Workshop on Frequency Assignment Problems

Sádek (Třebíč), September 23–27, 2007

organized by DIMATIA and ITI, Charles University

Basic information

The field of frequency assignment problems is an intensively studied research area. The task is to assign radio frequencies to transmitters in wireless networks without causing interference. These problems are usually combinatorially hard and the chance to reach an optimal solution is beyond the scope of today computational resources. On the other hand, depending on the topology of the network (GSM, ad-hoc sensor network, etc.), on the underlying technology (e.g., analog vs digital data) and other parameters of the problem, various strategies may be designed to obtain a reasonable and applicable solution.

The interest for this class of problems has grown up in the last decade, because of the rapid development of new wireless services. Like with all scarcely available resources, the cost of frequency-use provides the need for economic use of the available frequencies. Reuse of frequencies within a wireless communication network can offer considerable economies. However, reuse of frequencies may also lead to a loss of quality of communication links. The use of (almost) the same frequency for multiple wireless connections can cause an interference between the signals that is unacceptable. The frequency assignment problem balances the economies of reuse of frequencies and the loss of quality in the network.

There is considerable literature devoted to the study of the frequency assignment problems, following many different approaches, including graph theory and combinatorics, simulated annealing, genetic algorithms, tabu search, and neural networks.

When preparing the second international workshop on FAP, we have not intended to cover contributions in all these areas. In particular, we have aimed at bringing together scientists, engineers and practitioners of the field in order to have a forum for sharing and exchanging their experience, for discussing fundamental challenges, reporting state-of-the-art and in-progress research, identifying future perspectives, and exchanging ideas about ongoing research to keep up with the rapid evolution and increasing complexity of such systems.

This workshop continued the tradition that had started at the first workshop held in Certosa di Pontignano (Siena), Italy in October 2005. Apart of the scientific topic, one more link kept us connected with the preceding workshop. Recalling the rich taste of beautiful Montepulciano wines we got to taste in Siena, we have organized our workshop in a winery Sádek, the Northernmost vineyard of the South Moravian Znojmo region, known mostly for excellent white wines. The participants got not only a chance to taste some of the Moravian wines, but could walk through the vineyard anytime and feast both visually and gastronomically on the ripening grapes of different sorts. You can witness this on the official workshop photo.

The workshop started with a problem session on Sunday evening. To allow ample time to solving these (and other) problems and to informal working discussions, we have invited just three keynote lecturers and did not plan any other talks. J. Griggs presented a survey of the state-of-the-art of the real valued frequency assignment problem. A. Eisenblatter presented a thorough exposition of the current approaches and challenges from the applied point of view. And J. S. Sereni gave as detailed as possible overview of the recent solution of the so called Delta-squared conjecture for $L(2,1)$-labelings. Thus every day enough time was reserved for discussing fundamental challenges, future perspectives, and new ideas, allowing people from different research areas to communicate and interact. The present booklet brings abstracts of the keynote talks, summarizes the open problems posed during the inaugural problem session, and provides progress reports about the research carried on during the workshop.

We hope the participants enjoyed the event at least as much as the organizers, and we all look forward to the future workshops in this series.

Jiří Fiala, Dan Král' and Jan Kratochvíl

Program

September 23 (Sunday)

16:00	Transfer from Prague to Sádek
18:30 – 19:30	Problem session
19:30	Welcome party

September 24 (Monday)

8:00 – 9:00	Breakfast
9:00 – 11:00	Jerrold Griggs: *Real Number Graph Labellings with Separation Conditions*
11:00 – 11:30	Coffee break
11:30 – 13:00	Problem solving
13:00 – 14:00	Lunch
14:00 – 15:00	Guided tour through the vineyard
15:00 – 18:00	Problem solving
16:00 – 16:30	Coffee break
16:30 – 18:00	Problem solving
18:30	Dinner

September 25 (Tuesday)

8:00 – 9:00	Breakfast
9:00 – 11:00	Andreas Eisenblatter: *Automatic Frequency Planning in GSM networks* *How to replenish the spectrum*
11:00 – 11:30	Coffee break
11:30 – 13:00	Problem solving
13:00 – 14:00	Lunch
14:00	Departure for the conference trip to Telč
15:00 – 16:00	Excursion in Telč castle
18:30	Dinner in Telč
19:30	Wine tasting in Louka monastery in Znojmo

September 26 (Wednesday)

8:00 – 9:00	Breakfast
9:00 – 11:00	Jean-Sebastian Sereni: *Griggs and Yeh's Conjecture and the Probabilistic Method*
11:00 – 11:30	Coffee break
11:30 – 13:00	Problem solving
13:00 – 14:00	Lunch
14:00 – 18:00	Problem solving
16:00 – 16:30	Coffee break
16:30 – 18:00	Problem solving
18:30	Dinner

September 27 (Thursday)

8:00 – 9:00	Breakfast
9:00 – 11:00	Problem solving
11:00 – 11:30	Coffee break
11:30 – 13:00	Problem solving and progress reports
13:00 – 14:00	Lunch
14:30	Transfer to Prague

The conference photo taken in the vineyard of the conference center.

Abstracts of the keynote talks

Minimum Span Real Number Graph Labellings with Separation Conditions

Jerrold Griggs, University of South Carolina, Columbia, SC USA

In 1988 Roberts described a problem posed to him by Lanfear concerning the efficient assignment of channels to a network of transmitters in the plane. To understand this problem, Griggs and Yeh introduced the theory of integer vertex λ-labellings of a graph G. To prevent interference, labels for nearby vertices must be separated by specified amounts k_i depending on the distance i, $1 \leq i \leq p$. One seeks the minimum span of such a labelling. The $p = 2$ case with $k_1 = 2$ and $k_2 = 1$ has attracted the most attention, particularly the tantalizing conjecture that if G has maximum degree $\Delta \geq 2$, then the minimum span is at most Δ^2.

In order to gain more insights for general k_i, it is natural to expand the model to allow real number labels and separations, as well as infinite graphs with $\Delta < \infty$. Griggs and Jin showed that in this case there is a labelling of minimum span in which all of the labels have the form $\sum_{i=1}^{p} a_i k_i$, where the a_i's are integers ≥ 0. Moreover, they conjectured that the minimum span as a function of the separations k_i is piecewise linear with finitely many pieces.

Babilon et al. introduced λ-graphs, in which every edge has weight k_i for some i, where reals $k_1, \ldots, k_p \geq 0$ are fixed. This is a more general model, which better describes the interference restrictions for an irregular transmitter network in the plane. Král' proved the Piecewise Linearity Conjecture in this more general setting. Networks used in practice often correspond to regular infinite lattices in the plane, and with two levels of interference, the λ-labelling model with $p = 2$ is appropriate. Griggs and Jin determined the minimum span of such labellings for general k_1, k_2 for the square and hexagonal lattices. They also solved the triangular lattice for $k_1 \geq k_2$, and Král' and Škoda recently completed the remaining cases when $k_1 < k_2$.

Our lecture is intended as an overview of this labelling project, with an emphasis on directions for future research.

Automatic Frequency Planning in GSM networks: How to replenish the spectrum

Andreas Eisenblaetter, ZIB FU Berlin, Germany

The traditional model of (extended) graph coloring or graph k-partitioning for GSM frequency assignment uses a rather coarse notion of interference. Potential interference is given for pairs of cells (i.e., the transmitters emitting the cells' signals), and interference occurs in case transmitters from two cells receive the same or directly adjacent frequencies. Depending on the interference threshold used in the computation of potential interference, this approach systematically under- or over-estimates interference. In the live network, however, harmful interference occurs in case the ratio between the strength of the serving cell's signal (carrier) and the sum of all interferencing signals (carrier-to-interference ratio) falls below a certain threshold. Relevant interference may originate from the use of the same frequency as well as from adjacent frequencies up to a distance of three.

This talk addresses the difference between interference accounting using the traditional and using a more realistic system model. The more refined accounting model is used as the basis of an alternative mathematical optimization model that aims at minimizing interference in cases of a significant shortage of available frequencies (by the standards of the traditional model). The refined model is also suited to optimize frequency allocation in the presence of (slow) frequency hopping. Computational results illustrate the difference in the complexity as well as in the capabilities of the two approaches.

Griggs and Yeh's Conjecture and the Probabilistic Method

Jean-Sebastian Sereni, Charles University, Prague, Czech Republic

In 1992, Griggs and Yeh introduced the definition of an $L(p,q)$-labelling of a graph to model a channel assignment problem. Formally, given a graph $G = (V, E)$, an $L(p,q)$-*labelling* is a mapping $f : V \to \mathbb{N}^*$ such that (i) $|f(x) - f(y)| \geq p$ whenever $xy \in E$; and (ii) $|f(x) - f(y)| \geq q$ whenever $\mathrm{dist}(x, y) = 2$ where dist is the distance in the graph G. The $\lambda_{p,q}$-*number* $\lambda_{p,q}(G)$ of G is the smallest k such that G admits an $L(p,q)$-labelling into $\{1, 2, \ldots, k\}$.

The case where $p = 2$ and $q = 1$ attracted the most attention, due to the following conjecture of Griggs and Yeh: if G has maximum degree Δ, then $\lambda_{2,1}(G) \leq \Delta^2 + 1$. We will see that the following approximate version of this conjecture is true: *There exists a constant C such that $\lambda_{2,1}(G) \leq \Delta^2 + C$ for every graph G of maximum degree Δ.* To this end, we will use the probabilistic method to prove that Griggs and Yeh's conjecture holds for Δ large enough. We will actually consider a slightly more general setting, which is a bit more flexible regarding the channel assignment problem. This result generalizes to the $\lambda_{p,1}$-number for every positive integer p (with C depending on p).

This talk is not an introduction to the probabilistic method. However, its goal — more than proving the result — is to show how to create the right environment to be able to use the probabilistic method in our setting, with the hope that some may find it useful for other problems. Some related open questions will be given. This is joint work with Frédéric Havet and Bruce Reed.

Open problems suggested by the workshop participants

Kelly-width 3 problem

Given a digraph G, decide if it has Kelly-width at most 3. Is this problem solvable in polynomial-time?

Kelly-width has been proposed by Kreutzer and Hunter as a natural generalization of tree-width to directed graphs. There are several equivalent characterizations. The simplest is maybe the following: G has Kelly-width at most $k+1$ iff G can be reduced to the trivial graph by iteratively reducing by a vertex of out-degree at most k, where reducing a vertex v implies adding edges from all in-neighbors of v to all out-neighbors of v before removing v. The digraphs of Kelly-width 1 are precisely the directed acyclic graphs. At WG'07 an efficient algorithm was given to decide if a digraph had Kelly-width at most 2.
(*submitted by Martin Vatshelle*)

Our favorite FAP problem concerns $L(h, k)$ coloring of particular classes of graphs.

Given two non negative integers h and k, an $L(h, k)$-labeling of a graph $G = (V, E)$ is a map from V to a set of labels such that adjacent vertices receive labels at least h apart, while vertices at distance at most 2 receive labels at least k apart. The goal of the $L(h, k)$-labeling problem is to produce a legal labeling that minimizes the largest label used. Since the decision version of the $L(h, k)$ -labeling problem is NP-complete, it is important to investigate classes of graphs for which the problem can be solved efficiently.

Along this line of though, we are investigating superclasses of classes for which polynomial solution are known. Currently we are trying to extend the solutions known for the subclass of matrogenic graphs to the super class of unigraphs, i.e. graphs univocally determined by means of their own degree sequence. Another interesting class are co-comparability graphs. In the literature there are no results concerning the $L(h, k)$-labeling of general co-comparability graphs; it is neither known whether the problem remains NP-complete when restricted to this class. Since, with S. Olariu and S. Caminiti, we provide, in a constructive way, the first upper bounds on the $L(h, k)$-number of co-comparability graphs, we are interested to investigate what is

possible to say about the $L(h, k)$-labeling of this class of graphs. Since their $L(1, 1)$-labeling is polynomially solvable as they are perfect and the square of a comparability graph is still a comparability graph, the first answer is : does the $L(2, 1)$-labeling remain polynomially solvable?
(*submitted by Tiziana Calamoneri and Rossella Petreschi*)

On-line coloring of graphs

Intuitively, an on-line coloring algorithm properly colors the vertices of a graph one by one, consistently using a fixed strategy, depending only on the subgraph induced by the revealed vertices and the colors that have been assigned to them by the algorithm, according to an externally determined ordering of the presented vertices. See [H.A. Kierstead, *Coloring graphs on-line*, in: Online algorithms: the state of the art, Lecture Notes in Computer Science 1442 (1998) , Springer Verlag, 281–305] for more details.

We denote the (finite) set of all on-line coloring algorithms for a graph G by $AOL(G)$. Let $\Pi(G)$ denote the set of all permutations of the vertices of G. If $A \in AOL(G)$ and $\pi \in \Pi(G)$, we denote by $\chi_A(G, \pi)$ the number of colors used by A when the vertices of G are presented according to π. The largest number of colors used by the on-line algorithm A for G is called the A-*chromatic number* of G and denoted by $\chi_A(G)$. Hence

$$\chi_A(G) = \max_{\pi \in \Pi(G)} \chi_A(G, \pi).$$

The smallest number of colors used by an on-line algorithm for G is the *on-line chromatic number* of G, and denoted by $\chi_{OL}(G)$. Hence

$$\chi_{OL}(G) = \min_{A \in AOL(G)} \chi_A(G).$$

Let $\mathcal{G}$ denote a (possibly infinite) family of graphs. If $A \in AOL(G)$ for every $G \in \mathcal{G}$, we say that A is an on-line coloring algorithm for $\mathcal{G}$ and write $A \in AOL(\mathcal{G})$. An algorithm $A \in AOL(\mathcal{G})$ is said to be *competitive* for $\mathcal{G}$ if there exists a function f such that $\chi_A(G) \leq f(\chi(G))$ for every $G \in \mathcal{G}$. One of the main open problems concerning on-line competitive coloring algorithms [A. Gyárfás, Z. Király, J. Lehel, *On-line competitive coloring algorithms*, Technical report TR-9703-1 (1997)] is to decide whether for every k there exists a competitive coloring algorithm for the family of graphs with on-line chromatic number k. This is even open for bipartite graphs for $k = 4$, whereas it has been solved for general graphs for $k \leq 3$ [A. Gyárfás, Z. Király, J. Lehel, *On-line 3-chromatic graphs. II. Critical graphs*, Discrete Mathematics 177 (1997), 99–122] and [K. Kolossa, *On the on-line chromatic number of the family of on-line 3-chromatic graphs*, Discrete Mathematics 150 (1996), 205–230]. It is proven that for the family of graphs with on-line chromatic number 3 at most 4 colors are needed.

There are many families of graphs for which it has been proved that no competitive algorithms exist: two examples given in [A. Gyárfás, J. Lehel, *On-line and first-fit colorings of graphs*, Journal of Graph Theory 12 (1988), 217–227]. are the family of trees and the family of P_6-free bipartite graphs. These negative results have led to the definition of a weaker form of competitiveness in [A. Gyárfás, Z. Király, J. Lehel, *On-line competitive coloring algorithms*, Technical report TR-9703-1 (1997)], namely on-line competitiveness, although results of this type have been obtained before the term was formally introduced. An algorithm $A \in AOL(\mathcal{G})$ is *on-line competitive* for a family $\mathcal{G}$ if $\chi_A(G) \leq f(\chi_{OL}(G))$ for every $G \in \mathcal{G}$. It is shown in [A. Gyárfás, J. Lehel, *First fit and on-line chromatic number of families of graphs*, Ars Combinatorica 29C (1990), 168–176] that FF is on-line competitive for trees; it is even optimal for trees, in the sense that if FF uses k colors, then the on-line chromatic number of the tree is also k. In [H.J. Broersma, A. Capponi, D. Paulusma, *A new algorithm for on-line coloring bipartite graphs*, to appear in Siam Journal on Discrete Mathematics] an on-line competitive algorithm is given for the class of P_7-free bipartite graphs. The *girth* of a graph G is the number of edges of a smallest cycle in G. If G has girth at least five, then *First-Fit* (FF) uses at most $\chi_{FF}(G) \leq \binom{2^{\chi_{OL}(G)}}{2}$ colors [A. Gyárfás, Z. Király, J. Lehel, *On-line competitive coloring algorithms*, Technical report TR-9703-1 (1997)]. Open problems are:

- Does there exists a class of graphs for which there is no on-line competitive algorithm?
- Does there exist an on-line competitive algorithm for the class of H-free bipartite graphs for $H = C_6$ or $H = P_8$?
- Does there exist an on-line competitive algorithm for the class of graphs with girth four? Can we find a better algorithm than FF for graphs with girth at least five?

(*submitted by Daniël Paulusma*)

Algorithmic aspects of the packing chromatic number

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that the vertex set of G can be partitioned into packings with pairwise different widths. What is the computational complexity of the problem on trees, more precisely, whether the problem is polynomial? More generally, what is the complexity on graphs with bounded tree-width? The question about trees is due to Goddard et al. [W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris, D.F. Rall, *Broadcast chromatic numbers of graphs*, Ars Combin., in press.].
(*submitted by Sandi Klavžar*)

Transforming colorings by recolorings

The problem of reassigning frequencies to the nodes in a network can sometimes be modelled using recolouring of graphs. Here is a simple graph recolouring model: A *k-colouring* of a graph G is a proper vertex colouring using colours $\{1, \ldots, k\}$. If α is a proper k-colouring of G, then a *recolouring* of α is a new k-colouring β that differs in exactly one vertex from α. Given a graph G, the number of colours k, and two k-colourings α and β, many questions can be asked:

1. Is it possible to go from α to β via a sequence of recolourings?
2. If it is possible, how many steps are required?
3. Given α, what k-colourings can be obtained from α via a sequence of recolourings?
4. Etc.

Some of these questions have been asked before, and many (partial) answers are known. Here is an intriguing open problem.

Let the *degeneracy* $\deg(G)$ (also known as *maximin degree* or *colouring number*) be the maximum of the minimum degree of the subgraphs of G: $\deg(G) = \max_{H \leq G} \delta(H)$. It is not so hard to prove that if $k \geq \deg(G) + 2$, then any two k-colourings can be obtained from one another. But how many recolourings would be needed?

Conjecture. *Let G be a graph, $k \geq \deg(G) + 2$, and α and β be two k-colourings of G. Then it is possible to go from α to β using a number of recolourings that is polynomial in $|V(G)|$.*

The conjecture is known to be true for $k \geq 2\deg(G) + 1$ (in that case $O(|V(G)|^2)$ steps are always sufficient). If true, the result would also be best possible: for all $k \geq 4$, there exist infinitely many graphs G with $\deg(G) + 1 = k$ and two k-colourings α, β of G, so that to recolour α to β takes a number of steps that is superpolynomial in $|V(G)|$.
(*submitted by Jan van den Heuvel*)

Complexity of channel assignment problem for graphs of tree-width two

Let G be a weighted graph with positive edge weights, and λ be a positive integer. The *Channel assignment problem* asks about existence of channel assignment function $\phi: V(G) \to \{0, 1, \ldots, \lambda\}$ such that $|\phi(x) - \phi(y)| \geq l(xy)$ for every edge $xy \in E(G)$, where $l(xy)$ is the weight of xy.

McDiarmid and Reed (*Channel assignment on graphs of bounded treewidth*, Discrete Math., 273 (2003), 183–192) proved that the channel assignment problem is NP-complete for graphs of tree-width 3 (path-width 4). Since for bipartite graphs this problem has a trivial solution, the only case for which complexity is unknown is the case of tree-width two graphs.

For some special cases of tree-width two graphs the problem can be solved polynomially. Recall that graph is called *cactus* if every edge belongs to no more than one cycle. For cacti the channel assignment problem can be solved polynomially by dynamic programming.

It can be noted also that for the circular metric it is possible to give exact boundary between polynomial and NP-complete cases. The *Circular channel assignment problem* asks about existence of channel assignment function $\psi: V(G) \to \{0, 1, \ldots, \lambda\}$ such that $\min\{|\psi(x) - \psi(y)|, \lambda - |\psi(x) - \psi(y)|\} \geq l(xy)$ for every edge $xy \in E(G)$. This problem can be solved polynomially for trees (path-widths 2 graph), and it becomes NP-complete for graphs of tree-width two (path-width 3). For cacti this problem also can be solved polynomially.
(*submitted by Petr A. Golovach*)

Backbone coloring of graphs

Let $G = V, E$ be a simple connected graph, and let $H = (V, E_H)$ be a spanning subgraph of G. A vertex coloring $f : V \to \{1, 2, \ldots\}$ of G is proper if and only if $|f(x) - f(y)| \geq 1$ for every edge $xy \in E$. A vertex coloring f is a *backbone coloring* for (G, H) if it is proper and additionally $|f(x) - f(y)| \geq 2$ for every edge $xy \in E_H$. The backbone coloring number $\textsc{bbc}(G, H)$ of (G, H) is the smallest ℓ such that there exists a backbone coloring $f : V \to \{1, \ldots, \ell\}$.

The notion of backbone coloring was introduced by Broersma, Fomin, Golovach, and Woeginger in [H. J. Broersma, F. V. Fomin, P. A. Golovach, G. J. Woeginger, *Backbone colorings for networks*, in: Proceedings of the 29th International Workshop on Graph-Theoretic Concepts in Computer Science (WG), LNCS 2880 (2003) 131–142]. It models the frequency assignment problem in a network of transmitters, where a certain substructure of the network (the backbone) is more important than the rest of the network.

If we take a proper coloring $c : V \to \{1, \ldots, k\}$ of G and replace every color i by $2i - 1$, we obtain a backbone coloring, since every edge $xy \in E$ verifies $|f(x)-f(y)| \geq 2$. So, for any subgraph H of G, we have $\textsc{bbc}(G, H) \leq 2\chi(G)-1$, where $\chi(G)$ is the chromatic number of G. In [H. J. Broersma, F. V. Fomin, P. A. Golovach, G. J. Woeginger, *Backbone Colorings for Graphs: Tree and Path Backbones*, J. Graph Theory 55 (2007), 137–152], the authors consider the case where H is a spanning tree of G. They prove that for every k, there exists a graph G with chromatic number k and a spanning tree T of G such that $\textsc{bbc}(G, T) = 2k - 1$. They also give lower and upper bounds when T is a Hamiltonian path of G (in which case $\textsc{bbc}(G, T) \leq \frac{3k}{2} + O(1)$).

These results on paths do not seem to be very hard to extend to spanning trees with maximum degree bounded by d : we can prove that in this case, $\textsc{bbc}(G, T) \leq 2k \cdot \frac{d+1}{d+2} + C$, where k is again the chromatic number of G, and C is some small constant. Moreover, for every k and d, we can construct graphs and spanning trees for which this bound is achieved (joint work with F. Huc).

In [H. J. Broersma, F. V. Fomin, P. A. Golovach, G. J. Woeginger, *Backbone Colorings for Graphs: Tree and Path Backbones*, J. Graph Theory 55 (2007), 137–152], the authors prove that given a graph G and a spanning tree T, deciding whether $\textsc{bbc}(G, T) \leq \ell$ is polynomial if $\ell \leq 4$, and NP-complete if $\ell \geq 5$ and T is a Hamiltonian path of G. They conclude with some nice open problems:

- It can be shown that $\textsc{bbc}(G, P) \leq \chi(G) + 4$ when P is a hamiltonian path of G and G is chordal. Can we prove that there exists a constant c such that for any chordal graph G and any spanning tree T of G, we have $\textsc{bbc}(G, T) \leq \chi(G) + c$?
- Same question if G is triangle-free instead of being chordal.
- The Four-Color Theorem implies that $\textsc{bbc}(G, T) \leq 7$ whenever G

is planar, and we can construct examples verifying $\textsc{bbc}(G, P) = 6$, where P is a hamiltonian path of G. Can 7 be improved to 6? Can 7 be proved easily without the Four-Color Theorem?

- Can we still obtain tight results if we replace 2 by some λ in the definition of backbone coloring? This question has been investigated in [H. J. Broersma, J. Fujisawa, L. Marchal, D. Paulusma, A. N. M. Salman, K. Yoshimoto, λ-*Backbone colorings along pairwise disjoint stars and matchings*, submitted (2006)] in the case where T is a matching or a collection of disjoint stars.

We can add a few more questions:

- Given a graph G, find a necessary or sufficient (non trivial) condition on H so that $\textsc{bbc}(G, H) = 2\chi(G) - 1$.
- Can the results of Broersma et al. be extended to the case where H is a (spanning) partial k-tree? a planar graph?

(*submitted by Louis Esperet*)

Coloring with distance requirements

Given a simple graph $G = (V, E)$, a set of colors C, a distance function between colors $d(x, y) \to N_+$, $x, y \in C$, $d(x, y) = d(y, x)$ and $d(x, y) = 0$ implies $x = y$, a distance requirement $r(v, w)$ for each edge (v, w) from E, find an assignment $c(v)$ in C for every node v from V such that for every edge $(v, w) \in E$, $d(c(v), c(w)) \geq r(v, w)$.
(*submitted by Andreas Eisenblätter*)

Algorithmic aspects of $L(2, 1)$-labellings for planar graphs

What is the computational complexity of deciding $L_{2,1}(G) \leq 4$ (or $\leq k$ for a constant $k < 8$) for planar inputs G?
(*submitted by Jan Kratochvíl*)

Reports on the progress in solving some of the problems during the workshop

The packing chromatic number of some infinite product graphs

For a graph G and a positive integer d, a subset X of vertices of G is a *d-packing* if vertices of X are pairwise at distance more than d. The integer d is called the *width* of the packing X. Note that packings of width one are precisely the independent sets of G. Clearly, a packing of width d is also a packing of width $d-1$.

The *packing chromatic number* $\chi_\rho(G)$ of a graph G is the smallest integer k such that $V(G)$ can be partitioned into k packings $X_1, \ldots, X_k$ with pairwise different widths. Note that we can assume that X_i is an i-packing for each i.

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. The 2-way infinite path will be denoted by P_∞.

The following interesting phenomena was the starting point for our investigations. The packing chromatic number of the 2-dimensional integer lattice ($P_\infty \,\square\, P_\infty$) is finite, more precisely, it lies between 9 and 22; see [W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris and D.F. Rall, *Broadcast chromatic numbers of graphs*, Ars Combin., in press]. On the other hand, the packing chromatic number of the 3-dimensional integer lattice ($P_\infty \,\square\, P_\infty \,\square\, P_\infty$) is not bounded as proved in [A. Finbow and D.F. Rall, *On the packing chromatic number of some infinite graphs*, manuscript, 2007]. So where does a step from a finite number to the infinity occur?

We showed that graph from three copies of the square lattice has already unbounded packing chromatic number.

Theorem 1 *For any $m \geq 3$, $\chi_\rho(P_m \,\square\, P_\infty \,\square\, P_\infty) = \infty$.*

Moreover we observed that the Cartesian product of any finite graph and P_∞ has bounded packing chromatic number.

Theorem 2 *For any $n \geq 1$, $\chi_\rho(K_n \boxtimes P_\infty) < 4^n$.*

Corollary 1 *For any finite graph G, $\chi_\rho(G \square P_\infty) < \infty$.*

We were able to significantly improve an infinity bound from 3-dimensional integer lattice to just three slices of square lattice. But case of two slices of square lattice ($P_2 \square P_\infty \square P_\infty$) remains open. We think that there is no upper bound on two slices of square lattice but we do not have any proof yet.
(*submitted by Jiří Fiala, Sandi Klavžar and Bernard Lidický*)

$L(2,1)$-labeling of the random graph

During the workshop we studied the minimum span $\lambda_{2,1}(\mathbf{G}_{n,p})$ of L(2,1)-labelings of the random graph $\mathbf{G}_{n,p}$ with n vertices and edge probability p. It turns out that the behavior of $\lambda_{2,1}(\mathbf{G}_{n,p})$ is different in the various ranges of $p = p(n)$. For example, if p and $1 - p$ exceed the threshold probability for diameter two and for hamiltonicity, respectively, then $\lambda_{2,1}(\mathbf{G}_{n,p})$ almost surely is equal to $n-1$; this is the expected clique number of $(\mathbf{G}_{n,p})^2$ minus 1, and can also be written as $n - 2$ plus the linear arboricity of $\mathbf{G}_{n,1-p}$. In fact, the latter formula remains valid if we drop the condition on $1-p$, and also if p is very close to zero, but not if p is 'in between', e.g. when $p = n^{-c}$ for a suitably chosen constant c.
(*submitted by Jerrold Griggs, Jean-Sebastian Sereni and Zsolt Tuza*)

Recoloring of list colorings

We consider the algorithmic problem whose input is a graph $G = (V, E)$ with two of its proper colorings α, β, and the question is whether there exists a transformation from α to β such that just one vertex is recolored at a time and a proper coloring is constructed in each step. It is known from earlier results that for 4-colorings this problem is PSPACE-complete, while for 3-colorings it is solvable in polynomial time.

We initiate the analogous problem for list colorings; that is, a list L_v of allowed colors is given for each vertex v, and colorings φ are assumed to satisfy $\varphi(v) \in L_v$ for all $v \in V$ and $\varphi(u) \neq \varphi(v)$ for all $uv \in E$ in every step of the transformation. During the workshop we proved that this list coloring version is much harder than the original problem without lists: it is PSPACE-complete already for lists of size three.
(*submitted by Jan van den Heuvel and Zsolt Tuza*)

$L_{2,1}$ labelling of planar graphs

We looked into the problem of finding an $L_{2,1}$ labelling of a planar graph using labels 0,1,2,3,4. We call it $L_{2,1}[4]$ labelling. The aim is to construct a polytime algorithm. It is known to be polynomial for trees, so we looked into cycles. We found some interesting reduction-rules.

Let E be the set of all proper $L_{2,1}[4]$ labellings of an edge. Note that if there is a solution which labels an edge (a, b) then there is also a solution labelling that edge $(4 - a, 4 - b)$; we call this the dual labelling.

Theorem 3 *For every $S \subseteq E$, where a labelling is in S iff the dual labelling is in S, we can construct a tree T with an edge (u, v) such that:*

1. *v is a leaf of T*
2. *for every $(a, b) \in E$, T has an $L_{2,1}[4]$ labelling such that (u, v) is labeled (a, b), if and only if $(a, b) \in S$.*

Let G be a planar graph. We found the following interesting property. If G has a face where the cycle C bounding that face has length at most 6, then we can construct a graph G' with fewer faces and a constant number of new vertices such that G has a $L_{2,1}[4]$ labelling iff G' has an $L_{2,1}[4]$ labelling.

Short cycles can be removed, but can we make long cycles shorter?
Since every planar graph has a vertex of degree at most 5 we know that there is a set of at most 5 edges cutting the graph in two parts S and L such that S has an unique cycle. There are exactly 2 paths in S between any 2 vertices adjacent to a cutting edge. We now consider a pair of cutting edges with a path which is adjacent to no other cutting edge. This path has bounded length. The easy upper limit is 2^{12}, namely all subsets of proper labellings of an edge. If two identical color-configurations occur on the path we can contract the whole part between them. We tried to find better bounds, of course there exists some, but no good results were found.

We can get rid of small faces and we can make large faces smaller, so all graphs have a face of bounded length. Even if there exist reduction rules for all faces of bounded length, we are not sure we could find them and it certainly will not lead to a simple and practical algorithm. The fact that we can construct trees which enforce any combination of labels on an edge could be really interesting if it could be extended to force any labelling on a set of edges.
(*submitted by Martin Vatshelle, Jozef Miškuf and Tomáš Vyskočil*)