On Ramsey-type Positional Games

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Abstract

Beck introduced the concept of Ramsey games by studying the game versions of Ramsey and van der Waerden theorems. We contribute to this topic by investigating games corresponding to structural extensions of Ramsey and van der Waerden theorems—the theorem of Brauer, structural and restricted Ramsey theorems.

1 Introduction

Ramsey theory deals with the statements of the following type: For every partition $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$ of the set $\binom{C}{A}$ of all substructures of C which are isomorphic to A, there exists a substructure B of C such that the set $\binom{B}{A}$ belongs to one class of the partition. This definition of course assumes that we make precise notions of the structure and of the substructure. The validity of the previous statement is denoted by $C \to (B)^A_k$. Every $B' \in \binom{C}{B}$ is called a *copy* of B in C.

The classical Ramsey theorem in this setting claims that for all integers k, n, p there exists an integer N such that

$$K_N \to (K_n)^{K_p}_k.$$

The previous statement is shortly denoted by $N \to (n)^p_k$, which is the original Erdős-Rado partition arrow.

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In Ramsey theory one tries to prove the validity of statement $C \to (B)^A_k$ for various combinatorial, number theoretical and geometrical structures. For a good survey on this topic, see eg. [9] or [11].

Another question, which is intensively studied, is motivated by efforts to find, for a given A, B and k, the *minimal size* of the structure C satisfying $C \to (B)^A_k$. Denote by *$\mathcal{C}$-Ramsey number* $r_{\mathcal{C}}(A, B, k)$ the minimal size of $C \in \mathcal{C}$ which satisfies $C \to (B)^A_k$ in a fixed class $\mathcal{C}$ of structures where all the objects A, B, C are considered (we tacitly assume $C \in \mathcal{C}$ exists; otherwise we put $r_{\mathcal{C}}(A, B, k) = \infty$).

These questions seem to be very difficult even in the simplest instances, such as Ramsey theorem. In this case

$$2^{n/2} \leq r_{\mathcal{K}}(K_2, K_n, 2) \leq 2^{2n},$$

where $\mathcal{K}$ is the class of all complete graphs. This leads to tower function growth for numbers $r_{\mathcal{K}}(K_p, K_n, k)$.

For other structures, such as other classes of graphs, hypergraphs (with induced subgraphs and subhypergraphs), arithmetics progressions (Van der Waerden theorem), combinatorial cubes (Hales-Jewett theorem), the situation is much less satisfactory and in most instances one is satisfied with the existence of an object C, without trying to optimise its size, which seems to be extremely large.

Let $\Delta = (\delta_i;\ i \in I)$ be an integer sequence called *type*. An *ordered relational structure S of type* Δ is a tuple $S = (X, (R_i;\ i \in I))$ where X is an ordered set and $R_i \subseteq X^{\delta_i}$ (i.e. R_i is a δ_i-ary relation); we denote $V(S) = X$. A structure $S' = (X', Y')$ is a *substructure* of structure $S = (X, Y)$ if $X' \subseteq X$, $Y' \subseteq Y$, $Y' \subseteq 2^{X'}$ and X' preserves the ordering of X. A class $\mathcal{C}$ of structures is called *Ramsey class* if for every $A, B \in \mathcal{C}$ and every k there exists $C \in \mathcal{C}$ such that $C \to (B)^A_k$. Let us list few examples of Ramsey classes.

For a fixed Δ, we shall consider the class $\mathrm{Rel}(\Delta)$ of all finite ordered relational structures of type Δ. A structure $A = (X, (R_i;\ i \in I))$ of type Δ is called *irreducible,* if for every pair $x, y \in X$ there exist $i \in I$ and $R \in R_i$ such that $x, y \in R$. Let $\mathcal{F}$ be a (possibly infinite) set of structures of type Δ. Denote by $\mathrm{Forb}_\Delta(\mathcal{F})$ the class of all ordered structures A of type Δ which do not contain any member of $\mathcal{F}$ as a substructure (not necessarily induced).

Theorem 1.1. *(Nešetřil, Rödl [13, 15]) Let Δ be a type and let $\mathcal{F}$ be a (possibly infinite) set of irreducible structures of type Δ. Then the classes*

$\mathrm{Rel}(\Delta)$ *and* $\mathrm{Forb}_\Delta(\mathcal{F})$ *are Ramsey.*

Jószef Beck initiated a systematic study of Ramsey numbers in a setting of combinatorial games. He showed that the game versions of Ramsey number are much more easier to estimate. Particularly, for the case of Ramsey theorem and Van der Waerden theorem, he obtained asymptotically optimal results ([2], [3], see also [4]).

Let us now turn a general Ramsey-type theorem into a game. (This transformation is contained already in one of the earliest paper of Ramsey theory [10] by Hales and Jewett where the authors interpreted Hales-Jewett theorem.) We consider a structure C as a board. There are two players, **I** and **II**. **I** is called *maker*, **II** is called *breaker*. The players alternately pick substructures $A' \in \binom{C}{A}$. **I** wins if and only if he succeeds to find $B' \in \binom{C}{B}$ such that the whole set $\binom{B'}{A}$ is claimed by him. Otherwise **II** wins. We call this game *Ramsey* (A, B)*-game* on C and we denote the fact that **I** wins by

$$C \xrightarrow{g} (B)^A.$$

It follows by a general Strategy Stealing argument (Theorem 2.1) that **I** wins providing $C \to (B)^A_2$. However, this is far from necessary. To clarify this, let us denote by $r^g_{\mathcal{C}}(A, B)$ the minimal size of $C \in \mathcal{C}$ measured by $|V(C)|$ satisfying that **I** wins Ramsey (A, B)-game on C (providing such a C exists). It appears that in most cases we can claim that $r^g_{\mathcal{C}}(A, B)$ has a moderate size. This phenomenon was already exhibited in 1981 by Beck in a landmark paper [2] and in 2002 in [3]:

Theorem 1.2. *Consider the game version of Van der Waerden theorem and let* $r^g(AP(n))$ *denote the minimum size* N *such that the player* **I** *wins the game of building a arithmetic progression of length* n *on the set* $\{1, \dots, N\}$*. Then*

$$\lim_{n\to\infty} \sqrt[n]{r^g(AP(n))} = 2.$$

Theorem 1.3. *Let us consider the Ramsey* (E_p, K_n)*-game in the class* $\mathcal{K}^p$ *of all* p*-uniform complete hypergraphs, where* E_p *is a hypergraph edge, and let the board be the hypergraph* $K_N \in \mathcal{K}^p$*. In case* $p = 2$ *(graphs), if*

$$n \geq 2\log_2 N - 2\log_2\log_2 N + 2\log_2 e - 1 + o(1),$$

then breaker has an explicit winning strategy. On the other hand, if

$$n \leq 2\log_2 N - 2\log_2\log_2 N + 2\log_2 e - \frac{10}{3} + o(1),$$

then maker has an explicit winning strategy. In case $p \geq 3$, breaker wins if

$$n \geq (p! \log_2 N)^{\frac{1}{p-1}} + o(1),$$

and maker wins if

$$n \leq (p! \log_2 N)^{\frac{1}{p-1}} - O(1).$$

This should be compared with the bound for Ramsey function mentioned earlier. For Theorem 1.2, let us just recall that the Van der Waerden function is known to be primitive recursive (as shown first by Shelah [17]) and the two tower function bound was obtained more recently by Gowers [8]. Nevertheless, the lower bound is exponential only. Thus the game Ramsey function may be drastically smaller than the Ramsey version.

In this paper we generalise the results of Beck to $\mathcal{C}$-Ramsey numbers for relational structures. The main message of these results is that the game version of $\mathcal{C}$-Ramsey numbers may be essentially smaller than the extremely large $\mathcal{C}$-Ramsey numbers. And sometimes game Ramsey numbers exist even in situation when Ramsey-type results are not true.

Let $S(X, (R_i;\ i \in I))$ be a structure of type Δ. The *inflation of $x \in X$ by a factor k* is the structure $S' = (X', (R'_i;\ i \in I))$ of type Δ defined as follows:

$$\begin{aligned} X' &= \{y \in X;\ y \neq x\} \cup V_x, \quad V_x = \{y^x_1, y^x_2, \ldots, y^x_k\}, \\ R'_i &= \{\{y^x, y_1, \ldots, y_{\delta_i - 1}\};\ y^x \in V_x,\ \{y, y_1, \ldots, y_{\delta_i - 1}\} \in R_i\}, \quad i \in I. \end{aligned}$$

The set V_x is called *multivertex*. The *inflation of S by a factor k* is the structure S^k such that every $x \in X$ was inflated by k.

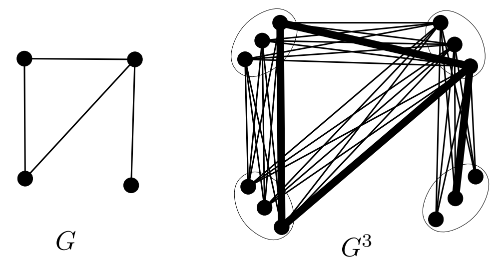

Figure 1: Example of inflating the graph G by factor 3.

We show the following.

Theorem 1.4. *Let $\mathcal{C}$ be a class of structures which is closed on vertex inflation. Then for every $B \in \mathcal{C}$ and every $A \subset B$, $|V(A)| < |V(B)|$, there exists $C \in \mathcal{C}$ such that $C \xrightarrow{g} (B)^A$. Moreover, $|V(C)| \leq 2^u \cdot u \cdot |V(B)|$ where $u = \left|\binom{B}{A}\right|$. Particularly, $r^g_{\mathcal{C}}(A, B) \leq 2^u \cdot u \cdot |V(B)|$.*

The condition of inflation holds for any class $\mathcal{C}$ of structures which is determined by a finite set of forbidden homomorphisms:

$$\mathcal{C} = \{G;\ F_i \not\to G,\ i = 1, \ldots, t\}$$

This covers for example class of K_k-free graphs, which is known to be a Ramsey class, and thus C satisfying $C \to (B)^A_k$ can be applied in Theorem 1.4.

As mentioned before, Theorem 1.4 covers also cases which are known to be not Ramsey. Denote by $\mathcal{G}$ the class of all undirected graphs. For example there is no (unordered) graph G satisfying

$$G \to (C_5)^{K_{1,2}}_2,$$

and thus $r_{\mathcal{G}}(K_{1,2}, C_5, 2) = \infty$. On the other hand, $r^g_{\mathcal{G}}(K_{1,2}, C_5)$ exists and it is $r^g_{\mathcal{G}}(K_{1,2}, C_5) \leq 800$.

Analogously with the definition of Ramsey classes, we can define game Ramsey classes. A class $\mathcal{C}$ of structures is called *game Ramsey class* if for every $A, B \in \mathcal{C}$ there exists $C \in \mathcal{C}$ such that $C \xrightarrow{g} (B)^A_k$.

Corollary 1.5. *Let $\mathcal{C}$ be a class of structures which is closed on vertex inflation. Then $\mathcal{C}$ is game Ramsey class.*

Proof. Given $A, B \in \mathcal{C}$, let B' be an large enough inflation of B such that $|V(A)| < |V(B')|$. Then we can apply Theorem 1.4 on A, B' and since $B \subset B'$, the corollary follows. □

In fact, $r_{\mathcal{G}}(G, H)$ exists for any graphs G, H and thus $\mathcal{G}$ is game Ramsey class.

This should serve as a warm up to Theorem 1.4 and further examples of structural Ramsey theorem whose game versions we shall consider. Our examples include restricted Ramsey theorems for set systems and extended versions of Van der Waerden's theorem (Brauer's theorem).

This also leads to challenging problems. Perhaps the most interesting is the question whether these results can be modified to obtain results for strong Ramsey theory games. A strong game is defined by a change of the winning criterion: the first player wins who achieves a monochromatic copy of structure B. This game may result in a draw.

The strong game is much harder to analyse and presently there is no analogy of Theorem 1.4 for strong games. Nevertheless, we show a peculiar result:

Theorem 1.6. *There exists a graph G with the following properties:*

1. *G does not contain K_4,*
2. *the first player wins strong (K_2, K_3)-game on G,*
3. *G has 8 vertices.*

This result is more interesting in its context than its proof (a case analysis, which seems to be typical for the analysis of strong games). The question of existence of K_4-free graph G satisfying $G \rightarrow (K_3)_2^{K_2}$ was fully solved by Nešetřil and Rödl [12]. Erdős asked whether there is such a G of size less than 10^{10}. Spencer [18] has shown the currently best known upper bound on size of G, which is less than 3.10^8.

2 Game theory facts

We define in general two kinds of *positional games*, weak games and strong games, on a finite hypergraph $H = (V, F)$, $F \subseteq 2^V$. We call the set V the *board* of the game. The edges F we prefer to call the *winning sets*.

A *strong game* on H is the following game. Players **I** and **II** alternately occupy previously unoccupied points of the board V, one point per move. That player wins who occupies all points of some edge $A \in F$ *first;* otherwise the play ends in a draw.

A *weak game* on H is similar to strong game, with the exception of the winning criterion: The player **I** wins if he occupies all points of some edge $A \in F$; otherwise player **II** wins. Note that **II** can completely occupy some winning set, but this is not considered as victory. Also note that draw is impossible in a weak game. The player **I** is usually called *maker* and **II** is called *breaker,* weak games are thus also called *maker-breaker games.*

A *strategy* for the player **I** (**II**) formally means a function S such that the domain of S is a set of even (odd) length subsequences of different elements of the board V, and the range is V. A *winning* (or *drawing*) *strategy* S for **I**

(**II**) means that in all possible plays where **I** (**II**) follows S to find his next move is a win for him (a win or a draw).

There are only three possible outcomes of a strong game: either the player **I** has a winning strategy, or the player **II** has a winning strategy, or both of them have a drawing strategy. For the class of weak games, there are only two outcomes: either **I** (maker) has a winning strategy or **II** (breaker) has a winning strategy.

The following important theorem is quite surprising; it states that in the class of strong games the first player cannot lose.

Theorem 2.1. *(Strategy Stealing [1]) Let $H = (V, F)$ be an arbitrary finite hypergraph. Then playing the strong game on H, first player can force at least a draw, i.e. a draw or possibly a win.*

Strategy Stealing implies the following.

Proposition 2.2. *Assume H is a hypergraph such that $\chi(H) > 2$ and assume two players play strong or weak game on H. Then the first player wins.*

The following two results are our basic tool in analysis of Ramsey games.

Theorem 2.3. *([6]) If $H = (V, F)$ is an k-uniform hypergraph and $|F| < 2^{k-3}$, then the player **II** can force a draw in the strong game on H.*

For a hypergraph H, let

$$\Delta_2(H) = \max_{u,v \in V(H)} \{F \in E(H);\ \{u, v\} \subseteq F\}.$$

Theorem 2.4. *(Weak Win Criterion [1]) Assume that we are playing the weak game on a k-uniform hypergraph $H = (V, F)$. If $|F| > 2^{k-3} \cdot \Delta_2(\mathcal{F}) \cdot |V|$, then the maker has a weak win in H.*

We mention that the proofs of Theorem 2.3 and Theorem 2.4 are constructive, that means they provide an explicit strategy description. We shall also need the following lemma.

Lemma 2.5. *Let $H = (V, F)$ be an arbitrary finite hypergraph. Assume there exists a hypergraph $H' = (V', F')$, $V' \subseteq V$, $F' \subseteq F$ such that the weak game on H' is win for the first player. Then the first player has a winning strategy also in the weak game on H.*

Proof. Consider the weak game on the hypergraph H' and the appropriate winning strategy S of the first player. Then apply S in the weak game on H. Clearly, if we restrict the winning lines on F' and the first player still wins, the second player is unable to block him on the set F. $\square$

Note that Lemma 2.5 does not hold for the class of strong games.

3 Structural Ramsey games

Here we present the proof of Theorem 1.4.

Proof. Let $C = B^k \in \mathcal{C}$ be the inflation of the structure $B \in \mathcal{C}$ by a factor of $k = 2^{u-3} \cdot u + 1$. The game takes place on C. Let us define the hypergraph G as follows: The set of vertices $V(G)$ contains all copies of A in C with the exception of copies having two or more vertices in a single multivertex of B^k. The set of edges $E(G)$ contains all sets S of vertices $V(G)$ such that every S contains all copies of A in some B-substructure of C having at most one vertex in a single multivertex of B^k. Formally this means

$$\begin{aligned} V(G) &= \{A' \in \binom{C}{A};\ |A' \cap V_i| \leq 1 \text{ for all } i\}, \\ E(G) &= \left\{S \subseteq V(G);\ S = \binom{B'}{A},\ B' \in \binom{C}{B},\ |B' \cap V_i| \leq 1 \text{ for all } i\right\}. \end{aligned}$$

Note that the number of copies of A in C may be higher than $|V(G)|$, and similarly the number of copies of B in C may be higher than $|E(G)|$. However, due to Lemma 2.5, if player **I** wins the weak game on G, then he wins also the original Ramsey (A, B)-game on C.

The uniformity u of G equals to number of copies of A in B. Using the fact every substructure of B gets inflated by the same factor k, we have $|V(G)| = u \cdot k^{|V(A)|}$ and $|E(G)| = k^{|V(B)|}$. In order to compute $\Delta_2(G)$, we choose two arbitrary vertices $A_1, A_2 \in V(G)$ (i.e. copies of A in C). They intersect with at least $|V(A)| + 1$ multivertices of C. To extend $A_1 \cup A_2$ to a copy of B, one can choose the remaining vertices from at most $|V(B)| - |V(A)| - 1$ multivertices of C, so this can be done by at most $k^{|V(B)|-|V(A)|-1}$ ways. Therefore, as long as

$$\begin{aligned} |E(G)| = k^{|V(B)|} &> 2^{u-3} \cdot |V(G)| \cdot \Delta_2(G) = 2^{u-3} \cdot u \cdot k^{|V(B)|-1} \\ k &> 2^{u-3} \cdot u, \end{aligned}$$

due to Theorem 2.4 (Weak Win Criterion) player **I** wins. Furthermore, $|V(C)| = 2^{u-3} \cdot u \cdot |V(B)| + |V(B)|$, consequently $r^g_{\mathcal{C}}(A, B) \leq 2^u \cdot u \cdot |V(B)|$.□

3.1 Colouring vertices

We can easily adapt Theorem 1.4 for the vertex colouring, i.e. (K_1, B)-game. However, in this case we can easily analyse even strong vertex game:

Theorem 3.1. *Let $\mathcal{C}$ be a class of structures which is closed on inflation. Let $B \in \mathcal{C}$ and $p = |V(B)|$. Then there exists $C \in \mathcal{C}$ on $2p - 1$ vertices such that player* **I** *wins the strong Ramsey (K_1, B)-game on C. Moreover, the size $2p - 1$ cannot be improved.*

Proof. For $B = (\{w_1, \ldots, w_p\}, E)$, let us define the structure $C = (\{w_1\} \cup V_2 \cup \cdots \cup V_p, E(C))$ as the inflated B where each vertex $w \in V(B)$, $w \neq w_1$, gets inflated by factor 2. Observe that $C \in \mathcal{C}$.

The strategy of player **I** is following. In the first move occupy w_1. When player **II** takes one point from V_i, take the remaining point from V_i. Observe that after p moves **I** wins. Clearly, to colour a copy of B, C has to have at least $2p - 1$ vertices. □

By a *cycle* C_s we mean every hypergraph satisfying the following condition: there exists a sequence $(v_1, E_1, v_2, E_2, \ldots, v_s, E_s)$ such that all v_i and all E_i are distinct and $v_i \in V(C_s)$, $E_i \in E(C_s)$, and $v_i, v_{i+1} \in E_i$ for $i = 1, \ldots, s-1$ and $v_s, v_1 \in E_s$. For a hypergraph G, by girth(G) we mean the minimum s such that G contains a cycle C_s.

In the case when the class $\mathcal{C}$ contains only cycle-free structures, we have to use different technique. Our basic tool is the following lemma, which shows that there exist "dense" hypergraphs without short cycles. Its proof is application of probabilistic method and it follows Erdős and Spencer [7] where the original proof can be found. We use the approach presented by Nešetřil and Rödl [14].

Lemma 3.2. *For all positive integers k and s there exists a k-uniform hypergraph $G = (V, E)$, $|V| = n$, without cycles of length less than s and with $|E| > n^{1+1/s}$ edges for all n sufficiently large.*

We mention that the proof of Lemma 3.2 is not constructive, i.e. it gives the desired hypergraph G by purely existential argument.

Theorem 3.3. *Let $\mathcal{F}$ be a set of 2-connected hypergraphs and let $\mathcal{C} = \mathrm{Forb}(\mathcal{F})$. Let $B \in \mathcal{C}$, $p = |V(B)|$ and*

$$\ell = \max_{i=1,\dots,t} \mathrm{girth}(F_i).$$

Then there exists $C \in \mathcal{C}$ on $\mathcal{O}(2^{p\ell})$ vertices such that $C \xrightarrow{g} (B)^{K_1}$.

Proof. Let $|V| = n = 2^{(p-3)(\ell+1)}+1$. By Lemma 3.2 there exists a p-uniform hypergraph $C' = (V, E')$ such that C' does not contain a cycle of length less than $\ell+1$ and $|E'| > n^{1+1/(\ell+1)}$. Let Δ be the type of B. Let us define the structure $C = (V, \mathcal{M})$ of type Δ by taking C' and arbitrarily replacing each edge by a copy of B. That is, $C = \bigcup_{S \in E'} (S, \mathcal{M}_S)$ where $(S, \mathcal{M}_S) \simeq B$ for each $S \in E'$.

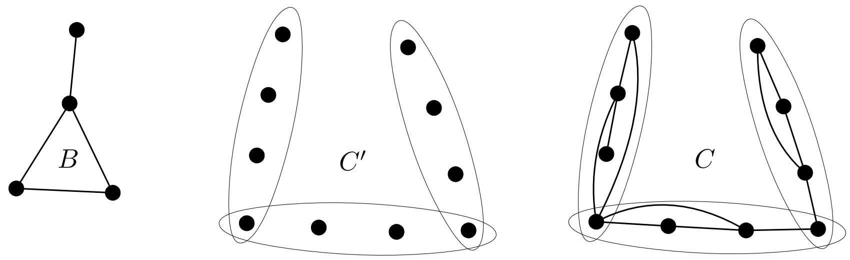

Figure 2: "Stuffing" the p-uniform hypergraph C' by copies of the graph B.

Let us show that C is F_i-free for $i = 1, \dots, t$, i.e. that $C \in \mathcal{C}$. For the sake of contradiction, assume there is an F_i-substructure in C. Clearly, since $F_i \not\subseteq B$, the vertices of F_i cannot be entirely contained in a single hyperedge of C'. Thus, let the vertices of F_i be incident with more than one hyperedge of C'. Then F_i must lie on a cycle in C' of length less or equal ℓ; otherwise F_i could not be 2-connected, since $|S \cap T| \le 1$ for any two distinct $S, T \in E(C')$ and the single common vertex would be a cut vertex of F_i. Due to the construction of C', there are no cycles shorter than ℓ, therefore C cannot contain a copy of F_i.

Let us construct a hypergraph G such that playing weak game on G is equivalent with the original game on C. That is, $V(G) = V(C)$ and

$$E(G) = \{S \subseteq V(C);\ C[S] \simeq B\},$$

where $C[S]$ is the substructure induced by S. That means each edge in $E(G)$ corresponds to a set of vertices on which there is a copy of B in C. Observe that $C' \subseteq G$; by the "stuffing" procedure, there are at least the edges of C' in G and maybe some more. Due to Lemma 2.5, we can restrict ourselves only to the weak game on C'; if we show **I** wins on C', then he wins on G and therefore also on C.

The hypergraph G has n vertices, at least $n^{1+1/(\ell+1)}$ edges, and $\Delta_2(G) = 1$ since it does not contain a 2-cycle. Provided the size n of C satisfies

$$n^{1+\frac{1}{\ell+1}} > 2^{p-3} \cdot n,$$

by Theorem 2.4 (Weak Win Criterion) there exists a winning strategy of **I**. $\square$

3.2 Strong Ramsey games

Here we prove Theorem 1.6.

Proof. We give an example of small K_4-free graph, where two players alternately colour the edges, trying to colour their own K_3 subgraph first. We show the winning strategy of player **I**.

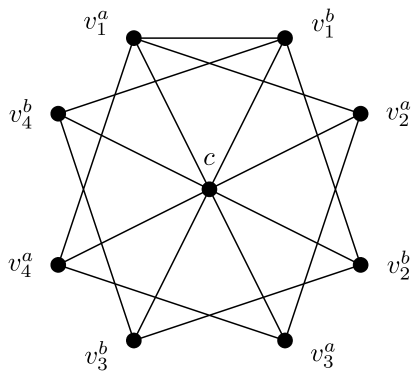

Figure 3: The K_4-free board.

The graph on Figure 3 does not contain K_4 (easy observation) and there exists an explicit winning strategy of **I** in the strong game. As the first move, **I** takes the edge $\{c, v_1^a\}$. Then **II** responds. Let us distinguish two cases:

1. **II**'s move was one of $\{c, v_i^a\}$ or $\{v_i^a, v_j^a\}$. Then **I** in the following four moves takes the edges $\{c, v_1^b\}$, $\{c, v_2^b\}$, $\{c, v_3^b\}$, $\{c, v_4^b\}$, respectively. **II** is forced to take the edges $\{v_1^a, v_1^b\}$, $\{v_1^b, v_2^b\}$, $\{v_2^b, v_3^b\}$, $\{v_3^b, v_4^b\}$, respectively, otherwise **I** takes them and wins. After the fourth move, the edge $\{v_4^b, v_1^b\}$ is left unoccupied, allowing **I** to win.

2. **II**'s move was one of $\{c, v_i^b\}$ or $\{v_i^b, v_j^b\}$ or $\{v_1^a, v_1^b\}$. Then **I** in the following three moves takes the edges $\{c, v_2^a\}$, $\{c, v_3^a\}$, $\{c, v_4^a\}$, respectively. **II** is forced to take the edges $\{v_1^a, v_2^a\}$, $\{v_2^a, v_3^a\}$, $\{v_3^a, v_4^a\}$, respectively, otherwise **I** takes them and wins. After the third move, the edge $\{v_4^a, v_1^a\}$ is left unoccupied, allowing **I** to win.

$\square$

4 Arithmetic progression games

The following theorem, conjectured by Schur and proved by Brauer [5], is an extension of van der Waerden theorem.

Theorem 4.1. *(Brauer, 1928) For a positive integer n, there exists a positive integer N such that in an arbitrary colouring of the set $[N]$ by r colours, we can find in one of the colour classes the arithmetic progression $a_0, a_0 + d, \dots, a_0 + nd$ together with the difference d.*

For two integers $k \geq 3$ and n, we define the *arithmetic progression game with difference* on the set $S = \{1, 2, \dots, n\}$ as follows. Maker and breaker alternately pick elements of S. Maker wins if he colours some $(k+1)$-tuple of S where k elements form an arithmetic progression P and the remaining element d denotes the difference of P. If he is unable to colour such set, breaker wins. By $r^g(APd(k))$ we mean the smallest n such that maker has a winning strategy.

The arithmetic progression game (i.e. based on the original van der Waerden theorem) was investigated by Beck [2] (see Theorem 1.2). We generalise the proof ideas of Beck to work on arithmetic progression games with difference. We also give a lower bound on the board size.

Theorem 4.2. *Let $k \geq 2$ be an integer. Assume maker and breaker play the k-term arithmetic progression game with difference. Then maker has a* *winning strategy on board of size $\mathcal{O}(2^k k^3)$ and breaker has a winning strategy on board of size $\Omega(2^{k/2}\sqrt{k})$, i.e.*

$$\Omega(2^{k/2}\sqrt{k}) \leq r^g(APd(k)) \leq \mathcal{O}(2^k k^3).$$

Proof. For a fixed n, let us define the following $(k+1)$-uniform hypergraph $H = (\{1,\ldots,n\}, F)$. The set F contains all $(k+1)$-element subsets $S \subseteq \{1,\ldots,n\}$ such that some k elements of S form an arithmetic progression P of length k, and the one remaining element d of S denotes the difference d of P. Clearly, playing the weak game on H is equivalent with the original weak arithmetic progression game with difference. We are going to find the smallest n such that the inequality $|F| > 2^{k-2} \cdot n \cdot \Delta_2(H)$ from Weak Win Criterion holds.

Let us fix two distinct points $a, b \in \{1,\ldots,n\}$, $a < b$, and we count the maximal number of edges incident both with a and b. Three cases are possible:

- The point a denotes the arithmetic progression difference. Therefore, b can lie on k positions of the arithmetic progression, so we get at most k possibilities.

- The point b denotes the difference. Similarly, there is at most k possibilities.

- Both a and b are members of the arithmetic progression. The number of possibilities is therefore at most $\binom{k}{2}$ as this is the number of all positions the two points can occupy in a k-term progression.

Thus we have $\binom{k}{2} + 2k \geq \Delta_2(H)$.

Observe there are $\Theta(n^2/k)$ arithmetic progressions of length k in $[n]$. By solving the inequality

$$c\frac{n^2}{k} > 2^{k-2}\left(\binom{k}{2} + 2k\right)n,$$

and due to Weak Win Criterion, maker has a winning strategy on H with $n = \mathcal{O}(2^k k^3)$ vertices, therefore also in the original game.

Let us now show the lower bound. Recall there are $\Theta(n^2/k)$ arithmetic progressions. By solving the inequality $cn^2/k < 2^k$, we get $n = \mathcal{O}(2^{k/2}\sqrt{k})$. Theorem 2.3 applied on H with n vertices proves the existence of player **II**'s drawing strategy. $\square$

Note that the lower bound (and the corresponding drawing strategy) of the previous theorem holds both for the strong and weak version of the game. It would be interesting to close the gap in Theorem 4.2. Particularly, we don't know whether $\lim_{k\to\infty} \left(r^g(APd(k))\right)^{2/k}$ exists. Another interesting question is to study the game, where the players keep taking arithmetic progressions of length ℓ and the goal of maker is to find arithmetic progression of length $k > \ell$ with all progression of his colour.

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