Eigenvalues of scale free graphs

Silvia Gago *††

Abstract

Scale free graphs can be found very often as models of real networks and are characterized by a power law degree distribution, that is, for a constant $\gamma \geq 1$ the number of vertices of degree d is proportional to $d^{-\gamma}$. Experimental studies show that the eigenvalue distribution also follows a power law for the highest eigenvalues. Hence it has been conjectured that the power law of the degrees determines the power law of the eigenvalues. In this paper we show that we can construct a scale free graph with non highest eigenvalue power law distribution. For $\gamma = 1$ we can construct a scale free graph with small spectrum and a regular graph with eigenvalue power law distribution.

1 Introduction

In general the distribution of the eigenvalues of a graph has no relation with the distribution of its degrees. We can find graphs with the same degree distribution and different eigenvalues and cospectral graphs with different degrees. However the recent discovery of power law distributions in real graphs has stimulated a new interest about their relation. A graph with a power law degree distribution is called scale free, that is, for a constant $\gamma \geq 1$ the number of vertices of degree d is proportional to $d^{-\gamma}$. In 1999 Faloutsos et al. made an experimental study of a part of the real Internet graph [4], finding a power law distribution for the degrees respect to their multiplicities and a power law distribution for the highest eigenvalues respect to their order.

*Universitat Politècnica de Catalunya, Department de Matemàtica Aplicada IV, Avda. Canal Olímpic s/n, 08860 Castelldefels, Barcelona, Spain, sgago@ma4.upc.edu

†Charles University, Institute for Theoretical Computer Science and Department for Applied Mathematics, Malostranské nám. 25, 118 00 Praha, Czech Republic, research supported by the European project AEOLUS

As a consequence, they conjectured a power law eigenvalue distribution for scale free graphs. The spectra of some random and deterministic models proposed for scale free graphs have been also studied, obtaining a power law distribution for the highest eigenvalues too [1, 2, 5, 7, 8]. Therefore, there seems to exist a natural relation between both distributions for scale free graphs.
However, in this paper we are going to see that this is not true for certain classes of scale free graphs. In Section 3 it is shown that the Cartesian product of a scale free graph with power law eigenvalue distribution with a regular graph is a scale free graph without eigenvalue power law distribution. Moreover, in Section 4 we construct a scale free graph with $\gamma = 1$ with a small spectrum (only three positive eigenvalues). In the last section, we construct a regular graph with an eigenvalue power law distribution, showing that the inverse of the conjecture is also false.

2 Notation and first results

If $G = (V, E)$ is a simple connected graph, let $\boldsymbol{A}$ denote its adjacency matrix. The spectrum of the graph is the spectrum of its adjacency matrix, and we denote it by $\mathrm{sp}(G) = \mathrm{sp}(\boldsymbol{A}) = \{\lambda_1^{[m_1]}, \ldots, \lambda_d^{[m_d]}\}$, where m_i are the corresponding multiplicities of the eigenvalues λ_i, $1 \le i \le d$. We are going to introduce a similar notation for the degree distribution of a graph G, considering the degrees in increasing order and together with their multiplicities:

$$DD(G) = \{d_1^{[n_1]}, \ldots, d_k^{[n_k]}\}.$$

The following results and notation can be found in [3]. The direct product $G \times H$ of the graphs G and H, is a graph with $V(G \times H) = V(G) \times V(H)$, and two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if u' is adjacent to v' and u is adjacent to v. It is also called tensor product, categorial product, cardinal product or Kronecker product. The adjacency matrix of the direct product of two graphs is $\boldsymbol{A}_{G \times H} = \boldsymbol{A}_G \otimes \boldsymbol{A}_H$, and if $\mathrm{sp}(G) = \{\lambda_1, \ldots, \lambda_r\}$ and $\mathrm{sp}(H) = \{\mu_1, \ldots, \mu_s\}$, then its spectrum is

$$\mathrm{sp}(G \times H) = \{\lambda_i \mu_j,\ 1 \le i \le r,\ 1 \le i \le s\}. \tag{1}$$

The Cartesian product $G \,\square\, H$ of the graphs G and H is the graph such that $V(G \,\square\, H) = V(G) \times V(H)$ and two vertices (u, u') and (v, v') are adjacent in $G \,\square\, H$ if and only if either $u = v$ and u' is adjacent to v' , or $u' = v'$ and

u is adjacent to v. The adjacency matrix of the Cartesian product is the sum of Kronecker of their matrices, $\boldsymbol{A}_{G\,\square\,H} = \boldsymbol{A}_H \otimes \boldsymbol{I}_r + \boldsymbol{I}_s \otimes \boldsymbol{A}_G$, and so if $\mathrm{sp}(G) = \{\lambda_1, \ldots, \lambda_r\}$ and $\mathrm{sp}(H) = \{\mu_1, \ldots, \mu_s\}$, then

$$\mathrm{sp}(G\,\square\,H) = \{\lambda_i + \mu_j,\ 1 \leq i \leq r,\ 1 \leq i \leq s\,\}. \tag{2}$$

The degree distributions of the direct and Cartesian products of two graphs are useful for constructing our scale free graphs.

Lemma 2.1. *Let* G_1, G_2 *be two graphs with* $DD(G_1) = \{d_1^{[n_1]}, \ldots, d_k^{[n_k]}\}$, $DD(G_2) = \{\delta_1^{[m_1]}, \ldots, \delta_l^{[m_l]}\}$ *respectively, then the graph* $G_1 \times G_2$ *has degree distribution*

$$DD(G_1 \times G_2) = \{(d_i \delta_j)^{[n_i m_j]},\ 1 \leq i \leq k,\ 1 \leq j \leq l\}. \tag{3}$$

And for the Cartesian product the resulting degree distribution is:

Lemma 2.2. *Let* G_1, G_2 *be two graphs with* $DD(G_1) = \{d_1^{[n_1]}, \ldots, d_k^{[n_k]}\}$, $DD(G_2) = \{\delta_1^{[m_1]}, \ldots, \delta_k^{[m_k]}\}$ *respectively, then the graph* $G_1\,\square\,G_2$ *has degree distribution*

$$DD(G_1\,\square\,G_2) = \{(d_i + \delta_j)^{[n_i m_j]},\ 1 \leq i \leq k,\ 1 \leq j \leq l\}. \tag{4}$$

These lemmas can be easily verified by summing the rows or the columns of the corresponding adjacency matrices.

3 Scale free graphs and Cartesian products

If we consider a scale free graph G with eigenvalue power law distribution $\mathrm{sp}(G) = \{\lambda_i = K' i^{-\gamma'},\ 1 \leq i \leq n\}$, and we take the Cartesian product of G with a regular graph H, the following proposition tell us that in some cases the power law distribution of the degrees is preserved in the resulting graph.

Proposition 3.1. *Let* G *be a scale free graph of order* n *and let* H *be a* δ*-regular graph of order* m*, then if* δ *is small enough the graph* $G\,\square\,H$ *is also scale free.*

Proof. If $DD(G) = \{d_i^{[n_{d_i}]} : n_{d_i} = K d_i^{-\gamma}, 1 \leq i \leq k\}$ and $DD(H) = \{\delta^{[m]}\}$, we can apply Lemma 2.2 to show that the degree distribution of the resulting graph is

$$DD(G\,\square\,H) = \{(d_1 + \delta)^{[m n_{d_1}]}, \ldots, (d_k + \delta)^{[m n_{d_k}]} :\ n_{d_i} = K d_i^{-\gamma},\ 1 \leq i \leq k\}$$

Suppose that $DD(G \square H)$ has a power law degree distribution. The multiplicities $n_{d_i+\delta} = K'(d_i+\delta)^{-\gamma'}$, $1 \le i \le k$ for suitable constants K' and γ'. As $n_{d_i+\delta} = mn_{d_i} = mKd_i^{-\gamma}$, we see that $K' = mK\big(1+\delta/d_i\big)^{\gamma}$ and $\gamma = \gamma'$. Therefore, if δ is small with respect to the largest degrees, the graph is scale free. $\square$

On the other hand, the power law distribution of the highest eigenvalues is not always preserved by the Cartesian product. If the distribution of eigenvalues of the regular graph is $\mathrm{sp}(H) = \{\mu_1, \ldots, \mu_s\}$, then applying (2) yields

$$\mathrm{sp}(G \square H) = \{K' i^{-\gamma'} + \mu_j,\ 1 \le i \le r,\ 1 \le j \le s\}.$$

The distribution of the Cartesian product depends on the distance between the eigenvalues of the regular graph, and thus the highest eigenvalues of the resulting graph might not to follow a power law. For instance, we consider a scale free graph G with $n = 10000$ vertices, power law eigenvalue distribution with $\gamma' = 2.3/2 = 1.15$ (which might happens in real scale free network) and $K' = 3000$, and we take the Cartesian product of G with a complete graph K_{30} (with $\mathrm{sp}(K_{30}) = \{29^{[1]}, -1^{[29]}\}$). The resulting graph has 300000 vertices. The power law distribution of the degrees has not change very much, so it is still a scale free graph. But in Figure 3 we can observe that the distribution of the highest eigenvalues does not follow a power law.

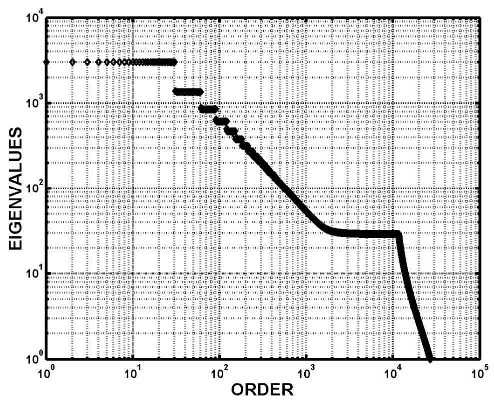

Figure 1: The highest eigenvalues of the Cartesian product of a scale free graph of $n = 10000$ vertices with a K_{30} have not power law distribution.

4 Scale free graphs with small spectrum

The construction of a scale free graph with parameter $\gamma = 1$ and small spectrum is based on the description of its adjacency matrix. First consider a graph SP_n by making the Cartesian product of star graphs $S_2, \ldots, S_n$, whose degree distribution follows exactly a power law. The adjacency matrix of this graph is $\boldsymbol{SP}_n = \boldsymbol{S}_n \otimes \ldots \otimes \boldsymbol{S}_2$. The degree distribution follows a perfect power law with $\gamma = 1$.

Proposition 4.1. *The degree distribution of SP_{n+1} is*

$$DD(SP_{n+1}) = \{d_\beta^{[n_{d_\beta}]} : d = 1^{\beta_1} 2^{\beta_2} \cdots n^{\beta_n},\ \beta = (\beta_1, \ldots, \beta_n),\ \beta_i \in \{0,1\}, \\ n_{d_\beta} = (2n!)/d_\beta\}. \quad (5)$$

Proof. The degree distribution of a star graph is $DD(S_{n+1}) = \{1^{[n]}, n^{[1]}\}$. Thus applying Lemma 2.1 we obtain

$$DD(SP_3) = DD(S_3 \,\square\, S_2) = \{1^{[2\cdot 2]}, 2^{[1\cdot 2]}\}.$$

Observe that each element $d^{[n_d]}$ of $DD(SP_n)$ gives rise to two elements $n \cdot d^{[n_d]}$ and $d^{[n \cdot n_d]}$ in $DD(SP_{n+1})$. Therefore, as $DD(S_n) = \{1^{[n]}, n^{[1]}\}$, the product of each degree by its multiplicity is constant. Moreover, as there are $2n!$ leaves in SP_{n+1}, this product is $2n!$. Hence we get the result. Note that the distribution follows a power law with $\gamma = 1$. $\square$

The spectrum of SP_n is easy to compute as it is the product of all the eigenvalues of the stars (Equation (1)), hence we get

$$\mathrm{sp}(SP_n) = \{\sqrt{n!}^{[2^n]}, -\sqrt{n!}^{[2^n]}, 0^{[n! - 2^{n+1}]}\}.$$

The degree distribution of this graph follows a perfect power law, but the problem is that this graph is not connected. To solve this problem, we connect the first $n! - 1$ vertices of the graph to the first one, forming a star. We obtain a connected graph SF_{n+1} whith adjacency matrix

$$\begin{pmatrix} \boldsymbol{S}_{n!} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix}, \quad \boldsymbol{B} = \overbrace{(\boldsymbol{SP}_n \mid \ldots \mid \boldsymbol{SP}_n)}^{n}.$$

When we calculate the degree distribution of the new graph to see if it preserves the power law, we get

$$DD(SF_{n+1}) = \begin{cases} (3n! - 1)^{[1]} & \text{if } \beta_i = 1\ \forall\, i, \\ (2^{\beta_2} \cdots n^{\beta_n})^{[n'_d]},\ n'_d = \frac{2n!}{d} + \beta_n & \text{otherwise.} \end{cases}$$

Observe that the distribution of the new graph is very similar to the previous one, and for large values of n the multiplicities $n'_d \approx n_d$. Therefore we can assure that SF_{n+1} is a scale free graph.
The spectrum of SF_{n+1} is characterized in the following theorem.

Theorem 4.2. *The characteristic polynomial of the graph SF_{n+1} is*

$$\Phi_{SF_{n+1}}(\lambda) = \lambda^{n!-2^n}[\lambda^2 - n \cdot n!]^{2^n-4}[\lambda^2 + \sqrt{n!-1}\lambda - n \cdot n!][\lambda^2 - \sqrt{n!-1}\lambda - n \cdot n!].$$

Proof. After $n+1$ steps of the inductive construction the adjacency matrix of the graph has order $(n+1)! \times (n+1)!$, and the characteristical system is

$$\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} \boldsymbol{S}_{n!} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix} \boldsymbol{x} = \lambda \boldsymbol{x}, \tag{6}$$

where $\boldsymbol{B} = (\boldsymbol{SP}_n, \dots, \boldsymbol{SP}_n)$ and $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n+1})$.
First of all we are going to prove that $\mathrm{Ker}(\boldsymbol{SP}_n) \subset \mathrm{Ker}(\boldsymbol{S}_{n!})$. The spectrum of a star graph on a vertices is $\mathrm{sp}(\boldsymbol{S}_a) = \{\sqrt{a}^{[1]}, -\sqrt{a}^{[1]}, 0^{[a-2]}\}$, and the kernel is formed by vectors characterized by having the first coordinate and the sum of the others equal to 0. A base of the kernel could be

$$\mathrm{Ker}(\boldsymbol{S}_a) = \{\boldsymbol{\phi}_i^a = [0, 1, 0, \dots, \overbrace{-1}^{i}, \dots, 0], \quad 3 \le i \le a\}.$$

The matrix $\boldsymbol{S}_a^2$ has the same eigenvectors as $\boldsymbol{S}_a$, and the eigenvalues of $\boldsymbol{S}_a^2$ are the square of the eigenvalues of $\boldsymbol{S}_a$. Therefore the kernels are equal, and $\boldsymbol{S}_a^2$ has only one non-zero eigenvalue a of multiplicity 2. On the other hand, the eigenvalues of a Kronecker product of n matrices $\boldsymbol{SP}_n^2 = \boldsymbol{S}_n^2 \otimes \cdots \otimes \boldsymbol{S}_2^2$ are all the possible products of the eigenvalues of the stars, and thus

$$\mathrm{sp}(\boldsymbol{SP}_n) = \{n!^{[2^n]}, 0^{[n!-2^n]}\}.$$

The corresponding eigenvectors are also the Kronecker product of the corresponding eigenvectors, so that

$$\mathrm{Ker}(\boldsymbol{SP}_n^2) = \{\boldsymbol{\psi}_j^n = \boldsymbol{\phi}_i^n \otimes \cdots \otimes \boldsymbol{\phi}_i^2,\ 3 \le i \le n,\ 2^n \le j \le n! - 2^n\}.$$

$$\boldsymbol{\psi}_j^n = \boldsymbol{\phi}_{i_n}^n \otimes \boldsymbol{\psi}_j^{n-1} = [\boldsymbol{O}_{(n-1)!}, \boldsymbol{\psi}_j^{n-1}, \boldsymbol{O}_{(n-1)!}, \dots, \overbrace{-\boldsymbol{\psi}_j^{n-1}}^{i_n}, \dots, \boldsymbol{O}_{(n-1)!}].$$

Observe that the first coordinate of $\boldsymbol{\psi}_j^n$ is zero and the sum of the orders too, therefore $\boldsymbol{\psi}_j^n \in \mathrm{Ker}(\boldsymbol{S}_{n!})$. So if $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{SP}_n) = \mathrm{Ker}(\boldsymbol{SP}_n^2)$, then $\boldsymbol{x}$ is a linear combination of $\boldsymbol{\psi}_j^n$. Therefore $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{S}_{n!})$.

Using this result is easy to verify that $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{A}) \Leftrightarrow \boldsymbol{x}_1 \in \mathrm{Ker}(\boldsymbol{SP}_n)$. Now consider the general equation (6) in the form

$$\begin{aligned} \boldsymbol{S}_{n!}\boldsymbol{x}_1 + \boldsymbol{SP}_n\Big(\sum_{i=2}^{n+1}\boldsymbol{x}_i\Big) &= \lambda\boldsymbol{x}_1 && (7) \\ \boldsymbol{SP}_n\boldsymbol{x}_1 &= \lambda\boldsymbol{x}_i, \qquad 2 \le i \le n+1. && (8) \end{aligned}$$

If $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{A})$, both equations [7] and [8] are zero, and from the second we get that $\boldsymbol{x}_1 \in \mathrm{Ker}(\boldsymbol{SP}_n)$. If $\boldsymbol{x}_1 \in \mathrm{Ker}(\boldsymbol{SP}_n) \subset \mathrm{Ker}(\boldsymbol{S}_{n!})$, then

$$\begin{aligned} \boldsymbol{SP}_n\Big(\sum_{i=2}^{n+1}\boldsymbol{x}_i\Big) &= \lambda\boldsymbol{x}_1 && (9) \\ 0 &= \lambda\boldsymbol{x}_i, \qquad 2 \le i \le n+1. && (10) \end{aligned}$$

Assuming that $\lambda \neq 0$, we can multiply the Equation (9) for λ, and use the second equation to get $0 = \lambda^2\boldsymbol{x}_1$. Hence λ must be zero, and therefore $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{A})$. Note also that $dim(\mathrm{Ker}(\boldsymbol{A})) = dim(\mathrm{Ker}(\boldsymbol{SP}_n)) = n! - 2^n$.
If $\boldsymbol{x} \notin \mathrm{Ker}(\boldsymbol{SP}_n)$, $\boldsymbol{x}$ must be an eigenvector associated to the eigenvalue $n!$ and from Equation (8) we can isolate each x_i and substitute it in Equation (7) to get

$$\lambda\boldsymbol{S}_{n!}\boldsymbol{x}_1 + n \cdot n!\boldsymbol{x}_1 = \lambda^2\boldsymbol{x}_1. \qquad (11)$$

Now we consider two cases. First suppose that $\boldsymbol{x}_1 \notin \mathrm{Ker}(\boldsymbol{S}_{n!})$, then $\boldsymbol{x}_1$ must be an eigenvector associated to either of the eigenvalues $\pm\sqrt{n!-1}$, and thus (11) can be arranged as

$$\lambda^2 \pm \sqrt{n!-1}\lambda - n \cdot n! = 0.$$

Note that from this equation we get four different eigenvalues. Second, suppose that $\boldsymbol{x}_1 \in \mathrm{Ker}(\boldsymbol{S}_{n!})$ the equation to solve is $n \cdot n!\,\boldsymbol{x}_1 = \lambda^2\boldsymbol{x}_1$, which leads to the two last equations

$$\lambda \pm \sqrt{n \cdot n!} = 0.$$

The multiplicity of each of these eigenvalues is $2^{n-1} - 2$. $\square$

5 Regular graphs with eigenvalues power law distribution

As we have seen in the previous section, the power law distribution of the degrees does not determine the power law distribution of the highest eigenvalues of the graph. As we will see now, the converse is also not true. To this end, we construct a regular graph with an eigenvalue power law distribution. The construction of this graph is very simple. We just consider direct products of complete graphs K_n. The degree distribution of a complete graph is $DD(K_n) = \{(n-1)^{[n]}\}$, and the spectrum is

$$\mathrm{sp}(K_n) = \{(n-1)^{[1]}, -1^{[n-1]}\}.$$

Proposition 5.1. *The graph $B_{n+1} = K_{n+1} \times \cdots \times K_2$ is $n!$-regular and its spectrum follows a power law with $\gamma = 1$. ($n \geq 2$)*

We omit this proof, as it is is very similar to the one of Lemma 4.1.

Acknowledgment

This research was supported by the Secretaria de Estado de Universidades e Investigación (Ministerio de Educación y Ciencia), Spain, and the European Regional Development Fund (ERDF) under project TEC2005-03575 and the European project AEOLUS.

References

References

[1] Chung, F., Lu, L., Vu, V.: Eigenvalues of random power law graphs. Annals of Combinatorics **7** (2003) 21–33.

[2] Comellas, F., Gago, S.: A star-based model for the eigenvalue power law of Internet graphs. *Phys. A* **351** (2005) 680–686

[3] Cvetkovic, D., Doob, M., Sachs, H.: Spectra of Graphs. *Academic Press* (1979) New York

[4] Faloutsos, M., Faloutsos, P., Faloutsos, C.: On power law relationships of the internet topology. *SIGCOMM* (1999) 251–262

[5] Flaxman, A., Frieze, A., Fenner, T.: High degree vertices and eigenvalues in the preferential attachment graph, *Internet Math.* Vol. 2 **1** (2005) 1–19.

[6] Godsil, C., Royle, G.: Algebraic Graph Theory. *Springer-Verlag* (2001) New York

[7] Goh, K.I., Kahng, B., Kim, D.: Spectra and eigenvectors of scale free networks. *Phys. Rev. E* **64** (2001) 051903

[8] Mihail, M., Papadimitriou, C. H.: On the eigenvalue power law. *Proc. of 6th Int. Workshop on Randomization and Approximation Techniques* (2002) 254–262