

# 6-critical graphs on the Klein bottle

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## Abstract

We provide a complete list of 6-critical graphs that can be embedded on the Klein bottle settling a problem of Thomassen [Problem 3, J. Comb. Theory Ser. B 70 (1997), 67–100]. The list consists of nine non-isomorphic graphs which have altogether 18 non-isomorphic 2-cell embeddings and one embedding that is not 2-cell.

## 1 Introduction

We study colorings of graphs embedded on surfaces. It is well-known [12] that the chromatic number of a graph embedded on a surface of Euler genus  $g$  is bounded by the Heawood number  $H(g) = \left\lfloor \frac{7+\sqrt{24g+1}}{2} \right\rfloor$ . Dirac Map Color Theorem [4, 5] asserts that a graph  $G$  embedded on a surface of Euler genus  $g \neq 0, 2$  is  $(H(g) - 1)$ -colorable unless  $G$  contains a complete graph of order  $H(g)$  as a subgraph. Dirac's theorem can be rephrased using

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the language of critical graphs as follows: the only  $H(g)$ -critical graph that can be embedded on a surface of Euler genus  $g \neq 0, 2$  is the complete graph of order  $H(g)$ . Recall that a graph  $G$  is  $k$ -critical if  $G$  is  $k$ -chromatic and every proper subgraph of  $G$  is  $(k - 1)$ -colorable.

In fact, Dirac [4] showed that there are only finitely many  $k$ -critical graphs,  $k \geq 8$ , that can be embedded on a fixed surface. The number of 7-critical graphs that can be embedded on a fixed surface is also finite by classical results of Gallai [10, 11] as pointed out by Thomassen in [15]. Later, Thomassen [17] established that the number of 6-critical graphs that can be embedded on any fixed (orientable or non-orientable) surface is finite (see also [9] for related results on 7-critical graphs). This result is best possible as there are infinitely many  $k$ -critical graphs,  $3 \leq k \leq 5$ , that can be embedded on any fixed surface different from the plane [8].

In this paper, we focus on 6-critical graphs on surfaces, motivated by Problem 3 from [17]. As every plane graph is 4-colorable [1, 2, 13], there are no 6-critical graphs in the plane. Dirac Map Color Theorem implies that the complete graph of order six is the only 6-critical projective planar graph. Thomassen [15] classified 6-critical toroidal graphs: the only 6-critical graphs that can be embedded on the torus are the complete graph  $K_6$ , the join of the cycles  $C_3$  and  $C_5$  (recall that the join of two graphs  $G_1$  and  $G_2$  is the graph obtained by adding all edges between  $G_1$  and  $G_2$ ), the graph obtained by applying Hajos' construction to two copies of  $K_4$  and then by adding  $K_2$  joined to all other vertices, and the third distance power of the cycle  $C_{11}$  (which is further denoted by  $T_{11}$ ). Thomassen posed as a problem [15, Problem 3] whether the toroidal 6-critical graphs distinct from  $T_{11}$  and the graph obtained by applying Hajos's to two copies of  $K_6$  are the only 6-critical graphs that can be embedded on the Klein bottle.

We refute Thomassen's conjecture by exhibiting the list of all nine 6-critical graphs that can be embedded on the Klein bottle (the graphs are depicted in Figure 7). The same result was independently of us established by Chenette, Postle, Streib, Thomas and Yerger [3]. Both the proofs use different approaches (and we thus agreed to publish two separate papers): Chenette et al. investigated the structure of 6-critical graphs on the Klein bottle, i.e., found 6-critical graphs that could possibly be embedded on the Klein bottle and ruled out those that cannot. Our approach is based on a systematic generating of all embeddings of 6-critical graphs on the Klein bottle from the complete graph  $K_6$ . Our proof is computer-assisted (unlike the proof of Chenette et al.) but it additionally yields the list of all non-isomorphic embeddings of 6-critical graphs on the Klein bottle. We

believe that our proof can be decomputerized (in the expense of massive case analysis) but we decided not to do so in the light of the proof of Chenette et al. which is significantly shorter than our decomputerized proof would be.

As we have mentioned, our proof is computer-assisted. In the paper, we outline the main concepts we use and explain the procedure used to generate all embeddings of 6-critical graphs on the Klein bottle. In order to verify the correctness of our programs, we have separately prepared two different programs implementing our procedures and compared their outputs. Further details of the implementation and the source code of our programs can be found at <http://kam.mff.cuni.cz/~bernard/klein>. In the paper, we establish the correctness of used algorithms and refer the reader for details on implementation to the web page. The outcome of our programs is summarized in Section 7 where we also briefly discuss the algorithmic corollaries of our results.

## 2 6-critical graphs

In this section, we observe basic properties of 6-critical graphs on the Klein bottle. Euler's formula implies that the average degree of a graph embedded on the Klein bottle is at most six. As Sasanuma [14] established that every 6-regular graph that can be embedded on the Klein bottle is 5-colorable, we have the following proposition (observe that no 6-critical graph contains a vertex of degree four or less):

**Proposition 1.** *The minimum degree of every 6-critical graph on the Klein bottle is five.*

Let  $G$  be a 6-critical graph on the Klein bottle and  $v$  a vertex of degree five in  $G$ . Further let  $v_i$ ,  $1 \leq i \leq 5$ , be the neighbors of  $v$  in  $G$ . If all vertices  $v_i$  and  $v_j$ ,  $1 \leq i < j \leq 5$ , are adjacent, the vertices  $v$  and  $v_i$ ,  $1 \leq i \leq 5$ , form a clique of order six in  $G$ . As  $G$  is 6-critical,  $G$  must then be a complete graph of order six. Hence, we can conclude the following:

**Proposition 2.** *Let  $G$  be a 6-critical graph embedded on the Klein bottle. If  $G$  is not a complete graph of order six, then  $G$  contains a vertex  $v$  of degree five that has two non-adjacent neighbors  $v'$  and  $v''$ .*

We now introduce the following *reduction*: let  $G$  be a 6-critical graph embedded on the Klein bottle that is not isomorphic to  $K_6$  and let  $v, v'$

and  $v''$  be three vertices as in Proposition 2.  $G|vv'v''$  is the graph obtained from  $G$  by removing all the edges incident with  $v$  except for  $vv'$  and  $vv''$  and contracting the edges  $vv'$  and  $vv''$  to a new vertex  $w$ . The obtained graph can have parallel edges but it does not have loops as the vertices  $v'$  and  $v''$  are not adjacent. Observe that the graph  $G|vv'v''$  is not 5-colorable: otherwise, consider a 5-coloring of  $G|vv'v''$  and color the vertices  $v'$  and  $v''$  with the color assigned to the vertex  $w$ . Next, extend the 5-coloring to  $v$ —this is possible since  $v$  has five neighbors and at least two of them ( $v'$  and  $v''$ ) have the same color. Hence, we obtain a 5-coloring of  $G$  contradicting our assumption that  $G$  is 6-critical. Since  $G|vv'v''$  has no 5-coloring, it contains a 6-critical subgraph—this subgraph will be denoted by  $|G|vv'v''|$  and we say that  $G$  can be *reduced* to  $|G|vv'v''|$ .

Observe that the reduction operation can again be applied to  $|G|vv'v''|$  until a graph that is isomorphic to  $K_6$  is obtained (the process eventually terminates since the order of the graph is decreased in each step).

We continue with a simple observation on the graph  $|G|vv'v''|$ .

**Proposition 3.** *Let  $G$  be a 6-critical graph embedded on the Klein bottle,  $v$  a vertex of degree five in  $G$  and  $v'$  and  $v''$  two non-adjacent neighbors of  $v$ . The graph  $|G|vv'v''|$  contains the vertex  $w$  obtained by contracting the path  $v'vv''$ . Moreover, the vertex  $w$  has a neighbor  $w'$  in  $|G|vv'v''|$  that is a neighbor of  $v'$  in  $G$  but not of  $v''$  and it also has a neighbor  $w''$  that is a neighbor of  $v''$  but not of  $v'$  in  $G$ .*

*Proof.* If  $|G|vv'v''|$  does not contain the vertex  $w$ , then  $|G|vv'v''|$  is a subgraph of  $G \setminus \{v, v', v''\}$ . Since both  $|G|vv'v''|$  and  $G$  are 6-critical graphs, this is impossible. Hence,  $|G|vv'v''|$  contains the vertex  $w$ .

Assume now that  $|G|vv'v''|$  contains no vertex  $w'$  as described in the statement of the proposition, i.e., all neighbors of  $w$  in  $|G|vv'v''|$  are neighbors of  $v''$  in  $G$ . This implies that  $|G|vv'v''|$  is isomorphic to a subgraph of  $G \setminus \{v, v'\}$  (view the vertex  $v''$  as  $w$ ) which is impossible since both  $G$  and  $|G|vv'v''|$  are 6-critical. A symmetric argument yields the existence of a vertex  $w''$ .  $\square$

The strategy of our proof is to generate all 6-critical graphs by reversing the reduction operation. More precisely, we choose a vertex  $w$  of a 6-critical graph  $G$  and partition the neighbors of  $w$  into two non-empty sets  $W_1$  and  $W_2$ . We next replace the vertex  $w$  with a path  $w_1ww_2$  and join the vertex  $w_i$ ,  $i = 1, 2$ , to all vertices in the set  $W_i$ . Let  $G[w, W_1, W_2]$  be the resulting graph. We say that  $G[w, W_1, W_2]$  was obtained by *expanding* the graph  $G$ .

By Proposition 3, the following holds (choose  $w$  as in the statement of the proposition):

**Proposition 4.** *Let  $G$  be a 6-critical graph embedded on the Klein bottle and let  $v$  be a vertex of degree five of  $G$  with two non-adjacent neighbors  $v'$  and  $v''$ . The graph  $G' = |G|vv'v''|$  contains a vertex  $w$  such that  $G'[w, W_1, W_2] \subseteq G$  for some partition  $W_1$  and  $W_2$  of the neighbors of the vertex  $w$ .*

### 3 Minimal graphs

Our plan is to generate all 6-critical graphs from the complete graph  $K_6$  by expansions and insertions of new graphs into faces. In this section, we describe the graphs we have to insert into the faces to be sure that we have generated all 6-critical graphs.

A plane graph  $G$  with the outer face bounded by a cycle  $C$  of length  $k$  is said to be  $k$ -minimal if for every edge  $e \in E(G) \setminus C$ , there exists a proper precoloring  $\varphi_e$  of  $C$  with five colors that cannot be extended to  $G$  and that can be extended to a proper 5-coloring of  $G \setminus e$  (the graph  $G$  with the edge  $e$  removed). Note that the precolorings  $\varphi_e$  can differ for various choices of  $e$ .

The cycle  $C_k$  of length  $k$  is  $k$ -minimal (the definition vacuously holds); we say that  $C_k$  is a *trivial*  $k$ -minimal graph. For  $k = 3$ , it is easy to observe that  $C_3$  is the only 3-minimal graph since the colors of the vertices of  $C_3$  must differ and every planar graph is 5-colorable. Similarly,  $C_4$  and the graph obtained from  $C_4$  by adding a chord are the only 4-minimal graphs. As for  $k = 5$ , Thomassen [15] showed that if  $G$  is a plane graph with the outer face bounded by a cycle  $C$  of length five and  $C$  is chordless, then a precoloring of  $C$  with five colors can be extended to  $G$  unless  $G$  is the 5-wheel and the vertices of  $C$  are precolored with all five colors. Hence,  $C_5$ ,  $C_5$  with one chord,  $C_5$  with two chords and the 5-wheel are the only 5-minimal graphs. The analogous classification result of Thomassen [15] implies that the only 6-minimal graphs (up to an isomorphism) are those depicted in Figure 1.

The following lemma justifies the use of  $k$ -minimal graphs in our considerations:

**Lemma 5.** *Let  $G$  be a 6-critical graph embedded on the Klein bottle. If  $C$  is a contractible cycle of  $G$  of length  $k$ , then the subgraph  $G'$  of  $G$  inside the cycle  $C$  ( $G'$  includes the cycle  $C$  itself) is  $k$ -minimal.*

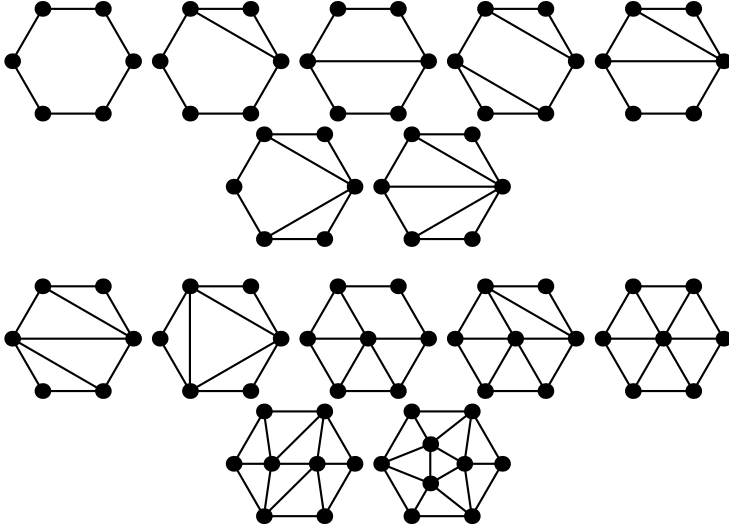


Figure 1: The list of all 6-minimal graphs.

*Proof.* We verify that  $G'$  is  $k$ -minimal. Let  $e$  be an edge of  $G'$  that is not contained in  $C$ . Since  $G$  is 6-critical,  $G \setminus e$  has a 5-coloring  $c$ . Let  $\varphi_e$  be the coloring  $c$  restricted to the cycle  $C$ . If  $\varphi_e$  could be extended to  $G'$ , then the extension of  $\varphi_e$  to the subgraph  $G'$  combined with the coloring  $c$  outside  $C$  would yield a 5-coloring  $G$ . This establishes the existence of a precoloring  $\varphi_e$  as in the definition of  $k$ -minimal graphs and the proof of the lemma is now finished.  $\square$

In the light of Lemma 5, our next goal is to find all  $k$ -minimal graphs for small values of  $k$ . The following proposition enables us to systematically generate all  $k$ -minimal graphs for any fixed  $k$  from the lists of  $k'$ -minimal graphs for  $3 \leq k' < k$ .

**Proposition 6.** *If  $G$  is a non-trivial  $k$ -minimal graph,  $k \geq 3$ , with the outer cycle  $C$ , then either the cycle  $C$  contains a chord or  $G$  contains a vertex  $v$  adjacent to at least three vertices of the cycle  $C$ . In addition, if  $C'$  is a cycle of  $G$  of length  $k'$  and  $G'$  is the subgraph of  $G$  bounded by the cycle  $C'$  (inclusively), then  $G'$  is a  $k'$ -minimal graph.*

*Proof.* First assume that  $C$  is chordless and each vertex  $v$  of  $G$  is adjacent to at most two vertices of  $C$ . Let  $G'$  be the subgraph of  $G$  induced by the vertices not lying on  $C$ . We consider the following list coloring problem: each vertex of  $G'$  not incident with the outer face receives a list of all five available colors and each vertex incident with the outer face is given a list of the colors distinct from the colors assigned to its neighbors on  $C$  in  $G$ . By our assumption, each such vertex has a list of at least three colors. A classical list coloring result of Thomassen [16] on list 5-colorings of planar graphs yields that  $G'$  has a coloring from the constructed lists. Hence, every precoloring of the boundary of  $G$  can be extended to the whole graph  $G$  and thus  $G$  cannot be  $k$ -minimal. This establishes the first part of the proposition. The proof of the fact that every cycle of length  $k'$  bounds a  $k'$ -minimal subgraph is very analogous to that of Lemma 5 and omitted.  $\square$

Proposition 6 suggests the following algorithm for generating  $k$ -minimal graphs. Assume that we have already generated all  $\ell$ -minimal graphs for  $\ell < k$  and let  $M_\ell$  be the list of all  $\ell$ -minimal graphs. Note that we have explicitly described the lists  $M_3, M_4, M_5$  and  $M_6$ . The list  $M_k$  is then generated by the following procedure (the vertices of outer boundary are denoted by  $v_1, \dots, v_k$ ):

```

M_k := { the cycle C_k on v_1, ..., v_k }
repeat
  M' := M_k
  forall 1 <= a < b <= k with b-a>=2 do
    G := the cycle C_k on v_1, ..., v_k with the chord v_av_b
    forall G_1 in M_{b-a+1} and G_2 in M_{k+a-b+1} do
      H := G with G_1 and G_2 pasted into its faces
      if H is k-minimal and H is not in M_k then
        add H to M_k
    endfor
  endfor
forall 1 <= a < b < c <= k do
  G := the cycle C_k on v_1, ..., v_k with the vertex v
    adjacent to v_a, v_b and v_c
  forall G_1 in M_{b-a+2}, G_2 in M_{c-b+2} and
    G_3 in M_{k+a-c+2} do
    H := G with G_1, G_2 and G_3 pasted into its faces
    if H is k-minimal and H is not in M_k then
      add H to M_k

```

$k$	3	4	5	6	7	8	9	10
$ M_k $	1	2	4	14	46	291	2124	19876
$n_k$	0	0	1	3	4	6	7	9

Table 1: The numbers of non-isomorphic  $k$ -minimal graphs for  $3 \leq k \leq 10$  and the largest number  $n_k$  of internal vertices of a  $k$ -critical graph.

```

endfor
endfor
until M_k = M'

```

Proposition 6 implies that the list  $M_k$  contains all  $k$ -minimal graphs after the termination of the procedure: if  $G$  is a  $k$ -minimal graph, it contains either a chord or a vertex  $v$  adjacent to three vertices on the outer cycle and the graphs inside the faces of the skeleton formed by the outer cycle and the chord / the edges adjacent to  $v$  are also minimal. The verifications whether the graph  $G$  is isomorphic to one of the graphs in  $M_k$  and whether  $G$  is  $k$ -minimal are straightforward and the reader can find the details in the program available at <http://kam.mff.cuni.cz/~bernard/klein>.

The numbers of non-isomorphic  $k$ -minimal graphs for  $3 \leq k \leq 10$  can be found in Table 1. We finish this section by justifying our approach with showing that the number of  $k$ -minimal graphs is finite for every  $k$ ; in particular, the procedure always terminates for each value of  $k$ .

**Proposition 7.** *The number of  $k$ -minimal graphs is finite for every  $k \geq 3$ .*

*Proof.* Let  $A_k$  be the number of  $k$ -minimal graphs and  $A_{k,\ell}$  the number of  $k$ -minimal graphs  $G$  such that exactly  $\ell$  precolorings of the boundary of  $G$  with five colors can be extended to  $G$ . Clearly,  $A_{k,\ell} = 0$  for  $\ell > 5 \cdot 4^{k-1}$  since there are at most  $5 \cdot 4^{k-1}$  proper precolorings of the boundary of  $G$ . We prove that the numbers  $A_{k,\ell}$  are finite by the induction on  $5^k + \ell$ . More precisely, we establish the following formula:

$$A_{k,\ell} \leq k \cdot \sum_{i=3}^{k-1} 4i(k+2-i)A_i A_{k+2-i} + \quad (1)$$

$$k \cdot \sum_{i=4}^{k-1} \sum_{i'=4}^{k+3-i} 8ii'(k+6-i-i')A_i A_{i'} A_{k+6-i-i'} + \quad (2)$$

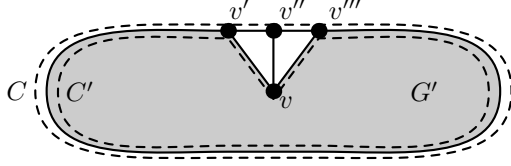


Figure 2: The notation used in the proof of Proposition 2.

$$k \sum_{i=1}^{\ell-1} 2k A_{k,i} \quad (3)$$

Fix  $k$  and  $\ell$ . By Proposition 6, every  $k$ -minimal graph  $G$  with  $\ell$  extendable precolorings of its boundary cycle  $C$  either contains a chord or a vertex  $v$  adjacent to three vertices on  $C$ . In the former case, the cycle  $C$  and the chord forms cycles of length  $i$  and  $k + 2 - i$ . Since these cycles bound  $i$ -minimal and  $(k + 2 - i)$ -minimal graphs by Proposition 6, the number of such  $k$ -minimal graphs is at most  $A_i A_{k+2-i}$ . After considering at most  $k$  possible choices of the chord (for fixed  $i$ ) and  $2i$  and  $2(k + 2 - i)$  possible rotations and/or reflections, we obtain the term (1).

Let us analyze the case that  $G$  contains a vertex  $v$  adjacent to three vertices on  $C$ . If the neighbors of  $v$  are not three consecutive vertices of  $C$ , then the edges between  $v$  and its neighbors delimit cycles of lengths  $i \geq 4$ ,  $i' \geq 4$  and  $k + 6 - i - i'$ . These cycles bound  $i$ -minimal,  $i'$ -minimal and  $(k + 6 - i - i')$ -minimal graphs and their number (including different rotations and reflections) is estimated by the term (2).

Assume that the neighbors of  $v$  on  $C$  are consecutive. Let  $v'$ ,  $v''$  and  $v'''$  be the neighbors of  $v$  and  $G'$  the subgraph of  $G$  inside the cycle  $C'$  where  $C'$  is the cycle  $C$  with the path  $v'v''v'''$  replaced with the path  $v'vv'''$  (see Figure 2). Fix a precoloring  $\varphi_0$  of the vertices of  $C$  except for  $v''$ . Let  $\alpha$  be the number of ways in which  $\varphi_0$  can be extended to  $v$  that also extends to  $G'$ . Similarly,  $\alpha'$  the number of ways in which  $\varphi_0$  can be extended to  $v''$  that also extends to  $G$ .

We show that  $\alpha \leq \alpha'$ . If  $\alpha = 0$ , then  $\alpha' = 0$ . If  $\alpha = 1$ , then  $\alpha' > 1$ . Finally, if  $\alpha > 1$ , then  $\alpha \leq \alpha'$  as any extension of  $\varphi_0$  to  $C$  also extends to  $G$  (note that  $\alpha'$  is 3 or 4 depending on  $\varphi_0(v')$  and  $\varphi_0(v''')$ ). We conclude that the number of precolorings of  $C'$  that can be extended to  $G'$  does not exceed the number of precolorings of  $C$  extendable to  $G$ .

Let  $\varphi$  be the precoloring of  $C$  that cannot be extended to  $G$  but that can

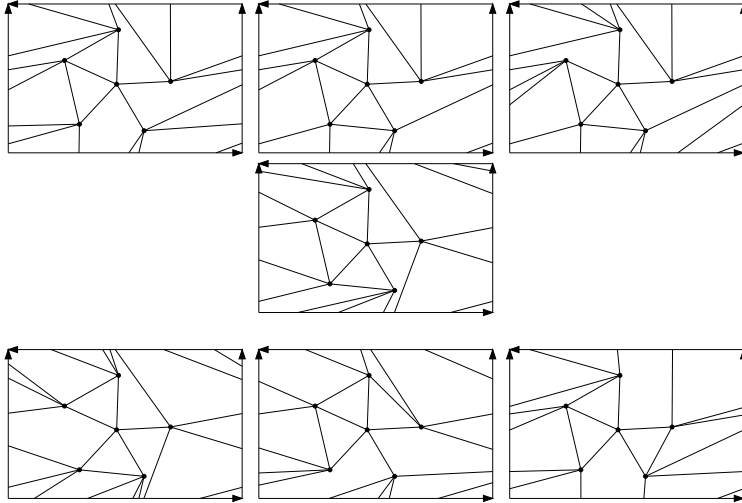


Figure 3: The list of all seven non-isomorphic 2-cell embeddings of  $K_6$  on the Klein bottle.

be extended to  $G \setminus vv''$  and let  $\varphi_0$  be the restriction of  $\varphi$  to  $C \setminus v''$ . It is easy to infer that the value of  $\alpha$  for this particular precoloring  $\varphi_0$  must be equal to one and consequently  $\alpha' > 1$  for  $\varphi_0$ . Hence, the number of precolorings of  $C'$  that can be extended to  $G'$  is strictly smaller than the number of precolorings of  $C$  that can be extended to  $G$ . Since  $G'$  is a  $k$ -minimal graph with fewer precolorings of the boundary that can be extended to  $G'$  than the number of precolorings of  $C$  extendable to  $G$ , the number of  $k$ -minimal graphs  $G$  with a vertex  $v$  with three consecutive neighbors on  $C$  including their possible rotations and reflections is estimated by (3). This finishes the proof of the inequality and thus the proof of the whole proposition.  $\square$

## 4 Embeddings of $K_6$ on the Klein bottle

Subsequent applications of our reduction procedure to a 6-critical graph on the Klein bottle eventually lead to an embedding of the complete graph  $K_6$ . The resulting embedding of  $K_6$  is either a 2-cell embedding or not. Recall that an embedding is said to be *2-cell* if every face is homeomorphic to a

disc.

If the resulting embedding of  $K_6$  is not 2-cell, the embedding must be isomorphic to the embedding obtained from the unique embedding of  $K_6$  in the projective plane by inserting a cross-cap into one of its faces. Otherwise, the embedding is isomorphic to one of the seven embeddings of  $K_6$  depicted in Figure 3. All 2-cell embeddings of  $K_6$  on the Klein bottle can be easily generated by a simple program that ranges through all 2-cell embeddings of  $K_6$  on surfaces: for each vertex  $v$  of  $K_6$ , the program generates all cyclic permutations of the other vertices (corresponding to the order in which the vertices appear around  $v$ ) and chooses which edges alter the orientation. Each such pair of cyclic permutations and alterations of orientations determines uniquely both the embedding and the surface. It is straightforward to compute the genus of the surface and test whether the constructed embedding is not isomorphic to one of the previously found embeddings. The source code of the program can be found at <http://kam.mff.cuni.cz/~bernard/klein>.

## 5 Expansions of 2-cell embeddings of $K_6$

In this section, we focus on embeddings of 6-critical graphs that can be reduced to a 2-cell embedding of  $K_6$ . All such 6-critical graphs can easily be generated, using the expansion operation and Lemma 5, by the following procedure:

```
G_1, G_2, G_3, G_4, G_5, G_6, G_7 :=
  non-isomorphic embeddings of K_6 on the Klein bottle
k := 7
i := 1
while i <= k do
  for all vertices w of G_i do
    for all partitions of N(w) into W_1 and W_2 do
      H_0 := G[w,W_1,W_2]
      for all H obtained from H_0 by pasting
        minimal graphs into its faces do
        if H is not isomorphic to any of G_1, ..., G_k then
          k := k + 1; G_k := H
        endfor
      endfor
    endfor
  endfor
endfor
```

```

i := i + 1
done { while }
output G_1, ..., G_k

```

The source code of the program implementing the above procedure can be found at <http://kam.mff.cuni.cz/~bernard/klein>. The program eventually terminates outputting 11 embeddings of 6-critical graphs on the Klein bottle, which are depicted in Figure 4, in addition to the seven 2-cell embeddings of  $K_6$ . Hence, Proposition 4 and Lemma 5 now yield:

**Lemma 8.** *Let  $G$  be an embedding of a 6-critical graph on the Klein bottle that is distinct from  $K_6$ . If  $G$  can sequentially be reduced to a 2-cell embedding of  $K_6$  on the Klein bottle, then  $G$  is isomorphic to one of the eleven embeddings depicted in Figure 4.*

## 6 Expansions of non-2-cell embedding of $K_6$

As we have already analyzed embeddings of 6-critical graphs that can be reduced to a 2-cell embedding of  $K_6$  on the Klein bottle, it remains to analyze 6-critical graphs that can be reduced to a non-2-cell embedding of  $K_6$ . We eventually show that all such embeddings are isomorphic to one of those depicted in Figure 4.

**Lemma 9.** *Let  $G$  be a 6-critical graph embedded on the Klein bottle. If  $G$  can be reduced to a non-2-cell embedding of  $K_6$ , then  $G$  is isomorphic to one of the embeddings depicted in Figure 4.*

*Proof.* Let  $G$  be a 6-critical graph on the Klein bottle with the smallest order that can be reduced to a non-2-cell embedding of  $K_6$  and that is not isomorphic to any of the embeddings in Figure 4. Observe that the choice of  $G$  implies that any possible reduction of  $G$  yields a non-2-cell embedding of  $K_6$  on the Klein bottle (otherwise, Lemma 8 yields that the reduced graph is a smaller graph missing in Figure 4 which contradicts our choice of  $G$ ).

Let  $H$  be the unique embedding of  $K_6$  in the projective plane and  $w$  a vertex of  $H$ . By Proposition 4,  $G$  contains  $H[w, W_1, W_2]$  for some partition of the neighborhood of  $w$  into non-empty sets  $W_1$  and  $W_2$ . By symmetry,  $|W_1| = 1$  or  $|W_1| = 2$ . We first analyze the case that  $|W_1| = 1$ , i.e.,  $G$  contains the embedding drawn in the middle of Figure 5 as a subgraph. The face which is not 2-cell is drawn using the gray color.

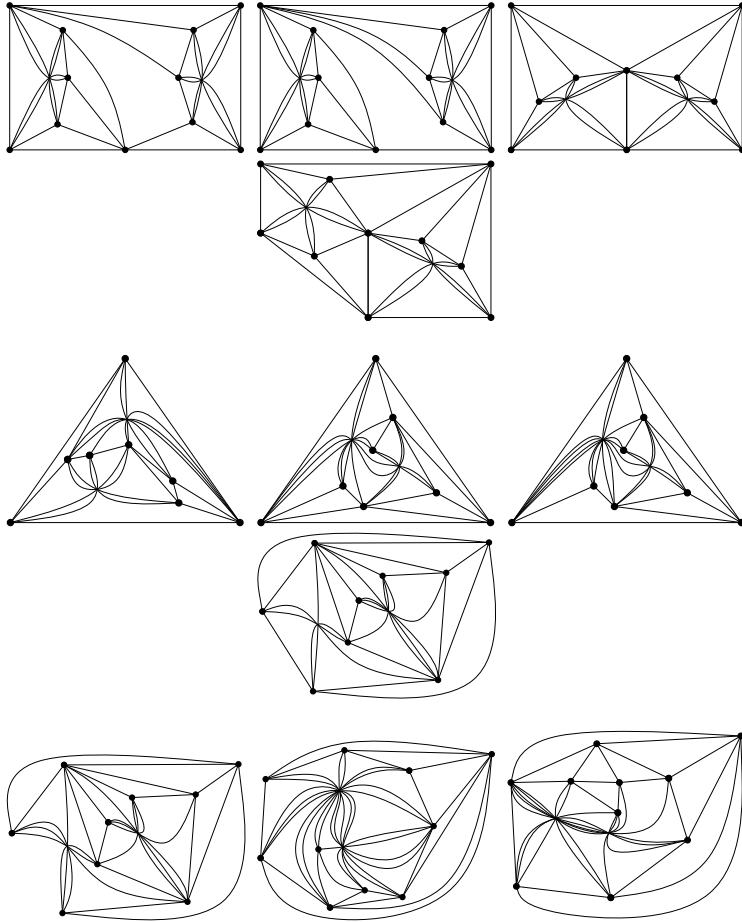


Figure 4: The list of 11 non-isomorphic embeddings of 6-critical graphs on the Klein bottle that are distinct from  $K_6$ . The graphs are drawn in the plane with two cross-caps.

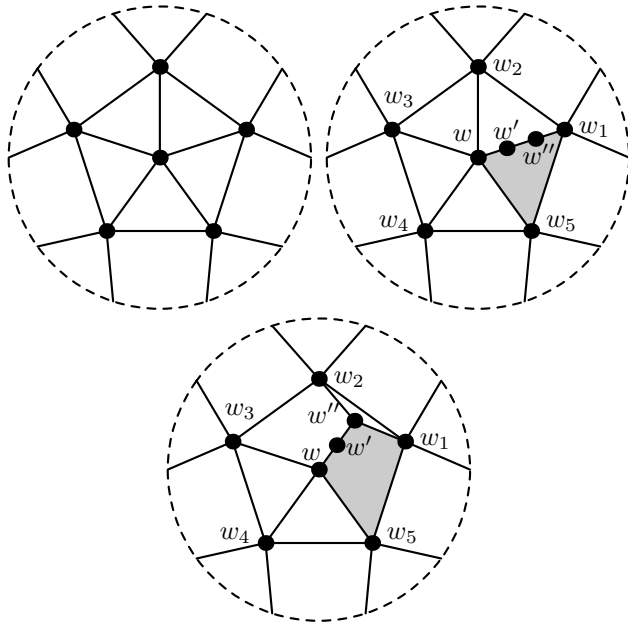


Figure 5: The unique embedding of  $K_6$  in the projective plane and its two possible expansions.

Let  $G_{15}$  be the subgraph of  $G$  contained inside the cycle  $C_{15} = ww'w''w_1w_5$  and  $G_{12}$  the subgraph contained inside the cycle  $C_{12} = ww'w''w_1w_2$ . By Lemma 5,  $G_{12}$  is either the cycle  $C_{12}$  with zero, one or two chords or a 5-wheel bounded the cycle  $C_{12}$ . The interiors of the remaining 2-cell faces of  $H[w, W_1, W_2]$  must be empty (since they are triangles).

Assume that  $G_{12} \neq C_{12}$ . The graph  $G$  without the interior of the cycle  $C_{12}$  is 5-colorable since  $G$  is 6-critical. Observe that the vertices  $w$  and  $w_1$  must get the same color in any such 5-coloring (since adding an edge  $ww_1$  to  $G$  would form a clique of order six). However, it is always possible to permute the colors of the vertices of  $G_{15}$  preserving the colors of  $w$ ,  $w_1$  and  $w_5$  in such a way that the 5-coloring can be extended to  $G_{12}$ . This contradicts our assumption that  $G$  is 6-chromatic. Hence,  $G_{12} = C_{12}$ .

Since  $G$  is 6-critical, the graph  $G_{15}$  is 5-colorable. Moreover, the vertices  $w$  and  $w_1$  receive distinct colors in every 5-coloring of  $G_{15}$ : if the vertices  $w$  and  $w_1$  have the same color, the 5-coloring of  $G_0$  can be extended to the whole graph  $G$ .

Let  $G'$  be the graph obtained from  $G_{15}$  by identifying the vertices  $w$  and  $w_1$ . Since  $G_{15}$  can be drawn in the projective plane with the cycle  $C_{15}$  bounding a face,  $G'$  can also be drawn in the projective plane. As no 5-coloring assigns the vertices  $w$  and  $w_1$  the same color,  $G'$  contains  $K_6$  as a subgraph. On the other hand, since  $G$  is 6-critical,  $G$  does not contain  $K_6$  as a subgraph and thus the subgraph of  $G'$  isomorphic to  $K_6$  contains the vertex obtained by the identification of  $w$  and  $w_1$ . In addition,  $G'$  does not contain any edges except for the edges of the complete graph and the path  $ww_5w_1$  (removing any additional edge from  $G$  would yield a graph that is also not 5-colorable contrary to our assumption that  $G$  is 6-critical). We conclude that  $G_{15}$  is comprised of

1. the path  $ww_5w_1$ , a complete graph on a 5-vertex set  $X$  such that  $\{w', w''\} \subset X$  and  $w_5 \notin X$ , and such that  $N(w)$  and  $N(w_1)$  partition  $X$ , or
2. the path  $ww_5w_1$ , a complete graph on a 5-vertex set  $X$ ,  $\{w', w'', w_5\} \subset X$ , such that  $N(w) \setminus \{w_5\}$  and  $N(w_1) \setminus \{w_5\}$  partition  $X \setminus \{w_5\}$ .

In the former case, the graph  $G$  is isomorphic to the first or the second embedding on the first line in Figure 4; in the latter case,  $G$  is isomorphic to the third or the fourth embedding on the first line in the figure. This finishes the analysis of the case that  $|W_1| = 1$ .

We now assume that  $|W_1| = 2$ , i.e.,  $G[w, W_1, W_2]$  is the graph depicted in the right part of Figure 5. We can also assume that  $w$  is not adjacent to  $w_2$  in  $G$  since otherwise we could choose  $W_1 = \{w_1\}$  which would bring us to the previous case. Similarly, the vertices  $w, w_1$  and  $w_5$  do not form a triangular face of  $G$ . Let  $C_{15}$  be the cycle  $ww'w''w_1w_5$ ,  $C_{23}$  the cycle  $ww'w''w_2w_3$ ,  $G_{15}$  the subgraph of  $G$  inside the cycle  $C_{15}$ , and  $G_{23}$  the subgraph inside the cycle  $C_{23}$ . As in the previous case,  $G_{23}$  is either the cycle  $C_{23}$  with zero, one or two chords or a 5-wheel bounded by  $C_{23}$ .

It is straightforward (but tedious) to check that any coloring  $c$  of  $G_{15}$  with five colors extends to a coloring of  $G$  unless:

- the vertices  $w$  and  $w''$  are assigned the same color in  $c$ , or
- all the five vertices  $w, w', w'', w_1$  and  $w_5$  are assigned mutually distinct colors and  $G$  contains edges  $w_3w'$  and  $w_3w''$  (see the embedding in the left part of Figure 6).

The reader is asked to verify the details him-/her-/itself.

We first show that there is a coloring of  $G_{15}$  of the latter type. Let  $G'$  be the graph obtained from  $G_{15}$  by adding the edge  $ww''$ . Assume that  $G'$  contains a complete graph of order six as a subgraph. If  $G_{23}$  contains an inner edge  $e$ , consider a 5-coloring of  $G \setminus e$  which exists since  $G$  is 6-critical. The coloring must assign the vertices  $w$  and  $w''$  the same color (since otherwise,  $c$  restricted to  $G_{15}$  would also be a proper coloring of  $G'$ ). Consequently, none of the vertices  $w_i, 1 \leq i \leq 5$ , can be assigned the common color of  $w$  and  $w''$  which is impossible since the vertices  $w_i, 1 \leq i \leq 5$ , form a clique. We conclude (under the assumption that  $G'$  contains  $K_6$  as a subgraph) that  $G_{23}$  is formed by the cycle  $C_{23}$  only. As in the previous case, we can now establish that  $G'$  is formed by a subgraph isomorphic to  $K_6$  and the path  $ww_5w_1w''$  (which need not to be disjoint); in particular, the vertex  $w'$  is contained in the subgraph isomorphic to  $K_6$ . It is now easy to verify that the embedding of  $G$  must be isomorphic to the first or the last embedding in the first line in Figure 4.

Since  $G'$  does not contain  $K_6$  as a subgraph, there is a coloring of  $G_{15}$  with five colors which assigns the vertices  $w$  and  $w''$  distinct colors. Since  $G$  is not 5-colorable,  $G_{15}$  has a coloring assigning all the vertices  $w, w', w'', w_1$  and  $w_5$  distinct colors and  $G$  must be of the type depicted in the left part of Figure 6. Since the vertices  $w$  and  $w_2$  are not adjacent in  $G$  and the degree of  $w_4$  is five, we can consider the graph  $|G|w_4ww_2|$ ; let  $G_0$  be this graph. By the choice of  $G$ ,  $G_0$  is a non-2-cell embedding of  $K_6$  in the

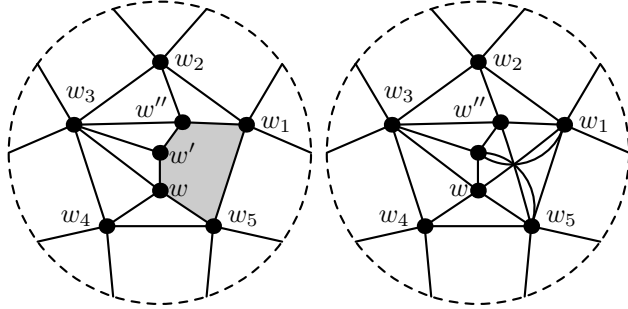


Figure 6: The embeddings obtained in the analysis in the proof of Lemma 9.

projective plane and Proposition 3 implies that  $G_0$  contains the vertex  $w_0$  obtained by contracting the path  $ww_4w_2$  in  $G$ .

If  $G_0$  does not contain the vertex  $w_3$ , consider a coloring of  $G_{15}$  assigning the vertices  $w, w', w'', w_1$  and  $w_5$  five distinct colors. This coloring restricted to  $G_0$  is a proper coloring of  $G_0 = K_6$  with five colors since  $G_0$  can contain only the edges  $w_0w''$  and  $w_0w_1$  in addition to those contained in  $G_{15}$  (viewing the vertices  $w$  and  $w_0$  to be the same vertex). Hence,  $G_0$  contains the vertex  $w_3$ . Since the only neighbors of  $w_3$  in  $G|w_4ww_2$  are the vertices  $w_0, w', w'', w_1$  and  $w_5$ , the vertex set of  $G_0$  must be  $\{w_0, w', w'', w_1, w_5, w_3\}$ . In particular, the vertex  $w_5$  is adjacent to  $w'$  and  $w''$  in  $G$ . A symmetric argument applied to  $|G|w_2w''w_4|$  implies that the vertex  $w_1$  is adjacent to  $w'$  and  $w''$  in  $G$ . This brings us to the embedding depicted in the right part of Figure 6, which is isomorphic to the third embedding on the second line in Figure 4.  $\square$

## 7 Main result

We now wrap the results obtained in the previous sections. The discussion in Section 4 and Lemmas 8 and 9 yield the following:

**Theorem 10.** *There are nine non-isomorphic 6-critical graphs that can be embedded on the Klein bottle which are depicted in Figure 7. The graphs have altogether a single non-2-cell embedding and 18 non-isomorphic 2-cell embeddings on the Klein bottle, which are depicted in Figures 3 and 4.*

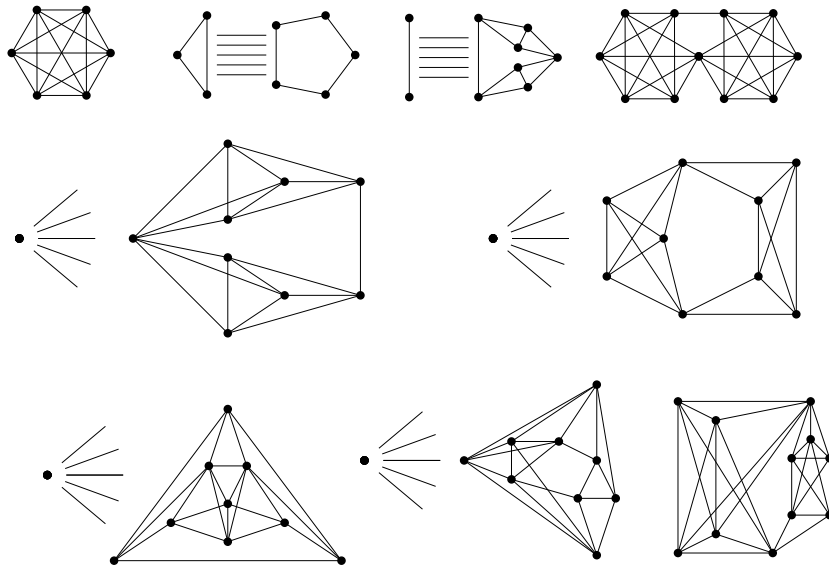


Figure 7: The list of all nine 6-critical graphs that can be embedded in the Klein bottle. Some of the edges are only indicated in the figure: the straight edges between two parts represent that the graph is obtained as the join of the two parts and the vertices with stars of edges leaving them are adjacent to all vertices in the graph.

Immediate corollaries of Theorem 10 are:

**Corollary 11.** *Let  $G$  be a graph that can be embedded on the Klein bottle.  $G$  is 5-colorable unless it contains one of the nine graphs depicted in Figure 7 as a subgraph.*

**Corollary 12.** *Let  $G$  be a graph embedded on the Klein bottle.  $G$  is 5-colorable unless it contains a subgraph with embedding isomorphic to one of the 19 embeddings depicted in Figures 3, 4 and 5.*

Eppstein [6, 7] showed that testing the existence of a subgraph isomorphic to a fixed graph  $H$  of a graph embedded on a fixed surface can be solved in linear time. As we have found the explicit list of 6-critical graphs on the Klein bottle, we also obtain the following corollary:

**Corollary 13.** *There is an explicit linear-time algorithm for testing whether a graph embedded on the Klein bottle is 5-colorable.*

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## References

- [1] K. Appel, W. Haken: Every planar map is four colorable, Part I. Discharging, Illinois J. Math. **21** (1977), 429–490.
- [2] K. Appel, W. Haken, J. Koch: Every planar map is four colorable, Part II. Reducibility, Illinois J. Math. **21** (1977), 491–567.
- [3] N. Chenette, L. Postle, N. Streib, R. Thomas, C. Yerger: Five-coloring graphs on the Klein bottle, submitted.
- [4] G. A. Dirac: Map colour theorems related to the Heawood colour formula, J. London Math. Soc. **31** (1956), 460–471.
- [5] G. A. Dirac: A theorem of R. L. Brooks and a conjecture of H. Hadwiger, Proc. London Math. Soc. **17** (1957), 161–195.

- [6] D. Eppstein: Subgraph isomorphism in planar graphs and related problems, in: Proc. 6th ACM-SIAM Symp. Discrete Algorithms (SODA), 1995, 632–640.
- [7] D. Eppstein: Subgraph isomorphism in planar graphs and related problems, J. Graph Algorithms and Applications **3(3)** (1999), 1–27.
- [8] S. Fisk: The non-existence of colorings, J. Combin. Theory Ser. B **24** (1978), 247–248.
- [9] S. Fisk, B. Mohar: Coloring graphs without short non-bounding cycles, J. Combin. Theory Ser. B **60** (1994), 268–276.
- [10] T. Gallai: Kritische Graphen I, Publ. Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192.
- [11] T. Gallai: Kritische Graphen II, Publ. Math. Inst. Hungar. Acad. Sci **8** (1963), 373–395.
- [12] P. J. Heawood: Map colour theorem, Quart. J. Pure Appl. Math. **24** (1890), 332–338.
- [13] N. Robertson, D. P. Sanders, P. D. Seymour R. Thomas: The four colour theorem, J. Combin. Theory Ser. B **70** (1997), 2–44.
- [14] N. Sasanuma: Chromatic numbers of 6-regular graphs on the Klein bottle, to appear in Discrete Mathematics.
- [15] C. Thomassen: Five-coloring graphs on the torus, J. Combin. Theory Ser. B **62** (1994), 11–33.
- [16] C. Thomassen: Every planar graph is 5-choosable, J. Combin. Theory Ser. B **62** (1994), 180–181.
- [17] C. Thomassen: Color-critical graphs on a fixed surface, J. Combin. Theory Ser. B **70** (1997), 67–100.