Chromatic number for a generalization of Cartesian product graphs

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Abstract

Let $\mathcal{G}$ be a class of graphs. The *d-fold grid over $\mathcal{G}$*, denoted $\mathcal{G}^d$, is the family of graphs obtained from d-dimensional rectangular grids of vertices by placing a graph from $\mathcal{G}$ on each of the lines parallel to one of the axes. Thus each vertex belongs to d of these subgraphs. Let $f(\mathcal{G};d) = \max_{G\in\mathcal{G}^d} \chi(G)$. If each graph in $\mathcal{G}$ is k-colorable, then $f(\mathcal{G};d) \le k^d$. We show that this bound is best possible by proving that $f(\mathcal{G};d) = k^d$ when $\mathcal{G}$ is the class of all k-colorable graphs. We also show that $f(\mathcal{G};d) \ge \left\lfloor \sqrt{\frac{d}{6\log d}} \right\rfloor$ when $\mathcal{G}$ is the class of graphs with at most one edge, and $f(\mathcal{G};d) \ge \left\lfloor \frac{d}{6\log d} \right\rfloor$ when $\mathcal{G}$ is the class of graphs with maximum degree 1.

1 Introduction

The *Cartesian product* of graphs $G_1, \ldots, G_d$ is the graph with vertex set $V(G_1) \times \cdots \times V(G_d)$ in which two vertices $(v_1, \ldots, v_d)$ and $(v'_1, \ldots, v'_d)$ are adjacent if they agree in all but one coordinate, and in the coordinate where

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they differ the values are adjacent vertices in the corresponding graph. The product can be viewed as a rectangular grid with copies of $G_1, \ldots, G_d$ placed on vertices forming lines parallel to the d axes. It is well-known (and easy to show) that the chromatic number of the Cartesian product of $G_1, \ldots, G_d$ is the maximum of the chromatic numbers of $G_1, \ldots, G_d$.

In this paper, we consider bounds on the chromatic number of graphs in a family resulting from a more general graph operation. Instead of placing copies of the same graph G_i on all the lines parallel to the i-th axis, we may place different graphs from a fixed class. Let $[n]$ denote $\{1, \ldots, n\}$. For a class $\mathcal{G}$ of graphs, the *d-fold grid over $\mathcal{G}$*, denoted $\mathcal{G}^d$, is the family of all graphs formed by choosing a vertex set of the form $[n_1] \times \cdots \times [n_d]$ and letting each set of vertices where all but one coordinate is fixed induce a graph from $\mathcal{G}$. For example, a Cartesian product of graphs in $\mathcal{G}$ belongs to $\mathcal{G}^d$.

The study of the chromatic number and independence number of graphs in $\mathcal{G}^d$ is motivated by an application in computational geometry. Frequency assignment problems for transmitters in the plane are modeled by coloring and independence problems on certain graphs (see [2, 6]). These graphs arise from sets of points using the Euclidean metric. When such problems are studied using the Manhattan metric (see [4]), results about grid structures give bounds for the number of frequencies needed. Motivated by such problems, Szegedy [13] posed the following open problem at the workshop "Combinatorial Challenges":

> *What is the maximum chromatic number of a graph $G \in \mathcal{G}^d$ when $\mathcal{G}$ is the class $\mathcal{B}$ of all bipartite graphs or when $\mathcal{G}$ is the class $\mathcal{S}$ of graphs containing at most one edge?*

If each graph of $\mathcal{G}$ is k-colorable, then every graph in $\mathcal{G}^d$ has chromatic number at most k^d, since it is the union of d subgraphs, each of which is k-colorable. In particular, all graphs in $\mathcal{B}^d$ are 2^d-colorable. We show that this bound is sharp, which is somewhat surprising since Cartesian products of bipartite graphs are bipartite. More generally, let $f(\mathcal{G}; d) = \max_{G \in \mathcal{G}^d} \chi(G)$. We show that if $\mathcal{G}$ is the class of all k-colorable graphs, then $f(\mathcal{G}; d) = k^d$. We prove the existence of k^d-chromatic graphs in $\mathcal{G}^d$ probabilistically, but an explicit construction can then be obtained by building, for each n, a graph in $\mathcal{G}^d$ that is "universal" in the sense that it contains all graphs in $\mathcal{G}^d$ with vertex set $[n]^d$. This settles the first part of Szegedy's question.

Determining $f(\mathcal{S}; d)$ is more challenging. Since the maximum degree of a graph in $\mathcal{S}^d$ does not exceed d, and these graphs do not contain K_{d+1}, Brooks' Theorem [3] implies that each graph in $\mathcal{S}^d$ is d-colorable (when $d \geq 3$). Also graphs in $\mathcal{S}^2$ are bipartite, since cycles in such a graph alternate between horizontal and vertical edges.

When d is large, we can use a refinement of Brook's Theorem obtained by Reed et al. [7, 10, 11, 12] to improve the upper bound.

Theorem 1 (Molloy and Reed [11]). *There exists a constant* D_0 *such that for all* $D \geq D_0$ *and* k *satisfying* $k^2 + 2k < D$*, every graph* G *with maximum degree* D *and* $\chi(G) > D - k$ *has a subgraph* H *with at most* $D + 1$ *vertices and* $\chi(H) > D - k$.

Theorem 1 implies that $f(\mathcal{S}; d) \leq d - \sqrt{d} + o(1)$. A still better upper bound follows from another result.

Theorem 2 (Johansson [9]). *The chromatic number of a triangle-free graph with maximum degree* D *is at most* $O(D/\log D)$.

This result, which was further strengthened by Alon et al. [1], implies that $f(\mathcal{S}; d) \in O(d/\log d)$.

We show that though the graphs contained in $\mathcal{S}^d$ are very sparse, and it is natural to expect that the graphs contained in $\mathcal{S}^d$ can be colored just with a few colors, $f(\mathcal{S}; d) \geq \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$. Our argument is again probabilistic. A similar argument yields $f(\mathcal{M}; d) \geq \left\lfloor \frac{d}{6 \log d} \right\rfloor$, where $\mathcal{M}$ is the class of all matchings (i.e., graphs with maximum degree 1). This lower bound is asymptotically best possible, since the discussion above yields $f(\mathcal{M}; d) \in O(d/\log d)$.

2 Preliminaries

In this section, we make several observations used in the proofs of our subsequent lower bounds on $f(\mathcal{G}; d)$ for various $\mathcal{G}$. We start by recalling the

Chernoff Bound, an upper bound on the probability that a sum of independent random variables deviates greatly from its expected value (see [8] for more details).

Proposition 3. *If X is a random variable equal to the sum of N independent identically distributed $0,1$-random variables having probability p of taking the value 1, then the following holds for every $0 < \delta \leq 1$:*

$$\mathrm{Prob}(X \geq (1+\delta)pN) \leq e^{-\frac{\delta^2 pN}{3}} \quad \textit{and} \quad \mathrm{Prob}(X \leq (1-\delta)pN) \leq e^{-\frac{\delta^2 pN}{2}}.$$

Next, we establish two technical claims. We begin with a standard bound on the number of subsets of a certain size.

Proposition 4. *For $\alpha > 2$, the number of N/α-element subset of an N-element set is at most $2^{\frac{N}{\alpha}(1+\log\alpha)}$.*

Proof. An N-element set has $\binom{N}{N/\alpha}$ subsets of size N/α. It is well known, see e.g. [5], that $\binom{N}{N/\alpha} \leq 2^{N \cdot H(1/\alpha)}$, where $H(p) = -p\log p - (1-p)\log(1-p)$ ($H(p)$ is called the entropy function). A simple calculation yields the upper bound:

$$\binom{N}{N/\alpha} \leq 2^{N\cdot(\frac{1}{\alpha}\log\alpha + \frac{\alpha-1}{\alpha}\log\frac{\alpha}{\alpha-1})} \leq 2^{N\cdot(\frac{1}{\alpha}\log\alpha + \frac{\alpha-1}{\alpha}\cdot\frac{1}{\alpha-1})} \leq 2^{\frac{N}{\alpha}(1+\log\alpha)}$$

□

The second claim is a straightforward upper bound on a certain type of product of expressions of the form $(1-\varepsilon)$:

Proposition 5. *If $a_1, \ldots, a_m$ are nonnegative integers with sum n, then*

$$\prod_{i=1}^{m}\left(1 - \binom{a_i}{2}\frac{1}{x}\right) \leq \left(1 - \frac{1}{x}\right)^{n-m}$$

for any positive real x such that $x \leq \binom{\max_i a_i}{2}$.

Proof. Since $\sum a_i = n$, it suffices to show that

$$\left(1 - \binom{a}{2}\frac{1}{x}\right) \leq \left(1 - \frac{1}{x}\right)^{a-1} \qquad (1)$$

for every nonnegative integer a. If $a \leq 1$, then the left side of (1) is 1 and the right side is at least 1. If $a \geq 2$, then (1) follows (by setting $k = a - 1$) from the well-known inequality

$$1 - \frac{k}{x} \leq \left(1 - \frac{1}{x}\right)^k,$$

which holds whenever $0 \leq k \leq x$. $\square$

3 Products of k-colorable graphs

In this section, we prove that $f(\mathcal{B}; d) = 2^d$. Note again that after the probabilistic proof of existence, we can construct such graphs explicitly as explained in Introduction. Even so, the argument that they are not $(2^d - 1)$-colorable remains probabilistic. We prove the result in the more general setting of k-colorable graphs.

Theorem 6. *For $d, k \in \mathbf{N}$, the d-fold grid over the class of k-colorable graphs contains a graph with chromatic number k^d.*

Proof. The claim holds trivially if $d = 1$, so we assume $d \geq 2$. The integers k and d are fixed, and N is a large integer to be chosen in terms of k and d. Consider the set $[N]^d$. For each $v \in [N]^d$, define a random d-tuple $X(v)$ such that $X(v)_i$ takes each value in $[k]$ with probability $1/k$, and the d coordinate variables are independent. Generate a graph G with vertex set $[N]^d$ by making two vertices u and v adjacent if they differ in exactly one coordinate and $X(u)_\ell \neq X(v)_\ell$, where ℓ is the coordinate in which u and v differ.

By construction, any set of vertices in G that all agree outside a fixed coordinate induce a complete multipartite graph with at most k parts. Hence G is in the d-fold grid over the k-colorable graphs. It will suffice to show that almost surely (as N tends to infinity) G does not have an independent set with more than $\frac{N^d}{k^d - 0.5}$ vertices. This yields $\chi(G) \geq k^d$, since otherwise some color class would be an independent set of size at least $\frac{N^d}{k^d - 1}$.

For an independent set A in G, let the *shade* of A be the function $\sigma\colon [d] \times [N]^{d-1} \to [k]$ defined as follows. For $z = (\ell; i_1, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_d) \in [d] \times$

$[N]^{d-1}$, consider the vertices in A of the form $(i_1, \dots, i_{\ell-1}, j, i_{\ell+1}, \dots, i_d)$. By the construction of G, the value of $X(v)_\ell$ is the same for each such vertex v, since vertices of A are nonadjacent. Let this value be $\sigma(z)$. If there is no vertex of A with this form, then let $\sigma(z) = 1$.

The union of independent sets with the same shade is an independent set. Hence for each function σ there is a unique maximal independent set in G with shade σ; denote it by A_σ. In order to have $v \in A_\sigma$, where $v = (i_1, \dots, i_d)$, the random variables $X(v)_1, \dots, X(v)_d$ must satisfy $X(v)_\ell = \sigma(\ell; i_1, \dots, i_{\ell-1}, i_{\ell+1}, \dots, i_d)$. Hence each v lies in A_σ with probability k^{-d}.

As a result, the expected size of A_σ is $(N/k)^d$. Since the variables $X(v)_\ell$ are independent for all v and ℓ, we can bound the probability that $|A_m| \geq \frac{N^d}{k^d - 0.5}$ using the Chernoff Bound (Proposition 3). Applied with $\delta = \frac{1}{2k^d - 1}$, this yields

$$\text{Prob}\left(|A_m| \geq \frac{N^d}{k^d - 0.5}\right) \leq e^{-\frac{N^d}{3(2k^d-1)^2 k^d}} \leq e^{-\frac{N^d}{12k^{3d}}}.$$

Since there are $k^{dN^{d-1}}$ possible shades, the probability that some independent set has more than $\frac{N^d}{k^d - 0.5}$ vertices is at most $k^{dN^{d-1}} \cdot e^{-N^d/12k^{3d}}$, and we compute

$$k^{dN^{d-1}} \cdot e^{-N^d/12k^{3d}} = e^{\log k dN^{d-1} - N^d/12k^{3d}} \to 0 \ .$$

If N is sufficiently large in terms of k and d, then the bound is less than 1, and there exists such a graph G with no independent set of size at least $\frac{N^d}{k^d - 0.5}$. □

4 Products of single-edge graphs

In this section, we prove the lower bound for the d-fold grid over the class of graphs with at most one edge.

Theorem 7. *For $d \geq 2$, there exists $G \in \mathcal{S}^d$ such that $\chi(G) \geq \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$.*

Proof. Let $k = \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$. For $k \leq 2$, the conclusion is immediate. Hence, we assume $k \geq 3$. We generate a graph G with vertex set $[2k]^d$. For

$(\ell; i_1, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_d) \in [d] \times [2k]^{d-1}$, choose a random pair of distinct integers j and j' from $[2k]$, and make the vertices $(i_1, \ldots, i_{\ell-1}, j, i_{\ell+1}, \ldots, i_d)$ and $(i_1, \ldots, i_{\ell-1}, j', i_{\ell+1}, \ldots, i_d)$ adjacent in G. The choices of j and j' are independent for all elements of $[d] \times [2k]^{d-1}$.

Since $G \in \mathcal{S}^d$, its chromatic number is at most d. To show that $\chi(G)$ is at least k with positive probability, it suffices to show that with positive probability, G has no independent set of size at least $(2k)^d/k$.

Consider a set A in $V(G)$ with size $(2k)^d/k$; we bound the probability that A is an independent set in G. Again we think of an element z in $[d] \times [2k]^{d-1}$ as designating a line in $[2k]^d$ parallel to some axis. Let $A[z]$ be the intersection of A with this line. By the construction of G, the probability that some two vertices in $A[z]$ are adjacent in G is

$$1 - \binom{|A[z]|}{2} \Big/ \binom{2k}{2},$$

which is at most $1 - \binom{|A[z]|}{2}\frac{1}{2k^2}$. By applying Proposition 5 with $x = 1/2k^2$, $n = |A| \geq \frac{(2k)^d}{k} = 2(2k)^{d-1}$, and $m = (2k)^{d-1}$, we conclude that the probability of all subsets of A lying along lines in a particular direction being independent in G is at most

$$\prod_{z \in \{\ell\} \times [2k]^{d-1}} \left(1 - \binom{|A[z]|}{2} \frac{1}{2k^2}\right) \leq \left(1 - \frac{1}{2k^2}\right)^{(2k)^{d-1}} .$$

Let p be the probability that A is an independent set in G. Since the edges in each of the d directions are added to G independently,

$$p \leq \left(1 - \frac{1}{2k^2}\right)^{d(2k)^{d-1}} \leq e^{-\frac{d(2k)^{d-1}}{2k^2}} \leq e^{-2d(2k)^{d-3}} \leq 2^{-2d(2k)^{d-3}} .$$

We want to show that with positive probability, G has no independent set of size $(2k)^d/k$. Let M be the number of subsets of $V(G)$ with size $(2k)^d/k$. By Proposition 4,

$$M \leq 2^{\frac{(2k)^d}{k} \cdot (1+\log k)} \leq 2^{2(2k)^{d-1} \cdot (1+\log k)} \leq 2^{3(2k)^{d-1} \log k}.$$

Therefore, we bound the probability that G has an independent set of size $(2k)^d/k$ by the following computation:

$$2^{3(2k)^{d-1} \log k} \cdot 2^{-2d(2k)^{d-3}} = 2^{2(2k)^{d-3}(6k^2 \log k - d)} < 1 .$$

The last inequality uses the fact that $6k^2 \log k - d$ is negative, by the choice of k. We conclude that some such graph G has no independent set of size at least $(2k)^d/k$. $\square$

5 Products of matchings

Finally, we consider the d-fold grid over the class $\mathcal{M}$ of matchings.

Theorem 8. *For $d \geq 2$, there exists $G \in \mathcal{M}^d$ such that $\chi(G) \geq \left\lfloor \frac{d}{6 \log d} \right\rfloor$.*

Proof. As the proof is similar to the proof of Theorem 7, we will give less detail and focus on the differences between the proofs. Set $k = \left\lfloor \frac{d}{6 \log d} \right\rfloor$ and assume $k \geq 3$. We randomly generate a graph G with vertex set $[2k]^d$. As before an element z in $[d] \times [2k]^{d-1}$ designates a line in $[2k]^d$ parallel to some axis. We place a random perfect matching on the $2k$ vertices in each such line. Hence, the resulting graph G is d-regular. It suffices to show that with positive probability, G has no independent set of size at least $(2k)^d/k$.

Consider a set A in $V(G)$ with size $(2k)^d/k$; we bound the probability that A is an independent set in G. Let $A[z]$ be the intersection of A with a line designated by $z \in [d] \times [2k]^{d-1}$. Let X be the random variable that is the number of edges in G induced by $A[z]$. By the construction of G, we have $\mathrm{E}(X) = \frac{1}{2k-1}\binom{|A[z]|}{2}$. When X is a nonnegative integer-valued random variable, $\mathrm{Prob}[X \geq 1] \geq \frac{\mathrm{E}(X)}{\max(X)}$. Since $A[z]$ cannot induce more than $|A[z]|/2$ edges, we bound the probability p that $A[z]$ contains an edge by computing

$$p \geq \frac{\frac{1}{2k-1}\binom{|A[z]|}{2}}{|A[z]|/2} = \frac{|A[z]| - 1}{2k-1} \geq \frac{|A[z]| - 1}{2k}.$$

Let q_ℓ denote the probability that all subsets of A lying along lines in direction ℓ are independent (each such line consists of d-tuples that agree outside the ℓth coordinate). We compute

$$q_\ell \leq \prod_{z \in \{\ell\} \times [2k]^{d-1}} \left(1 - \frac{|A[z]| - 1}{2k}\right) \leq \prod_{z \in \{\ell\} \times [2k]^{d-1}} e^{-\frac{|A[z]|-1}{2k}}$$

$$= e^{-\frac{(2k)^d/k-(2k)^{d-1}}{2k}} = e^{-(2k)^{d-2}}.$$

The probability P that A is independent can now be bounded as follows:

$$P \leq e^{-d(2k)^{d-2}} \leq 2^{-d(2k)^{d-2}}.$$

Finally, an upper bound on the probability that G has an independent set of size $(2k)^d/k$ is obtained by multiplying the bound on P and the bound on the number of $(2k)^d/k$-element subsets of vertices from Proposition 4.

$$2^{3(2k)^{d-1}\log k} \cdot 2^{-d(2k)^{d-2}} = 2^{(2k)^{d-2}(6k\log k-d)} < 1.$$

□

Acknowledgement

This research was conducted during the DIMACS, DIMATIA and Renyi tripartite workshop "Combinatorial Challenges" held in DIMACS in April 2006; we thank the DIMACS center for its hospitality. We also thank Stephen Hartke and Hemanshu Kaul for participating in lively early discussions at DIMACS about the problem.

References

[1] N. Alon, M. Krivelevich, B. Sudakov: Coloring graphs with sparse neighborhoods, J. Combinatorial Theory Ser. B 77(1) (1999), 73–82.

[2] M. de Berg, M. van Kreveld, M. Overmars, O. Schwarzkopf: Computational geometry, algorithms and applications, Springer-Verlag, Berlin, 2000.

[3] R. L. Brooks: On coloring of the nodes of a network, Proc. Cambridge Phil. Soc. 37 (1941), 194–197.

[4] X. Chen, J. Pach, M. Szegedy, G. Tardos: Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles, in: Proc. 19th annual ACM-SIAM symposium on Discrete algorithms (SODA) 2008, 94–101.

[5] T. M. Cover and J. A. Thomas: Elements of information theory, John Wiley & Sons, 1991.

[6] G. Even, Z. Lotker, D. Ron, S. Smorodinsky: Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks, SIAM J. Comput. 33 (2003), 94–136.

[7] B. Farzad, M. Molloy, B. Reed: $(\Delta - k)$-critical graphs, J. Combinatorial Theory Ser. B **93** (2005), 173–185.

[8] T. Hagerup, Ch. Rüb: A guided tour of Chernoff bounds. Inform. Process. Letters 33 (1989) 305–308.

[9] A. R. Johannson: Asymptotic choice number for triangle-free graphs, manuscript.

[10] M. Molloy, B. Reed: Colouring graphs whose chromatic number is near their maximum degree, in: Proc. 3rd Latin American Symposium on Theoretical Informatics (LATIN) 1998, 216–225.

[11] M. Molloy, B. Reed: Colouring graphs when the number of colours is nearly the maximum degree, in: Proc. 33rd annual ACM symposium on Theory of computing (STOC) 2001, 462–470.

[12] B. Reed: A strengthening of Brooks' theorem, J. Combinatorial Theory Ser. B **76** (1999), 136–149.

[13] M. Szegedy, presentation at open problem session of DIMACS, DIMATIA and Renyi tripartite workshop "Combinatorial Challenges", DIMACS, Rutgers, NJ, April 2006.