LC reductions yield isomorphic simplicial complexes

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Abstract

We say that a vertex v of a finite simplicial complex K is *LC-removable* if the link of v is a cone, and that K is *LC-irreducible* if it has no LC-removable vertices. Answering a question of Civan and Yalçın [*J. Comb. Theory* Ser. A(2007), doi:10.1016/j.jcta.2007.02.001], we prove that all LC-irreducible simplicial complexes that can be obtained from a given K by repeatedly deleting LC-removable vertices (plus all simplices containing them) are isomorphic.

When dealing with questions concerning finite simplicial complexes in computational topology, it is often desirable to first replace the possibly large given simplicial complex by a smaller one, using suitable local reductions that preserve the homotopy type. One type of such reductions was recently introduced by Civan and Yalçın [1] under the name LC-reduction, where the letters L and C stand for *linear coloring*. However, in view of an equivalent characterization (Theorem 6.2 in [1]), we can also read LC-reduction as *link-cone reduction*, and this is the point of view we adopt in the present note.

Let K be a finite simplicial complex with vertex set $V = V(K)$. We recall that the *link* of a simplex $\tau \in K$ is the subcomplex $\mathrm{lk}_K(\tau) = \{\sigma \in$

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$K : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in K\}$ (for $\tau = \{v\}$ we write just $\mathrm{lk}_K(v)$). Further let B be a simplicial complex and v a vertex not belonging to the vertex set of B. The *cone* with apex v and base B is the simplicial complex $v * B = B \cup \{\sigma \cup \{v\} : \sigma \in B\}$.

Let us say that a vertex v is *LC-removable in* K if $\mathrm{lk}_K(v)$ is a cone. We call K *LC-irreducible* if it has no LC-removable vertex.

If v is an LC-removable vertex of K, we say that the simplicial complex $K - v := \{\sigma \in K : v \notin \sigma\}$ is obtained from K by an *elementary LC-reduction*. Finally, K can be *LC-reduced* to L if there is a sequence of elementary LC-reductions transforming K into L.

We recall that two simplicial complexes K and K' are *isomorphic*, in symbols $K \cong K'$, if there is a bijection $\varphi\colon V(K) \to V(K')$ such that both φ and φ^{-1} are simplicial maps, i.e., $\varphi(\sigma) \in K'$ for every $\sigma \in K$ and $\varphi^{-1}(\sigma') \in K$ for every $\sigma' \in K'$.

Here is our main result, answering a question posed in [1].

Theorem 1 *Let K be a finite simplicial complex, and let K_1 and K_2 be LC-irreducible simplicial complexes such that K can be LC-reduced to both K_1 and K_2. Then K_1 and K_2 are isomorphic.*

This theorem allows us to define the *LC-core* of a given simplicial complex in an obvious way. Computing the LC-core may be a relatively fast way of reducing the number of simplices of a simplicial complex while preserving the homotopy type, and so it may be a useful preprocessing step in algorithms for computing topological invariants of simplicial complexes. Properties of LC-cores, fast algorithms for computing them, and perhaps also other notions of reductions are possible subjects for further research.

We begin the proof of Theorem 1 by stating the key lemma (a "diamond" lemma):

Lemma 2 *Let v_1, v_2 be LC-removable vertices in a simplicial complex K. Then there are simplicial complexes K_1' and K_2' such that $K - v_1$ can be LC-reduced to K_1', $K - v_2$ can be LC-reduced to K_2', and $K_1' \cong K_2'$.*

Proof of Theorem 1 assuming the lemma. Let us call K *bad* if it can be reduced to two nonisomorphic LC-irreducible simplicial complexes, and *good* otherwise. Assuming for contradiction that a bad simplicial complex exist, let us fix one with the smallest possible number of vertices and call it K.

Let $K_1 \not\cong K_2$ be LC-irreducible complexes such that K can be LC-reduced to both of them, let v_1 be the first removed vertex in an LC-reduction of K to K_1, and similarly for v_2. We have $v_1 \neq v_2$, for otherwise, $K - v_1$ would be a smaller bad complex.

By the lemma, $K - v_1$ and $K - v_2$ can be LC-reduced to isomorphic complexes K_1' and K_2'. Let us further LC-reduce K_1' to an LC-irreducible K_1'' and K_2' to an LC-irreducible K_2'' so that $K_1'' \cong K_2''$. In more detail, we consider a sequence S_1 of elemantary LC-reductions that LC-reduces K_1' to an LC-irreducible K_1''. Letting $\varphi\colon V(K_1') \to V(K_2')$ be an isomorphism of K_1' and K_2', we construct a sequence S_2 of elemantary LC-reductions of K_2', such that when a vertex v is LC-removed in S_1, the corresponding vertex $\varphi(v)$ is LC-removed in S_2. Then S_2 obviously transforms K_2' into an LC-irreducible complex, which we call K_2''.

Now $K - v_1$ is good and it can be LC-reduced to both K_1 and K_1'', which are both LC-irreducible, so they are isomorphic. Similarly $K_2 \cong K_2''$, and therefore, $K_1 \cong K_2$—a contradiction. $\square$

Proof of Lemma 2. If v_1 is LC-removable in $K - v_2$ and v_2 is LC-removable in $K - v_1$, then we can set $K_1' = (K - v_1) - v_2$ and $K_2' = (K - v_2) - v_1$.

Thus, we may assume that v_2 is not LC-removable in $K - v_1$, say. In this case we show that $K - v_1 \cong K - v_2$ (and hence $K_1' = K - v_1$, $K_2' = K - v_2$ will do). Actually we show that the mapping of $V(K) \setminus \{v_1\}$ onto $V(K) \setminus \{v_2\}$ sending v_2 to v_1 and leaving all other vertices in place is an isomorphism. This amounts to verifying that for every subset $\sigma \subseteq V(K) \setminus \{v_1, v_2\}$, we have $\sigma \cup \{v_1\} \in K$ if and only if $\sigma \cup \{v_2\} \in K$, or in other words, that $\mathrm{lk}_{K-v_1}(v_2) = \mathrm{lk}_{K-v_2}(v_1)$.

Since v_2 is LC-removable in K but not in $K - v_1$, $\mathrm{lk}_K(v_2)$ is a cone while $B_1 := \mathrm{lk}_{K-v_1}(v_2) = \mathrm{lk}_K(v_2) - v_1$ is not a cone. Since removing any vertex different from the apex from a cone again results in a cone, we have $\mathrm{lk}_K(v_2) = v_1 * B_1$.

Our goal is to show that $B_1 = \mathrm{lk}_{K-v_2}(v_1)$. If $\sigma \in B_1$, then $\sigma \cup \{v_1\} \in v_1 * B_1 = \mathrm{lk}_K(v_2) \subseteq K$. Thus $\sigma \in \mathrm{lk}_{K-v_2}(v_1)$ and we have the inclusion $B_1 \subseteq \mathrm{lk}_{K-v_2}(v_1)$.

If $\mathrm{lk}_K(v_1)$ were a cone with apex v_2, then a symmetric argument would show that $B_1 \supseteq \mathrm{lk}_{K-v_2}(v_1)$ as required. Hence we may assume that $\mathrm{lk}_K(v_1)$ is a cone with an apex $x \neq v_2$.

Since B_1 is not a cone, in particular it is not a cone with apex x, and hence there exists $\tau \in B_1$ with $\tau \cup \{x\} \notin B_1$ (possibly $\tau = \emptyset$). Then

$\tau \cup \{x, v_1, v_2\} \notin K$ (by the definition of B_1), while $\tau \cup \{v_1, v_2\} \in K$ (since $\tau \in B_1$, we have $\tau \cup \{v_1\} \in v_1 * B_1 = \mathrm{lk}_K(v_2)$, and so $\tau \cup \{v_1, v_2\} \in K$ as asserted).

Setting $\sigma := \tau \cup \{v_2\}$, we have $\sigma \in \mathrm{lk}_K(v_1)$, while $\sigma \cup \{x\} \notin \mathrm{lk}_K(v_1)$. This contradicts the assumption that $\mathrm{lk}_K(v_1)$ is a cone with apex x, thus finishing the proof of the lemma. □

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References

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