \Delta}$. We show that the probability $P(A_v) < \frac{1}{4\Delta^5}$. Moreover, the event A_v is mutually independent of all but at most Δ^4 other events, since two events A_u and A_v can be dependent only if $\mathrm{dist}(u,v) \le 4$. We proceed with two claims.

Claim 1. *For every vertex $v \in V(G)$, it holds*

$$\mathbf{E}(X_v) > \frac{2B}{e^4 \Delta}.$$

We focus on the random variable X'_v that counts number of colors of vertices so that it was assigned to exactly two vertices and retained by both of them. Trivially, $X_v \ge X'_v$, and hence $\mathbf{E}(X_v) \ge \mathbf{E}(X'_v)$.

Let $u, w \in N_G(v)$. The probability that both vertices u, w were assigned a color α is $p_1 = \frac{1}{c^2}$. The probability that no other vertex in $N_G(v)$ was assigned the color α is $p_2 > \left(1 - \frac{1}{c}\right)^{\Delta}$. The probability that the color α at u did not cause any conflict on $N_G(u)$ is $p_3 \ge \left(1 - \frac{3}{c}\right)^{d} \cdot \left(1 - \frac{1}{c}\right)^{\Delta - d} \ge$

$\left(1-\frac{1}{c}\right)^{\frac{3}{2}\Delta}$ since $d \le \frac{\Delta}{6}$ and there are at most d neighbors of u in H. A similar computation applies to p_4, the probability that the color α at w did not cause any conflict on $N_G(w)$. Now, by the fact $\Delta \ge 2c$ we have that the probability of the event that each of u, w is assigned and retained α and no other vertex in $N_G(v)$ was assigned α is at least

$$p_1\, p_2\, p_3\, p_4 \ge \frac{1}{c^2}\left(1-\frac{1}{c}\right)^{8c} \ge \frac{1}{e^8 c^2}.$$

There are c choices for α and at least B choices for u, w, therefore by linearity of expectation we infer

$$\mathbf{E}(X_v) \ \ge \mathbf{E}(X'_v) \ \ge \ c\, B\frac{1}{e^8 c^2} \ \ge \ \frac{2B}{e^8 \Delta}.$$

This establish Claim 1.

Claim 2. *For every vertex $v \in V(G)$, it holds*

$$P\left(|X_v - \mathbf{E}(X_v)| > \frac{1}{2}\mathbf{E}(X_v)\right) < \frac{1}{4\Delta^5}.$$

In order to prove the claim, we will consider two random variables Y_v and Z_v, where

- Y_v is the number of colors that were assigned (but maybe not retained) to at least two non-adjacent neighbors of v.
- Z_v is the number of colors that were assigned to at least two non-adjacent neighbors of v and removed from at least one of them.

We have $X_v = Y_v - Z_v$. Using Simple concentration bound, and the fact that assignment of a color to $u \in N(v)$ can change Y_v by at most 2, we have

$$P(|Y_v - \mathbf{E}(Y_v)| > t) < 2e^{-\frac{t^2}{8\Delta}}.$$

Similarly, assignment of a color to a vertex v can change Z_v by at most 2. Next, for $Z_v \ge s$ we can take for every color β of these s removed colors two vertices from the neighborhood of v assigned by β and one vertex that caused conflict in this color β, i.e. at most $3s$ vertices. Using Talagrand's Inequality with $c = 2$ and $r = 3$ we have

$$P(|Z_v - \mathbf{E}(Z_v)| > t) < 4\exp\left(-\frac{(t - 120\sqrt{3\mathbf{E}(Z_v)}\ )^2}{96\mathbf{E}(Z_v)}\right) < 4\exp\left(-\frac{t^2}{100\Delta}\right).$$

In order to obtain the second inequality, we use this facts $t \geq \sqrt{\Delta \log \Delta}$ and $\mathbf{E}(Z_v) \leq \Delta$.

By linearity of expectation, $\mathbf{E}(X_v) = \mathbf{E}(Y_v) - \mathbf{E}(Z_v)$. If $|X_v - \mathbf{E}(X_v)| > t$ then we must have either $|Y_v - \mathbf{E}(Y_v)| > \frac{t}{2}$ or $|Z_v - \mathbf{E}(Z_v)| > \frac{t}{2}$. By the concentration of Y_v, Z_v and the subadditivity of probability measure, we infer

$$\begin{aligned} P(|X_v - \mathbf{E}(X_v)| > t) &\leq P\left(|Y_v - \mathbf{E}(Y_v)| > \tfrac{t}{2}\right) + P\left(|Z_v - \mathbf{E}(Z_v)| > \tfrac{t}{2}\right) \\ &< 2e^{-\frac{t^2}{32\Delta}} + 4e^{-\frac{t^2}{400\Delta}} < 8e^{-\frac{t^2}{400\Delta}}. \end{aligned}$$

The maximum degree Δ of G is sufficiently large and using $t = \frac{1}{2}\mathbf{E}(X_v) \geq \sqrt{\Delta \log \Delta}$, we obtain

$$\begin{aligned} P(|X_v - \mathbf{E}(X_v)| > \tfrac{1}{2}\mathbf{E}(X_v)) &< 8e^{-\frac{B^2}{400e^{16}\Delta^3}} \leq 8e^{-\frac{3000\,e^{16}\log\Delta}{400e^{16}}} \\ &\leq 8\Delta^{-\frac{3000e^{16}}{400e^{16}}} < \tfrac{1}{4\Delta^5}. \end{aligned}$$

This establish Claim 2.

Probability that A_v occurs is at most $P(|X_v - \mathbf{E}(X_v)| > \frac{1}{2}\mathbf{E}(X_v))$ and this probability is bounded from above by $\frac{1}{4\Delta^5}$. Moreover, A_v is mutually independent of all but Δ^4 other events. Since $4\,\frac{1}{4\Delta^5}\,\Delta^4 < 1$, the assumption "$4pd$" of the Lovász Local Lemma is satisfied. So we conclude that none of A_v occurs with positive probability. This means that for every uncolored vertex v there are at least $\frac{B}{e^8\Delta}$ colors that are used on at least two neighbors of v. In other words for coloring of the vertex v it is enough to have

$$\Delta - \frac{B}{e^8\Delta} + (2\lambda - 1)d + 1$$

colors. This ends the proof of the theorem. $\square$

Corollary 5.2. *Let G be a graph of large enough maximum degree $\Delta >> \lambda$, $\lambda \in \mathbb{N}$, and H its subgraph of maximum degree $d << \Delta$. If the neighborhood $N_G(v)$ of any vertex $v \in V(G)$ contains at most $\binom{\Delta}{2} - B$ edges for $B \geq \Delta^{\frac{3}{2}}\sqrt{3000\,e^{16}\log\Delta}$, then*

$$\mathrm{BBC}_\lambda(G, H) \leq \Delta - \sqrt{\Delta}.$$

Proof. From the previous theorem we have

$$\begin{aligned}
\mathrm{BBC}_\lambda(G,H) &\leq \Delta + 1 + (2\lambda-1)d - \tfrac{B}{e^8\Delta} \\
&\leq \Delta + 1 + (2\lambda-1)d - \tfrac{\Delta^{\frac{3}{2}}\sqrt{3000\,e^{16}\,\log\Delta}}{e^8\Delta} \\
&= \Delta - \sqrt{\Delta} + \left(1 + (2\lambda-1)d - \sqrt{\Delta}\left(\frac{\sqrt{3000\,e^{16}\,\log\Delta}}{e^8} - 1\right)\right) \\
&\leq \Delta - \sqrt{\Delta}.
\end{aligned}$$

Notice that the last inequality holds since $d << \Delta$ and $\lambda << \Delta$. □

We end by showing that with some relaxation on the constraint $B \geq \Delta^{\frac{3}{2}}\sqrt{3000\,e^{16}\log\Delta}$, there are graphs which 2-backbone chromatic number is at least $\Delta - \frac{B}{\Delta} - 1$. Moreover this hold for any choice of the backbone.

Proposition 5.3. *For every $\Delta \in \mathbb{N}$ and $D \leq \Delta^{\frac{3}{2}} - \Delta$ there is a graph G in which the neighborhood of any vertex $v \in V(G)$ contains at most $\binom{\Delta}{2} - D$ edges such that*

$$\mathrm{BBC}_2(G,H) > \Delta - \frac{D}{\Delta} - 1,$$

for any backbone graph $H \subseteq G$.

Proof. Let G_1 be a complete graph on $k+1$ vertices, and G_2 be an edgeless graph on $\Delta - k$ vertices where $k = \lfloor y \rfloor$ and y is a solution of the equation

$$y^2 - y = \left(\Delta - \frac{D}{\Delta}\right)^2 - \left(\Delta + \frac{D^2}{\Delta^2}\right).$$

It is not hard to show that $y > \Delta - \frac{D}{\Delta} - 1$.

Now we define our graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup \{v_0 w \,|\, w \in V(G_2)\}$ for a fixed $v_0 \in V(G_1)$. Next let H be an arbitrary subgraph of G. The complete graph K_{k+1} is a subgraph of G, and hence $\chi(G) \geq k+1$. Any backbone coloring of a graph G is also a proper coloring of G and therefore

$$\mathrm{BBC}_2(G,H) \geq \chi(G) \geq k+1 > y > \Delta - \frac{D}{\Delta} - 1.$$

Now we show that our graph also satisfies the "neighborhood" condition. There are at most $\binom{k}{2}$ edges in the neighborhood of any vertex $v \in V(G)$. Since

$$\binom{k}{2} \leq \frac{y^2 - y}{2} = \frac{\left(\Delta - \frac{D}{\Delta}\right)^2 - \left(\Delta + \frac{D^2}{\Delta^2}\right)}{2} = \binom{\Delta}{2} - D,$$

we are done. $\square$

Notice that D is chosen such that the maximal complete graph on $k+1$ vertices satisfying the "neighborhood" condition with such a choice of D, has chromatic number exactly $\lceil \Delta - \frac{D}{\Delta} - 1 \rceil \geq \Delta - \frac{D}{\Delta} - 1$.

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