Solution set of complex linear interval systems of equations

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Abstract

We present a solution set description for the system of complex interval equations, where complex intervals have a rectangular form. The solution set is described by a system of nonlinear inequalities, which can be used to obtain very accurate approximation of interval hull of the solution set. In our numerical experiments we exploit this approximation to study overestimation for common complex interval equations solvers (Gauss elimination, Hansen-Bliek-Rohn-Ning-Kearfott method).

Keywords: *Complex interval systems, complex intervals, solution set.*

1 Introduction

Uncertainty which naturally appears in real-life problems can be treated by various ways. In interval analysis models we suppose that input data independently varies in some (given) compact intervals. Solving interval system of equations is a basic problem of interval analysis. Sometimes in practical problems (e.g. electrical circuits [9, 10]), complex variables can occur. Complex intervals can be defined as rectangles (see e.g. [1])

$$\boldsymbol{a} + \boldsymbol{b}i \equiv [a_1 + b_1\, i, a_2 + b_2\, i] = \{a + b\, i \in \mathbb{C} \mid a_1 \leq a \leq a_2,\ b_1 \leq b \leq b_2\},$$

or as circles (see e.g. [1])

$$\langle m, r\rangle \equiv \{z \in \mathbb{C} \mid |m - z| \leq r\}.$$

Polar form representation of complex intervals and it's arithmetic can be found in [3]. In this paper we study complex interval equations for rectangular complex intervals.

1.1 Notation

The vector $e_n = (1, \ldots, 1)^T$ is the n-dimensional vector of ones. An interval matrix is defined as

$$\boldsymbol{A} = [A_1, A_2] = \{A \in \mathbb{R}^{m\times n} \mid A_1 \leq A \leq A_2\},$$

where $A_1 \leq A_2$ are fixed matrices. The set of all $m \times n$ real and complex matrices will be denoted by $\mathbb{IR}^{m\times n}$ and $\mathbb{IC}^{m\times n}$, respectively; n-dimensional interval vectors can be regarded as interval matrices $n \times 1$. By

$$A^c \equiv \frac{1}{2}(A_1 + A_2), \quad A^\Delta \equiv \frac{1}{2}(A_2 - A_1)$$

we denote the midpoint and radius of $\boldsymbol{A}$, respectively. $\square S$ stands for the interval hull of he set S.

1.2 Complex interval equations

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n\times n}$, $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{IR}^n$. The complex interval system in question is the following

$$(\boldsymbol{A} + \boldsymbol{B}i)z = \boldsymbol{c} + \boldsymbol{d}i.$$

This can be rewritten with real variables x, y as

$$(\boldsymbol{A} + \boldsymbol{B}i)(x + yi) = \boldsymbol{c} + \boldsymbol{d}i. \tag{1}$$

The solution set of (1) is defined traditionally as

$$\begin{aligned}\Sigma =&\{x + yi \in \mathbb{C} \mid \exists A \in \boldsymbol{A}\ \exists B \in \boldsymbol{B}\ \exists c \in \boldsymbol{c}\ \exists d \in \boldsymbol{d} : \\ &(A + Bi)(x + yi) = c + di\}.\end{aligned} \tag{2}$$

The system (1) can be transformed into the real form as the set of variable $x, y \in \mathbb{R}^n$ satisfying

$$\begin{aligned} Ax - By &= c, \\ Bx + Ay &= d. \end{aligned} \tag{3}$$

for certain $A \in \boldsymbol{A}, B \in \boldsymbol{B}, c \in \boldsymbol{c}, d \in \boldsymbol{d}$. This is not a regular interval system of equations due to dependences in matrices $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$, but the interval system

$$\begin{aligned} \boldsymbol{A}x - \boldsymbol{B}y &= \boldsymbol{c}, \\ \boldsymbol{B}x + \boldsymbol{A}y &= \boldsymbol{d}. \end{aligned} \tag{4}$$

can be used to obtain an outer estimate of Σ via various solvers such as the Gauss elimination [1], the Hansen-Bliek-Rohn-Ning-Kearfott method [13] or the Householder method [2] (see Example 1). Other enclosures of Σ can be also obtained directed by the complex interval system solvers: the Gauss elimination [1], the complex Householder method [4]. We should note that the complex Householder method described in [4] is wrong, because of the missing the complex number ζ (cf. [11]). One can this easily see in their numerical experiments, where $[1,5] + [-1,1]i = \mathbf{A}_{1,1}^{(1)} \not\subseteq -\mathbf{H}_{1,1}^{(1)}.\alpha = -[-0.4306, 0.1348].[25.0199, 25.5147] = [-3.4394, 10.9867]$.

2 Solution set characterization

In this section we derive a description of the solution set Σ. We use the form (3) and exhibit the following theorem from [6] dealing with dependences in interval systems.

Theorem 1. *Let $\boldsymbol{M} \in \mathbb{IR}^{m \times n}$, $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{IR}^{m \times h}$, $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{IR}^m$. Then for certain $M \in \boldsymbol{M}$, $P \in \boldsymbol{P}$, $Q \in \boldsymbol{Q}$, $p \in \boldsymbol{p}$, $q \in \boldsymbol{q}$ vectors $u, v \in \mathbb{R}^n$, $w \in \mathbb{R}^h$ form a solution of the system*

$$Mu + Pw = p, \tag{5}$$

$$Mv + Qw = q \tag{6}$$

if and only if they satisfy the following system of inequalities

$$M^\Delta|u| + P^\Delta|w| + p^\Delta \geq |r_1|, \quad (7)$$

$$M^\Delta|v| + Q^\Delta|w| + q^\Delta \geq |r_2|, \quad (8)$$

$$P^\Delta|w||v|^T + Q^\Delta|w||u|^T + p^\Delta|v|^T + q^\Delta|u|^T + M^\Delta|uv^T - vu^T| \geq |r_1v^T - r_2u^T|, \quad (9)$$

where $r_1 \equiv -M^cu - P^cw + p^c$, $r_2 \equiv -M^cv - Q^cw + q^c$.

Due to this theorem we can simply give a description of the solution set Σ. By setting $\boldsymbol{M} \equiv (\boldsymbol{A}\ \boldsymbol{B})$, $\boldsymbol{P} \equiv \boldsymbol{Q} \equiv 0$, $\boldsymbol{p} \equiv \boldsymbol{c}$, $\boldsymbol{q} \equiv \boldsymbol{d}$, $u \equiv (x^T, -y^T)^T$, $v \equiv (y^T, x^T)^T$ we immediately have the following corollary.

Corollary 1. *The solution set* Σ *is described by*

$$A^\Delta|x| + B^\Delta|y| + c^\Delta \geq |r_1|, \quad (10)$$

$$A^\Delta|y| + B^\Delta|x| + d^\Delta \geq |r_2|, \quad (11)$$

$$c^\Delta|y|^T + d^\Delta|x|^T + A^\Delta|xy^T - yx^T| + B^\Delta| - yy^T - xx^T| \geq |r_1y^T - r_2x^T|, \quad (12)$$

$$c^\Delta|x|^T + d^\Delta|y|^T + A^\Delta|xx^T + yy^T| + B^\Delta| - yx^T + xy^T| \geq |r_1x^T + r_2y^T|, \quad (13)$$

where $r_1 \equiv -A^cx + B^cy + c^c$, $r_2 \equiv -A^cy - B^cx + d^c$.

The solution set for real interval equations represents a polyhedral set, which is convex in each orthant (see e.g. [5]). This is not generally true for complex interval systems. The solutions set Σ is neither a polyhedral set, nor necessary convex in each orthant, see Figures 5–8. Nevertheless, this is not surprising, cf. [12].

An approximation of $\square\Sigma$ can be obtain with an interval hull version of the procedure SIVIA from [7], see Table 1. This algorithm computes approximation of the set described by the system of nonlinear inequalities $f(z) \geq 0$; in our case the set is described by (10)–(13). SIVIA returns inner and outer enclosures $\underline{\boldsymbol{z}}$, $\overline{\boldsymbol{z}}$, respectively, such that $\underline{\boldsymbol{z}} \subseteq \square\Sigma \subseteq \overline{\boldsymbol{z}}$. The parameter ε determines how the approximation will be tight. The functions $\boldsymbol{l}(\boldsymbol{z})$ and $\boldsymbol{r}(\boldsymbol{z})$ divide the interval $\boldsymbol{x}$ into two parts along the widest components.

Table 1: Version of SIVIA for the set described by $f(z) \geq 0$.

Algorithm SIVIA(in:f,$\boldsymbol{z}$,ε, inout: $\underline{\boldsymbol{z}}$, $\overline{\boldsymbol{z}}$)

if $f(\boldsymbol{z}) \geq 0$ **then**
 $\quad \underline{\boldsymbol{z}} = \square(\underline{\boldsymbol{z}} \cup \boldsymbol{z})$; $\overline{\boldsymbol{z}} = \square(\overline{\boldsymbol{z}} \cup \boldsymbol{z})$; **return**;
 if $\exists i : f(\boldsymbol{z})_i < 0$ **return**;
 if $\boldsymbol{z}^\Delta < \varepsilon$ **then**
 $\quad \overline{\boldsymbol{z}} = \square(\overline{\boldsymbol{z}} \cup \boldsymbol{z})$; **return**;
 if $\boldsymbol{l}(\boldsymbol{z}) \not\subseteq \underline{\boldsymbol{z}}$ **then** SIVIA(f,$\boldsymbol{l}(\boldsymbol{z})$,$\varepsilon$, $\underline{\boldsymbol{z}}$, $\overline{\boldsymbol{z}}$);
 if $\boldsymbol{r}(\boldsymbol{z}) \not\subseteq \underline{\boldsymbol{z}}$ **then** SIVIA(f,$\boldsymbol{r}(\boldsymbol{z})$,$\varepsilon$, $\underline{\boldsymbol{z}}$, $\overline{\boldsymbol{z}}$);

Let $\underline{\boldsymbol{z}} = \underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}\,i$ and $\overline{\boldsymbol{z}} = \overline{\boldsymbol{x}} + \overline{\boldsymbol{y}}\,i$ be an (n-dimensional) inner and outer enclosure of $\square\Sigma$, respectively, and let $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}\,i$ be an outer approximation of $\square\Sigma$. The quality this approximation can be measured by various *approximation ratios*. For the sake of this paper we introduce the following one

$$\boldsymbol{\rho}(\boldsymbol{z}) \equiv \left[\frac{e_n^T \boldsymbol{x}^\Delta + e_n^T \boldsymbol{y}^\Delta}{e_n^T \overline{\boldsymbol{x}}^\Delta + e_n^T \overline{\boldsymbol{y}}^\Delta},\ \frac{e_n^T \boldsymbol{x}^\Delta + e_n^T \boldsymbol{y}^\Delta}{e_n^T \underline{\boldsymbol{x}}^\Delta + e_n^T \underline{\boldsymbol{y}}^\Delta} \right].$$

3 Numerical experiments

In this section we compute overestimation for some complex interval equation solvers.

Example 1. Let us consider the complex interval system from [1, 4]

$$\begin{pmatrix} [1,5] + [-1,1]\,i & 1 \\ 25 & [-1,1] + [-1,1]\,i \end{pmatrix} (x + y\,i) = \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}. \tag{14}$$

Hence

$$\boldsymbol{A} = \begin{pmatrix} [1,5] & 1 \\ 25 & [-1,1] \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} [-1,1] & 0 \\ 0 & [-1,1] \end{pmatrix},\ \boldsymbol{c} = \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix},\ \boldsymbol{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We will compare diverse solvers and compute their approximation ratios. The following computations were carried out in MATLAB 7.0.4 (R14.2) with help of the interval toolbox INTLAB v5.3 (see [14]). Solving (14) by the function `verifylss` leads to the solution

$$\boldsymbol{x}^1 = \big([-0.1453, 0.1453],\ [-1.7646, 1.7646]\big)^T,$$
$$\boldsymbol{y}^1 = \big([-0.1453, 0.1453],\ [-1.7646, 1.7646]\big)^T,$$

while the function `intgauss` (interval Gaussian elimination with mignitude pivoting) returns the enclosure

$$\boldsymbol{x}^2 = \big([-0.1373, 0.1373],\ [-1.7185, 1.7185]\big)^T,$$
$$\boldsymbol{y}^2 = \big([-0.1373, 0.1373],\ [-1.7185, 1.7185]\big)^T,$$

The Hansen-Bliek-Rohn-Ning-Kearfott method `hsolve` results with the same solution as `intgauss`

$$\boldsymbol{x}^3 = \big([-0.1373, 0.1373],\ [-1.7185, 1.7185]\big)^T,$$
$$\boldsymbol{y}^3 = \big([-0.1373, 0.1373],\ [-1.7185, 1.7185]\big)^T,$$

All the functions `intgauss`, `verifylss` and `hsolve` applied on real interval system (4) return the same intervals

$$\boldsymbol{x}^4 = \big([-0.1354, 0.1354],\ [-1.7724, 1.7724]\big)^T,$$
$$\boldsymbol{y}^4 = \big([-0.0954, 0.0954],\ [-0.6124, 0.6124]\big)^T,$$

The Householder method due to [4] returns

$$\boldsymbol{x}^5 = \big([-0.2086, 0.2086],\ [-2.1663, 2.1663]\big)^T,$$
$$\boldsymbol{y}^5 = \big([-0.11, 0.11],\ [-0.0607, 0.0607]\big)^T,$$

which is wrong (cf. Subsection 1.2), the width of $\boldsymbol{y}^5$ is too small. For instance $(-0.01, 0.1) \notin \boldsymbol{y}^5$, but

$$\begin{pmatrix} 5-i & 1 \\ 25 & 1-0.5\,i \end{pmatrix} \begin{pmatrix} 0.05-0.01\,i \\ -0.3+0.1\,i \end{pmatrix} = \begin{pmatrix} -0.06 \\ 1 \end{pmatrix}.$$

In our implementation of intervalization of complex Householder method we obtained $\boldsymbol{x}^6 = \mathbb{R}^2$, $\boldsymbol{y}^6 = \mathbb{R}^2$, hence the direct using of this method is very ineffective. The effective implementation producing narrower intervals is still open problem.

What is the result for *this* example? The best functions are `intgauss` and `hsolve`, but surprisingly the real outer approximation (4) is worse in only one variable and better is in three variables.

In the second part of this example we show how tight these enclosures are. We implemented the algorithm SIVIA (Table 1) in the programming language C++ supplemented by the C++ class library C-XSC 2.1.1 [8]. For $\varepsilon = 0.0001$ we obtained inner and outer approximations

$$\begin{aligned}
\underline{\boldsymbol{x}} &= \big([-0.1010, 0.1010],\ [-1.5338, 1.5338]\big)^T, \\
\underline{\boldsymbol{y}} &= \big([-0.0715, 0.0715],\ [-0.4200, 0.4200]\big)^T, \\
\overline{\boldsymbol{x}} &= \big([-0.1012, 0.1012],\ [-1.5363, 1.5363]\big)^T, \\
\overline{\boldsymbol{y}} &= \big([-0.0718, 0.0718],\ [-0.4221, 0.4221]\big)^T,
\end{aligned}$$

The interval approximation ratio (as introduced in Section 2) applied on $\boldsymbol{x}^3 + \boldsymbol{y}^3 i$ is

$$\boldsymbol{\rho}^3 = [1.7415, 1.7456],$$

and applied on $\boldsymbol{x}^4 + \boldsymbol{y}^4 i$ is as follows

$$\boldsymbol{\rho}^4 = [1.2272, 1.2301].$$

Hence solving the complex interval system (1) leads to the about 74% overestimation, the real form (4) gives only about 23% overestimation.

One could think that we can use the algorithm SIVIA directly to the complex interval system (14). But due to the equations, SIVIA generally does not return arbitrarily tight enclosures. For our example with $\varepsilon = 0.001$ and the starting enclosure $\boldsymbol{x}^4$, $\boldsymbol{y}^4$, we obtain very wide approximation $\overline{\boldsymbol{x}}^4 = \boldsymbol{x}^4$, $\overline{\boldsymbol{y}}^4 = \boldsymbol{y}^4$, $\underline{\boldsymbol{x}}^4 = \underline{\boldsymbol{y}}^4 = \emptyset$, i.e., no improvement happened (especially due to the noninterval vector $\boldsymbol{d}$).

Example 2. Figures 1 to 4 show real overestimation for four cases and selected methods; each case consists of seven computations. Vertical profile of the grey area represents an average solution interval computed by SIVIA (Table 1) with accuracy ε; dark grey belongs to $\underline{\boldsymbol{z}}$ and light grey to $\overline{\boldsymbol{z}}$. The difference between $\underline{\boldsymbol{z}}$ and $\overline{\boldsymbol{z}}$ is undistinguishable on the first two figures. The mark $*$ signifies the bounds of the average interval solution obtained by Hansen-Bliek-Rohn-Ning-Kearfott method applied on complex interval system (1). Hansen-Bliek-Rohn-Ning-Kearfott method applied on real interval system (4) results in average interval solution the bounds of which are marked by $\circ$. The corresponding average approximation ratios are denoted by $\boldsymbol{\rho}^*$, $\boldsymbol{\rho}^\circ$, respectively.

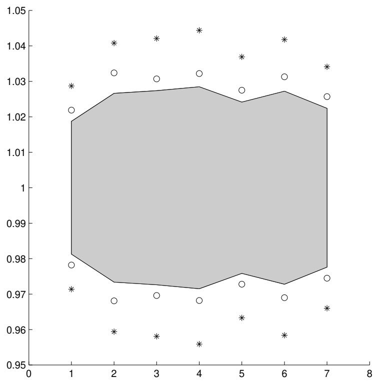


Figure 1: $n = 2$, $\varepsilon = 10^{-6}$, $\boldsymbol{\rho}^* = [1.5318, 1.5321]$, $\boldsymbol{\rho}^\circ = [1.1477, 1.1479]$.

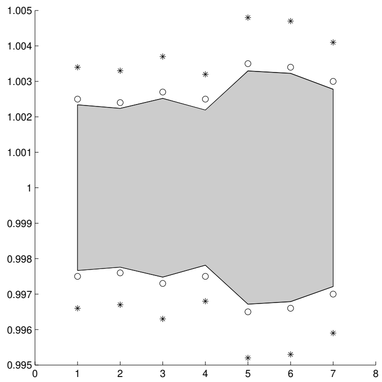


Figure 2: $n = 2$, $\varepsilon = 10^{-6}$, $\boldsymbol{\rho}^* = [1.4633, 1.4658]$, $\boldsymbol{\rho}^\circ = [1.0782, 1.0800]$.

By E_n we denote the $n \times n$ matrix of all ones, by I_n the identity matrix, and Random(n, n) stands for random $n \times n$ matrix whose elements are uniformly distributed in the interval (0,1). Then the interval matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and interval vectors $\boldsymbol{c}$, $\boldsymbol{d}$ are given in the following way.

Figure 1: $n = 2$, $\delta = 0.01$,
$A^c = \text{Random}(n, n) + 2E_n$, $B^c = \text{Random}(n, n) + 4E_n - 2I_n$,
$A^\Delta = B^\Delta = \delta E_n$,
$c^c = (A^c - B^c)\, e_n$, $d^c = (A^c + B^c)\, e_n$, $c^\Delta = d^\Delta = \delta e_n$;

Figure 2: Like situation 1., but $\delta = 0.001$.

Figure 3: $n = 4$, $\delta = 0.1$,
$A^c = \text{Random}(n, n) + E_n + 4I_n$, $B^c = \text{Random}(n, n) + 5I_n$,

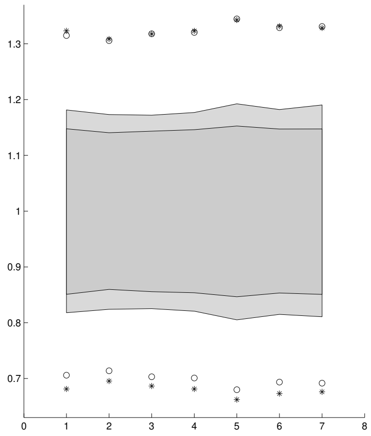


Figure 3: $n = 4$, $\varepsilon = 10^{-2}$, $\boldsymbol{\rho}^* = [1.7538, 2.1997]$, $\boldsymbol{\rho}^\circ = [1.6974, 2.1290]$.

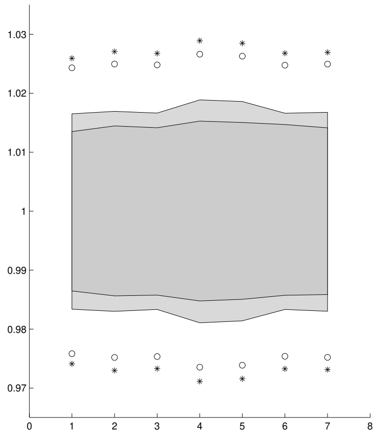


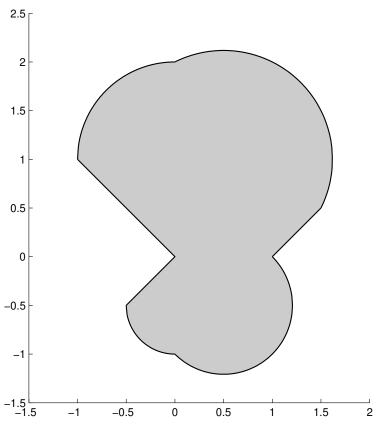
Figure 4: $n = 4$, $\varepsilon = 10^{-3}$, $\boldsymbol{\rho}^* = [1.5751, 1.8886]$, $\boldsymbol{\rho}^\circ = [1.4552, 1.7448]$.

$$A^\Delta = B^\Delta = \delta E_n,$$
$$c^c = (A^c - B^c)\, e_n,\ d^c = (A^c + B^c)\, e_n,\ c^\Delta = d^\Delta = \delta e_n;$$

Figure 4: Like situation 3., but $\delta = 0.01$.

The results were carried out on `PC i686, AMD Athlon(tm) 64 Processor 4400+, 2.2 GHz, 884 MB RAM, GNU/Linux`, and the source code was written again in C++ with C-XSC 2.1.1 [8]. In the first and second case, each example took from one to five hours CPU time. The last two examples were much more time consuming and we had to do some improvements: We used the SIVIA method only to compute $\overline{\boldsymbol{z}}$; the inner approximation $\underline{\boldsymbol{z}}$ was obtained by a Monte Carlo method. Nevertheless, each example took about ten days of CPU time (due to $\overline{\boldsymbol{z}}$).

Example 3. Figures 5 to 8 show solution sets images for 1×1 complex interval systems. The boundary of solution set is always formed by lines and circular arcs. Computations were carried out in MATLAB 7.0.4 (R14.2) with the toolbox INTLAB v5.3 [14].



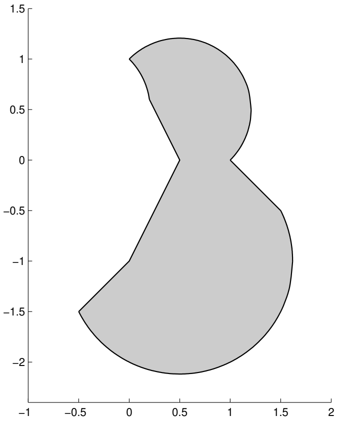


Figure 5: Solution set for the equation
$[1-i, 5+i]z = [-i, 1+2i]$.

Figure 6: Solution set for the equation
$[-1+i, 1+2i]z = [-1+i, 2+i]$.

4 Conclusion

We described the solution set for systems of complex interval systems of equations (rectangular case) and showed how it can be used for checking algorithmic efficiency. We presented several examples which imply that overestimation for the best algorithms is between 5% and 15% for the two dimensional cases, and between 50% and 110% for the four dimensional cases.

Numerical experiments also showed that the SIVIA method has a tremendous time complexity. Developing more efficient appoximation methods is a challenge for the future.

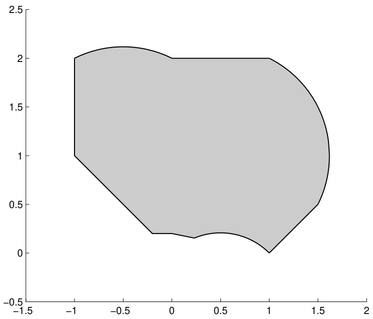


Figure 7: Solution set for the equation
$[1, 5+i]z = [-1+i, 1+2i]$.

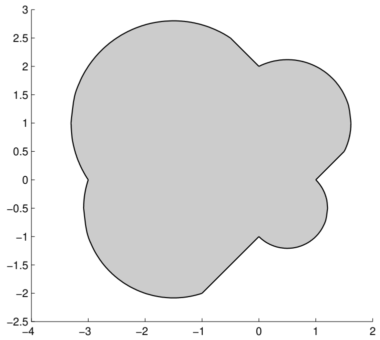


Figure 8: Solution set for the equation
$[1-i, 5+i]z = [-3-i, 1+2i]$.

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