Epimorphisms of metric frames

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Abstract

This paper deals with several aspects of epimorphisms in the category **MFrm** of metric frames and contractive homomorphisms. In particular, it is shown that

(i) the epicomplete metric frames are uniquely determined by the power-set lattices of sets,

(ii) episurjective is the same as Boolean,

(iii) a metric frame has an epicompletion iff it is spatial, and

(iv) the subcategory of epicomplete L in **MFrm** is reflective.

Moreover, we show that the counterpart of the latter does not hold for uniform frames.

It is a well-known fact that the epimorphisms in the category of frames can be very far from surjective. The same phenomenon is encountered also in categories of more special or enriched frames such as the paracompact and the uniform ones (see [2]).

In this paper we make first steps in investigating this phenomenon in the category of *metric frames and contractive homomorphisms.* The question whether there are metric frames L for which there exist epimorphisms $L \to M$ with arbitrarily large M remains still open (and seems to be rather difficult). However, we can prove that the metric frames L such that every epimorphism $L \to M$ is surjective (the episurjective objects) are very special, and those for which every epimorphic monomorphism $L \to M$ is an isomorphism (the epicomplete ones) are indeed very rare. Thus, unlike for mere frames, episurjectivity and epicompleteness do not coincide here.

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Further, we show that only spatial metric frames have epicompletions, but, on the other hand, the epicomplete objects do constitute a reflective subcategory in the whole of the category of metric frames; hence, each metric frame L has a canonical epimorphism $L \to M$ into an epicomplete one, very much in contrast, as we show, with the behaviour of epicomplete objects in the uniform case.

1 Preliminaries

1.1. As usual, a *concrete category* is understood to be a category the objects of which are sets endowed with structures, morphisms are (special) maps between these sets, with the standard composition of set maps as composition, and the identities are carried by the identity maps.

A morphism carried by an onto map will be referred to as a *surjection*, and if $f : A \to B$ is a surjection we call B a *weak image* of A.

1.2. Note that a surjection is always an epimorphism while the converse, as is familiar, need not hold.

We use the following terminology.

1. An *episurjective* object is an object A such that every epimorphism $A \to B$ is a surjection.
2. An *epicomplete* object A has the property that every one-one epimorphism $A \to B$ is an isomorphism.
3. An *epicompletion of* A is a one-one epimorphism $A \to B$ where B is epicomplete..

Note the following

1.2.1. Observation. *A weak image of an episurjective object is episurjective.*

1.3. As usual,

$$\mathbf{Frm}$$

will denote the category of frames and frame homomorphisms (viewed as a concrete category in the familiar way).

The lattice $\mathfrak{O}X$ of open sets of a topological space X is a frame, and if $f : X \to Y$ is a continuous map we have a frame homomorphism $\mathfrak{O}f$:

$\mathfrak{O}Y \to \mathfrak{O}X$ defined by $\mathfrak{O}f(U) = f^{-1}[U]$. Thus we have a contravariant functor

$$\mathfrak{O} : \mathbf{Top} \to \mathbf{Frm}.$$

It is adjoint on the right to the *spectrum* functor $\Sigma : \mathbf{Frm} \to \mathbf{Top}$ defined by the space ΣL with the homomorphisms $\xi : L \to \mathbf{2}$ as its points ($\mathbf{2}$ is the two-element frame $\{0 < 1\}$) and $\Sigma_a = \{\xi \mid \xi(a) = 1\}$ as its open sets; further, for any $h : L \to M$, $\Sigma h : \Sigma M \to \Sigma L$ is the continuous map defined by $(\xi \mapsto \xi \cdot h)$. The units of the adjunction $\varepsilon_L : L \to \mathfrak{O}\Sigma L$ and $\lambda_X : X \to \Sigma\mathfrak{O}L$ are given by $\varepsilon_L(a) = \Sigma_a$, and $\lambda_X(x)(U) = 1$ iff $x \in U$.

A frame L is called *spatial* if it is isomorphic to some $\mathfrak{O}X$, and one has the following criterion of spatiality:

L is spatial iff $\varepsilon_L : L \to \mathfrak{O}\Sigma L$ is one-one

Consequently we have

1.3.1. *Any subframe of a spatial frame is spatial.*

Indeed, let $j : M \to L$ be one-one; the equality $\varepsilon_L \cdot j = \mathfrak{O}\Sigma j \cdot \varepsilon_M$ makes ε_M one-one.

For more about frames see, e.g., [11], [17] or [18].

1.3.2. Complete Boolean algebras are obviously frames, called here *Boolean frames.*

For every frame L we have its *Booleanization*

$$\beta_L : L \to \mathfrak{B}L = \{a \mid a = a^{**}\}, \quad a \mapsto a^{**},$$

where a^* is the pseudocomplement of a. $\mathfrak{B}L$ is Boolean and β_L is a frame homomorphism onto, clearly dense, that is, $\beta(a) = 0$ implies $a = 0$.

1.4. Frame congruences are equivalence relations respecting all joins and all finite meets. For any frame L,

$$\mathfrak{C}L = \{E \mid E \text{ frame congruence on } L\},$$

ordered by inclusion, is a frame, and the mapping

$$\nu_L : L \to \mathfrak{C}L, \quad \nu_L(a) = \nabla_a = \{(x,y) \mid x \vee a = y \vee a\}$$

is a one-one frame homomorphism. Moreover:

1.4.1. *Each ν_L is an epimorphism, and it is surjective (and hence an isomorphism) iff L is Boolean.*

This initial step gives rise to the following sequence

$$L \to \mathfrak{C}L \to \mathfrak{C}^2 L \to \cdots \to \mathfrak{C}^\alpha L \to \dots$$

indexed by all ordinals, defined by transfinite induction

$$\begin{aligned} &\nu_L^1 = \nu_L : L \to \mathfrak{C}L = \mathfrak{C}^1 L, \\ &\nu^{\alpha+1} = \nu_{\mathfrak{C}^\alpha L}\nu_L^\alpha : \mathfrak{C}^\alpha L \to \mathfrak{C}^{\alpha+1} = \mathfrak{C}(\mathfrak{C}^\alpha L), \end{aligned}$$

and for limit ordinals α, $\mathfrak{C}^\alpha L$ is the colimit of the sequence

$$L \to \mathfrak{C}L \to \mathfrak{C}^2 L \to \cdots \to \mathfrak{C}^\beta L \to \dots, \quad \beta < \alpha.$$

The resulting transfinite sequence is referred to as the *tower of* L. There exists L such that this sequence never terminates. Thus, for such L one has epimorphisms $L \to M$ with arbitrarily large M ([10], see also [11]).

1.5. Proposition. *The following are equivalent for any L in* **Frm**.

(1) *L is episurjective,*

(2) *L is epicomplete,*

(3) *L is Boolean.*

Proof. (1)⇒(2) : A surjective one-one frame homomorphism is an isomorphism.

(2)⇒(3) : For epicomplete L, the one-one epimorphism $\nu_L : L \to \mathfrak{C}L$ of 1.4 is an isomorphism and hence L is Boolean by 1.4.1.

(3)⇒(1) : We use the familiar fact that, for any frame M, the composite

$$M \to \mathfrak{C}M \to \mathfrak{B}(\mathfrak{C}M), \quad a \mapsto \nabla_a \to \nabla_a^{**} = \nabla_a,$$

is a one-one homomorphism into a Boolean frame. Now, for any Boolean frame L, if $L \to M$ is an epimorphism in **Frm** then $L \to M \to \mathfrak{B}(\mathfrak{C}M)$ is an epimorphism in the category of Boolean frames, hence surjective by [12] and then $L \to M$ is also surjective. □

1.5.1. Remark. The result of [12] referred to above follows from the more general fact proved in [12] that the category of complete Boolean algebras has strong amalgamation. We note that the type of argument used above also shows that, for any *Boolean* subframe L of *arbitrary* frames M and N, M and N can be strongly amalgamated over L. In fact, this condition actually characterizes the Boolean frames.

2 Metric frames

2.1. A *cover* A of a frame L is a subset $A \subseteq L$ such that $\bigvee A = 1$. For covers A, B we write $A \leq B$ and say that A *refines* B if for each $a \in A$ there is a $b \in B$ such that $a \leq b$. Further we set $A \wedge B = \{a \wedge b \mid a \in A, b \in B\}$; note that this is a common refinement of A and B.

For a cover A and an $x \in L$, set

$$Ax = \bigvee\{a \in A \mid a \wedge x \neq 0\}$$

and for covers A, B write

$$AB = \{Ab \mid b \in B\}.$$

If $AA \leq B$ we say that A is a *star-refinement* of B and write $A \leq^* B$. If $\mathcal{A}$ is a set of covers define

$$x \lhd_{\mathcal{A}} y \quad \equiv_{\text{def}} \quad \exists A \in \mathcal{A},\ Ax \leq y.$$

A system of covers $\mathcal{A}$ of L is said to be *admissible* if

$$\forall x \in L,\ x = \bigvee\{y \mid y \lhd_{\mathcal{A}} x\}.$$

A uniformity on a frame L is a system of covers $\mathcal{A}$ such that

(U1) if $A \in \mathcal{A}$ and $A \leq B$ then $B \in \mathcal{A}$,

(U2) if $A, B \in \mathcal{A}$ then $A \wedge B \in \mathcal{A}$, and

(U3) for every $A \in \mathcal{A}$ there is a $B \in \mathcal{A}$ such that $B \leq^* A$.

A *uniform frame* is a couple $(L, \mathcal{A})$ where $\mathcal{A}$ is an admissible uniformity on L. A *uniform frame homomorphism* $h : (L, \mathcal{A}) \to (M, \mathcal{B})$ is a frame homomorphism $h : L \to M$ such that for every $A \in \mathcal{A}$, $h[A] \in \mathcal{B}$.

A uniformity is often presented by a *basis* $\mathcal{A}$, a system of covers satisfying

(U2') for any $A, B \in \mathcal{A}$ there is a $C \in \mathcal{A}$ such that $C \leq A, B$, and

(U3) for every $A \in \mathcal{A}$ there is a $B \in \mathcal{A}$ such that $B \leq^* A$.

Then $\widetilde{\mathcal{A}} = \{B \mid \exists A \in \mathcal{A},\ A \leq B\}$ is a uniformity. Note that in terms of bases the uniform frame homomorphisms are determined by the condition

$$\forall A \in \mathcal{A},\ \exists B \in \mathcal{B},\quad B \leq h[A].$$

For more about uniformities on frames see, e.g., [10], [6], [17] or [1].

2.2. A *diameter* on a frame L is a mapping $d : L \to \mathbb{R}_+$ (where $\mathbb{R}_+$ is the set of non-negative reals augmented by $+\infty$) such that

(D1) $d(0) = 0$,

(D2) $a \leq b \ \Rightarrow \ d(a) \leq d(b)$,

(D3) $a \wedge b \neq 0 \ \Rightarrow \ d(a \vee b) \leq d(a) + d(b)$, and

(D4) for each $\varepsilon > 0$, $U^d_\varepsilon = \{a \mid d(a) < \varepsilon\}$ is a cover.

We speak of a *metric diameter* if, moreover,

(M) for each $a \in L$ and each $\varepsilon > 0$ there are $u, v \leq a$ such that $d(u), d(v) \leq \varepsilon$ and $d(a) \leq d(u \vee v) + \varepsilon$.

A metric diameter expresses better the classical structure of a metric space; furthermore, for standard purposes, any diameter can be modified to a metric one. Therefore, to avoid unnecessary discussions, we will consider only metric diameters, and speak simply of diameters.

Obviously the system $\mathcal{U}^d = \{U^d_\varepsilon \mid \varepsilon > 0\}$ is a basis of a uniformity on L. If this is admissible we call (L, d) *metric frame.*

Note. Augmenting the non-negative reals by infinity to $\mathbb{R}_+$ is necessary even if we have in mind metric spaces with finite distances. But also in metric spaces (X, ρ) it is of advantage to consider possible infinite distances $\rho(x, y)$: for instance, free sums of spaces then make better sense.

2.3. Let $(L, d), (M, d')$ be metric frames. A homomorphism $h : L \to M$ is *contractive* if for each $\varepsilon > 0$ and each $b \in M$ with $d'(b) < \varepsilon$ there is an $a \in L$ with $d(a) < \varepsilon$ and $b \leq h(a)$; note this means $U^{d'}_\varepsilon \leq h[U^d_\varepsilon]$ for each $\varepsilon > 0$

The resulting category will be denoted by

$$\mathbf{MFrm}.$$

2.3.1. Observation. *Let d, d' be admissible diameters on a frame L. Then* $\mathrm{id} : (X, d) \to (X, d')$ *is contractive iff $d \leq d'$.*

2.4. Let (X, ρ) be a metric space (again, we admit infinite distances $\rho(x, y)$). For $U \in \mathfrak{O}(X, \rho)$ set, as usual,

$$\operatorname{diam}(U) = \operatorname{diam}_\rho(U) = \sup\{\rho(x, y) \mid x, y \in U\}.$$

Then $d = \operatorname{diam}$ is a (metric) diameter on $\mathfrak{O}X$ with $\mathcal{U}^d$ admissible, and hence we have a metric frame $(\mathfrak{O}X, \operatorname{diam}_\rho)$. A *spatial metric frame* is an (L, d) isomorphic (in **MFrm**) to an $(\mathfrak{O}X, \operatorname{diam}_\rho)$.

It is easy to check that

2.4.1. $\mathfrak{O}f : (\mathfrak{O}Y, \operatorname{diam}_\sigma) \to (\mathfrak{O}X, \operatorname{diam}_\rho)$ *is contractive if and only if* $f : (X, \rho) \to (Y, \sigma)$ *is contractive.*

The full subcategory of **MFrm** determined by the spatial metric frames will be denoted by **SpMFrm**.

2.5. Let (L, d) be a metric frame and $h : L \to M$ a surjective frame homomorphism. Define a mapping $\overline{d} : M \to \mathbb{R}_+$ by setting

$$\overline{d}(b) = \inf\{d(a) \mid b \leq h(a)\}.$$

By 2.11 in [16] we have

2.5.1. Proposition. $(M, \overline{d})$ *is a metric frame and* $h : (L, d) \to (M, \overline{d})$ *is contractive.*

(The contractivity follows from the definition of $\overline{d}$.)

Consider the coproduct $L \oplus M$ of frames (see, e.g., [11] or [17]). From [16] we can infer

2.5.2. Lemma. *The underlying frame of the coproduct*

$$(L_1, d_1) \oplus (L_2, d_2)$$

in **MFrm** *is the frame coproduct* $L_1 \oplus L_2$.

Proof. Recall from 1.1 in [16] that α^d_ε is defined as the right adjoint of $x \mapsto U^d_\varepsilon$, and the fact that the admissibility condition $a = \bigvee\{b \in L \mid b \lhd_d a\}$ is also expressed as $a = \bigvee\{\alpha^d_\varepsilon(a) \mid \varepsilon > 0\}$; furthermore note that

$$\varepsilon \leq \delta \quad \Rightarrow \quad \alpha^d_\varepsilon a \geq \alpha^d_\delta a.$$

Now by the description of the coproduct of metric frames in [16] the case of a pair $(L_1, d_1), (L_2, d_2)$ may be described as follows. The underlying frame of that coproduct is given by the down-sets $U \subseteq L_1 \times L_2$ for which

(R1) $\{a\} \times S \subseteq U$ implies $(a, \bigvee S) \in U$, and

$S \times \{b\} \subseteq U$ implies $(\bigvee S, b) \in U$,

(R2) $(\alpha_\varepsilon^{d_1} a, \alpha_\varepsilon^{d_2} b) \in U$ for all $\varepsilon > 0$ implies $(a, b) \in U$.

In our case, (R2) is actually already implied by (R1):

Let $(\alpha_\varepsilon^{d_1} a, \alpha_\varepsilon^{d_2} b) \in U$. Then for $\varepsilon \leq \delta$,

$$(\alpha_\varepsilon^{d_1} a, \alpha_\delta^{d_2} b) \in U$$

and by (R1) we have $(a, \alpha_\delta^{d_2} b)$ for all $\delta > 0$ so that, again by (R1), $(a, b) \in U$.

Finally, because (R1) defines the frame coproduct $L_1 \oplus L_2$, this proves the claim. □

Consequently, this lemma together with [16], 4.1, now proves

2.5.3. Proposition. *Let $(L_1, d_1), (L_2, d_2)$ be metric frames. Then there exists an admissible (metric) diameter d on $L = L_1 \oplus L_2$ such that the coproduct maps $\iota_i : (L_i, d_i) \to (L, d)$ are contractive.*

2.6. Proposition. *$h : (L.d) \to (M, d')$ is an epimorphism in* **MFrm** *if and only if $h : L \to M$ is an epimorphism in* **Frm**.

Proof. Obviously, if h is an epimorphism in **Frm** then it is an epimorphism in **MFrm**.

Now let $h : L \to M$ be epic in **MFrm** and $f, g : M \to N$ frame homomorphisms such that $fh = gh$. Further, in the diagram

$$\begin{tikzcd}
 & M \oplus M \arrow[rr, "\nu"] \arrow[dr, "k"] & & \mathrm{Im}(k) \arrow[dl, "l"] \\
L \arrow[r, "h"'] & M \arrow[u, shift left, "i"] \arrow[u, shift right, "j"'] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N &
\end{tikzcd}$$

let $M \oplus M$ be the frame coproduct with coproduct injections i and j, k such that $ki = f$ and $kj = g$, and $k = l\nu$ the image factorization. Now by 2.5.3 and 2.5.1 we can endow $M \oplus M$ and $\mathrm{Im}(k)$ with diameters such that i, j and ν are contractive. Since $fh = gh$ we have $l\nu ih = l\nu jh$, and since l is one-one, $\nu ih = \nu jh$. As $\nu i, \nu j$ are in **MFrm**, in which h is an epimorphism, we have $\nu i = \nu j$ and finally $f = l\nu i = l\nu j = g$. □

2.6.1. Notes. 1. By the same procedure one can prove that $h : (L, \mathcal{A}) \to (M, \mathcal{B})$ is an epimorphism in **UniFrm** iff $h : L \to M$ is an epimorphism in **Frm**, and the same also holds for the larger category of nearness frames (see [6]). Endowing the $M \oplus M$ and $\mathrm{Im}(k)$ with appropriate structures is (even) simpler.

2. Recall the complete uniform frames from (e.g.) [1], [6]. By [10] (see also [3]), a complete metric frame is always spatial. Hence all we can say about the corresponding category **CMFrm** (and the uniform variant) is included in the results on spatial frames. Anyway, 2.6 holds for **CMFrm** as well.

3 Epicompleteness and episurjectivity

3.1. In this section an important role will be played by by Boolean metric frames. Note that they are ubiquitous: for any (L, d), the Booleanization is a metric frame again. More precisely, since the frame homomorphism $\beta : L \to \mathfrak{B}L$ from 1.3.2 is onto we can endow $\mathfrak{B}L$ with a diameter using 2.5.1. Actually, this diameter turns out to be the restriction of d to $\mathfrak{B}L$, as shown in [4].

In particular we will be interested in the atomic Boolean algebras endowed with the largest diameter

$$d(a) = \begin{cases} 0 \text{ if } a \text{ is an atom or } 0, \\ +\infty \text{ otherwise.} \end{cases}.$$

These will be called *extremal metric frames.*

3.2. Lemma. *In any metric frame, $d(a) = 0$ for an $a > 0$ iff a is an atom.*

Proof. $\Rightarrow$: Let $a > 0$ and $d(a) = 0$. Then $a \in U_\varepsilon^d$ (recall 2.2) for any $\varepsilon > 0$ and hence $U_\varepsilon^d x \geq a$ for any x such that $0 < x \leq a$. Now if $b < a$ we have $b = \bigvee\{x \mid \exists\varepsilon > 0,\ U_\varepsilon^d x \leq b\}$ and hence the only x involved can be 0; consequently $b = 0$.

$\Leftarrow$: Let a be an atom. Suppose $d(a) > 0$. Then for any $\varepsilon > 0$, there exist $b, c \leq a$ such that $d(b), d(c) < \varepsilon$ and $d(b \vee c) > 0$. It follows that $b \vee c > 0$ so that $b > 0$ or $c > 0$, implying $a = b$ or $a = c$ since a is an atom, and hence $d(a) < \varepsilon$. Thus $d(a) = 0$, since ε was arbitrary, contradicting the assumption. $\square$

3.3. Proposition. *A metric frame is epicomplete in* **MFrm** *iff it is extremal.*

Proof. $\Rightarrow$: Let (L, d) be an epicomplete metric frame. Then $2d$ is also an admissible diameter, and the identity map $(L, d) \to (L, 2d)$ is contractive by 2.3.1 and hence an isomorphism. Then, however, we have to have, again by 2.3.1, $2d \leq d$ as well, and we see that

$$\forall a, \quad d(a) = 0 \ \text{ or } \ d(a) = +\infty. \tag{$*$}$$

Thus, for any $\varepsilon > 0$, U^d_ε consists, besides of 0, of atoms, and since it is a cover we have

$$\forall \varepsilon > 0, \quad U^d_\varepsilon = A = \{a \in L \mid a \text{ atom}\}.$$

Consequently, for each $x \in L$ we have $x = \bigvee\{y \mid Ay \leq x\} = \bigvee\{a \in A \mid a \leq x\}$, and L is an atomic Boolean algebra. Recalling $(*)$ again we see that it is extremal.

$\Leftarrow$: Let (L, d) be extremal and $h : (L, d) \to (M, d')$ a one-one epimorphism in **Frm**. Then h is onto because L is Boolean, and without loss of generality we can assume that $M = L$ and h is identical. Then $d \leq d'$ and hence $d'(a) = +\infty$ for all non-atoms a; for atoms a we have $d'(a) = 0$ by Lemma 3.1. $\square$

3.3.1. Since a weak image of an extremal metric frame is obviously extremal, we have

Corollary. *In* **MFrm** *a weak image of an epicomplete object is epicomplete.*

(This is a special fact. Unlike the situation with episurjectiveness in 1.2.1 there is no reason for weak images to inherit epicompleteness in a general concrete category.)

3.3.2. Note. The same holds for the categories **SpMFrm** and **CMFrm** (recall 2.6.1). In the latter case it suffices to realize that if (X, d) is complete then $(X, 2d)$ is complete as well.

3.4. Unlike in **Frm**, episurjectivity in **MFrm** does not coincide with epicompleteness. In fact, episurjectivity does not even imply epicompleteness (this implication was trivial in **Frm** because there a one-one onto homomorphism is always an isomorphism, which is not the case in **MFrm**).

3.4.1. Lemma. *For any non-Boolean metric frame* (L, d) *there is a non-surjective epimorphism* $h : (L, d) \to (M, d')$ *in* **MFrm**.

Proof. Let $u \in L$ be an element that is not complemented. View L as embedded into $\mathfrak{C}L$, take as v the complement of u and define M as the subframe of $\mathfrak{C}L$ generated by L and v; let $h : L \to M$ be the embedding. Obviously h is an epimorphism (since the value $f(v)$ of a frame homomorphism $f : M \to N$ is determined by $f(u)$). Further define B as the subframe of M consisting of $0, u, v$ and 1 and endow it with the diameter $d(0) = d(u) = d(v) = 0$ and $d(1) = 1$.

Consider the coproduct

$$L \xrightarrow{\ i\ } L \oplus B \xleftarrow{\ j\ } B$$

and the map $k : L \oplus B \to M$ satisfying $ki = h$, kj the identical embedding $B \to M$. Obviously k is surjective and by 2.4.1 and 2.4.2 we can endow $L \oplus B$ and M by metric diameters such that $h = ki$ is contractive. $\square$.

3.4.2. Proposition. *A metric frame is episurjective iff it is Boolean.*

Proof. $\Rightarrow$: If L is not Boolean it is not episurjective, by 3.4.1.

$\Leftarrow$: By 2.6, any epimorphism $h : (L, d) \to (M, d')$ in **MFrm** is an epimorphism in **Frm**. Thus, this implication follows from 1.5. $\square$

3.4.3. Note. The analogue of 3.4.2 also holds in **UniFrm**. In fact, the proof is even more straightforward for $\Rightarrow$ since we can use the embedding $L \to \mathfrak{C}L$ without any particular construction: just take $\mathfrak{C}L$ with its fine uniformity. – For $\Leftarrow$ recall 2.6.1.

4 Epicompletions and an epicomplete reflection

4.1. Let X be a set. The symbol $\mathfrak{P}X$ will be used

- in the frame context for the Boolean frame of all subsets of X, and
- in the metric context for the corresponding extremal metric frame.

For a map $f : X \to Y$ we will denote by $\mathfrak{P}f$ the mapping $(U \mapsto f^{-1}[U]) : \mathfrak{P}Y \to \mathfrak{P}X$. Thus,

$$\mathfrak{P}f = \mathfrak{O}f \quad \text{for the continuous} \quad f : (X, \mathfrak{P}X) \to (Y, \mathfrak{P}Y).$$

4.2. The spectrum adjunction (recall 1.3) can be modified to an adjunction

$$\mathfrak{O} : \mathbf{MSp} \to \mathbf{MFrm}, \quad \Sigma : \mathbf{MFrm} \to \mathbf{MSp}$$

(where **MSp** is the category of metric spaces and contractive mapping) with $\mathfrak{O}(X,\rho)$ enriched by the diameter as in 2.4, and $\Sigma(L,d)$ endowed with the distance $\rho_d(\alpha,\beta)=\inf\{d(a)\mid \alpha(a)=\beta(a)=1\}$. The units work as in 1.3, and in case of a spatial metric frame (X,d) (recall 2.4) the unit morphism $\varepsilon_{(L,d)}$ is an isometry, that is, preserves the diameter. For details see, e.g., [16].

In this section we will use the functors $\mathfrak{O},\Sigma$ in this sense.

4.3.1. Lemma. *For every T_1-space $(X,\mathfrak{O}X)$ the embedding $m:\mathfrak{O}X\to\mathfrak{P}X$ is a (frame) epimorphism.*

Proof. Let $fm=gm$ for frame homomorphisms $f,g:\mathfrak{P}X\to N$. Then for each $x\in X$, $f(X\smallsetminus\{x\})=g(X\smallsetminus\{x\})$ and hence, as frame homomorphism preserve complements, $f(\{x\})=g(\{x\})$ and the statement follows by taking unions. □

Note. By a more involved argument we can prove that this holds for any T_0-space. But we will need the fact for the T_1-case only.

4.3.2. Proposition. *A metric frame has an epicompletion iff it is spatial, and then the epicompletion is unique.*

Proof. $\Rightarrow$: Since an extremal metric space is spatial (consider (X,ρ) with $\rho(x,y)=+\infty$ for $x\neq y$) this implication follows from 1.3.1.

$\Leftarrow$: For a spatial $(L,d)\cong\mathfrak{O}(X,\rho)$ consider the identical embedding $\mathfrak{O}X\to\mathfrak{P}X$. Because of the extremal diameter in $\mathfrak{P}X$ this is contractive, and by 4.3.1 an epimorphism.

For uniqueness consider $(L,d)=\mathfrak{O}(X,\rho)$ and the epicompletion $m:\mathfrak{O}X\subseteq\mathfrak{P}X$ as above. Let $h:L\to M$ be any other epicompletion. We can assume that $M=\mathfrak{P}X$, and then $h[L]$ is a topology on X. Moreover,

this topology $h[L]$ is Hausdorff and hence sober.

(Since it is isomorphic to the topology of a metric space it is regular. Thus it suffices to prove it is T_0. Let $x\in U$ iff $y\in U$ for some $x\neq y$. Take the map $f:Y\to Y$ interchanging x and y and leaving all the other points intact. Then $\mathfrak{O}f\cdot h=\mathrm{id}\cdot h$ contradicting the epimorphism assumption.)

We have the isomorphism $\overline{h}=(a\mapsto h(a)):L\to h[L]$ and both (X,L) and $(Y,h[L])$ are sober. Thus (see, e.g., [11], [17]) there is a homeomorphism $g:Y\to X$ such that $\overline{h}=\mathfrak{O}g$. Now we have, for any $a\in L$, $\mathfrak{P}g(m(a))=\mathfrak{O}g(a)=\overline{h}(a)=h(a)$, hence

$$\mathfrak{P}g\cdot m=h$$

with an isomorphism $\mathfrak{P}g$. $\square$

4.4.1. Combining the (surjective, but generally not one-one) morphisms $\varepsilon_{(L,d)} : (L,d) \to \mathfrak{O}\Sigma(L,d)$ with the epimorphic $m_L : \mathfrak{O}\Sigma L \to \mathfrak{P}\Sigma L$ we obtain a transformation $\eta_{(L,d)} : (L,d) \to \mathfrak{P}\Sigma L$. Since, as we easily see, for an extremal metric frame (L,d), $\eta_{(L,d)}$ is an isomorphism, we have obtained

Proposition. *The subcategory of epicomplete metric frames (≡ extremal metric frames) is reflective (indeed, epireflective) in* **MFrm**.

4.4.2. In the category **UniFrm** of uniform frames the epicomplete objects are precisely the Boolean frames with the uniformities of allcovers. In contrast with the metric situation above, we have

Proposition. *The subcategory of epicomplete objects is not reflective in* **UniFrm**.

Proof. Suppose there is a reflection $\eta_L : L \to \mathcal{B}L$. Consider the free Boolean algebra F on a countable set S, with the embedding $j : S \to F$; further, take the frame $\mathfrak{I}F$ of ideals in F and the mapping

$$\delta = (a \mapsto {\downarrow} a) : F \to \mathfrak{I}F.$$

The frame $\mathfrak{I}F$ is obviously compact regular and hence all its covers constitute its unique uniformity ([10], [15]); it will be considered a uniform frame in this sense.

Now let B be any Boolean frame and $\phi : S \to B$ an arbitrary mapping. Then we have a unique Boolean homomorphism $\overline{\phi} : F \to B$ such that $\overline{\phi} \cdot j = \phi$. This in turn extends uniquely to a frame homomorphism $\widetilde{\phi} : \mathfrak{I}F \to B$ such that $\widetilde{\phi} \cdot \delta$, by the formula

$$\widetilde{\phi}(J) = \bigvee \{\overline{\phi}[a] \mid a \in J\}.$$

If we now endow B with the fine uniformity, we have a unique (uniform) frame – and hence complete Boolean – homomorphism $\psi : \mathcal{B}(\mathfrak{I}F) \to B$ satisfying $\psi \cdot \eta_{\mathfrak{I}F} = \widetilde{\phi}$. This, however, proves that

$$\eta_{\mathfrak{I}F} \cdot \delta \cdot j : S \to \mathcal{B}(\mathfrak{I}F)$$

constitutes a free *complete* Boolean algebra with countably many generators which is a contradiction ([8], [9], see also [11]). $\square$

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