Bounds for the real number graph labellings and application to labellings of the triangular lattice

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Abstract

We establish new lower and upper bounds for the real number graph labelling problem. As an application, we completely determine the optimum spans of $L(p, q)$-labellings of the infinite triangular plane lattice (solving an open problem of Griggs).

1 Introduction

One of the important applications of graph colorings lies in the area of the channel assignment problem. The significance of results on the channel assignment problem has increased with development of mobile phone networks—large activity in this area yielding numerous results is documented in several recent surveys [1, 5].

One of the well established models for the channel assignment problem is the distance constrained graph labelling, introduced by Griggs and Yeh [12]. This notion can be treated in a more general setting of real number graph labellings and λ-graphs introduced in a series of papers of Griggs and Jin [8, 9, 10] and further developed in [2, 15]. The reader can check a recent survey [13] for further details.

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Throughout this paper, we will use the concept of λ-graphs. A λ-*graph* G is a graph with k types of edges; two vertices of G could be joined by several edges of different types. Given the parameters $x_1, \ldots, x_k$, a labelling f of vertices of G by non-negative real numbers is called *proper* if the labels of every two vertices u and v joined by an edge of type i differ by at least x_i. The *span* of the labelling f is the supremum of the labels of all vertices. The infimum of the spans of proper labellings of G is denoted by $\lambda(G; x_1, \ldots, x_k)$. The values of $\lambda(G; x_1, \ldots, x_k)$ interpreted as a k-parameter function form the λ-*function of* G.

Note that throughout the article we often assume implicitly that for every choice of parameters $x_1, \ldots, x_k$ the value of $\lambda(G; x_1, \ldots, x_k)$ is finite. This is equivalent to the statement that G viewed as an ordinary graph can be colored by a finite number of colors.

Let us now recall the notion of *distance constrained labellings*. For a graph G and non-negative integers $p_1, \ldots, p_k$, an $L(p_1, \ldots, p_k)$-*labelling* of G is a labelling of the vertices of G with non-negative integers such that vertices at distance i, $1 \leq i \leq k$, receive labels that differ at least by p_i. This notion can be relaxed from integer labellings to real number graph labellings yielding a special class of λ-graphs: the corresponding λ-graph is the graph with the same vertex set where the vertices at distance i are connected by an edge of type x_i.

The most studied case both in the settings of λ-graphs and distance constrained labellings is $k = 2$. Since the so-called Scaling Property [8, 13] asserts

$$\alpha\lambda(G; x_1, \ldots, x_k) = \lambda(G; \alpha x_1, \ldots, \alpha x_k)$$

for all $\alpha \geq 0$, the λ-function could be viewed as a one parameter function $\lambda(G; x, 1)$ if $k = 2$. In Section 3, we prove several lemmas that allow to derive (matching) upper and lower bounds on $\lambda(G; x, 1)$ from the knowledge of $\lambda(G; x, 1)$ for specific values of x. To our best knowledge, these bounds are the first general bounds that can be used to deduce upper and lower bounds on $\lambda(G; x, 1)$ besides a direct application of Scaling Property (see [8, 9] for an example of such an application). As an application of our bounds, we were able to solve an out-standing open problem on optimal $L(p, q)$-labellings of the infinite triangular lattice which we now describe.

In practical applications, infinite graphs often provide a convenient model for the underlying topology of the network: instead of considering a finite graph, one can model a network as an infinite regular tiling of the plane. Hence, the infinite triangular lattice, the infinite square lattice and the infinite hexagonal lattice naturally appear in such applications. The optimum spans of $L(p, q)$-labellings of the infinite square and hexagonal lattice have been determined by Griggs and Jin [9], cf. Calamoneri [4]. The case of infinite triangular lattice seems to be more difficult.

The problem of determining the values of λ-function for infinite triangular lattice Γ_Δ was posed by Griggs [11] in the 2000 International Math Contest in Modeling (MCM). Five teams proved new results for particular choices of parameters [3, 6, 7, 17, 18]. In particular, Goodwin, Johnson, and Marcus [7] determined the value of $\lambda(\Gamma_\Delta; x, 1)$ for $x \geq 4$. Jin and Yeh [14], Zhu and Shi [19], and Calamoneri [5] studied further the values of $\lambda(\Gamma_\Delta; x, 1)$ for $x \geq 1$. Finally, Griggs and Jin [9] managed to determine the values of $\lambda(\Gamma_\Delta; x, 1)$ for all $x \notin (1/3, 4/5)$. As we have already said, we apply our bounds to complete the behavior of the function $\lambda(\Gamma_\Delta; x, 1)$—the reader can find the whole function $\lambda(\Gamma_\Delta; x, 1)$ in Figure 1.

2 Preliminary results

In this section, we recall basic results on λ-functions mainly from [2, 8, 15].

The results of Griggs and Jin [8] and Král' [15] characterize general behavior of λ-function that could be summarized into the following Theorem.

Theorem 1. *Let G be a (possibly infinite) λ-graph with k types of edges. The λ-function of G is continuous, non-decreasing and piecewise linear function of $x_1, \ldots, x_k$ on $[0, \infty)$ with finitely many linear parts.*

Additionally, the values taken by the λ-function are restricted to a subset $D(x_1, \ldots, x_p)$ of real numbers defined as follows: $D(x_1, \ldots, x_p)$ is a set of all linear combinations $\sum_{i=1}^{p} \alpha_i x_i$, where α_i are non-negative integers. One of theorems of Griggs and Jin [8] states that there exists a finite optimal labelling that uses only labels from set $D(x_1, \ldots, x_p)$.

Theorem 2 (Griggs and Jin [8]). *Let G be a λ-graph, possibly infinite, with k types of edges. For all non-negative real numbers $x_1, \ldots, x_k$, there is an optimal labelling f for the parameters $x_1, \ldots, x_k$ such that $f(v) \in D(x_1, \ldots, x_k)$ for each vertex v of G. In particular, $\lambda(G; x_1, \ldots, x_k)$ belongs to the set $D(x_1, \ldots, x_k)$.*

As we explained in Introduction, the infinite coverings of the plane plays a special role in the channel assignment problem. By Compactness Principle, every λ-graph G contains a finite subgraph with the same λ-function:

Theorem 3 (Král' [16]). *Every λ-graph G with k types of edges and a finite chromatic number contains a finite subgraph H such that*

$$\lambda(G; x_1, \ldots, x_k) = \lambda(H; x_1, \ldots, x_k)$$

for all $(x_1, \ldots, x_k) \in [0, \infty)^k$.

Note that it is easy to derive the existence of a subgraph H for a fixed k-tuple $(x_1, \ldots, x_k)$ such that $\lambda(G; x_1, \ldots, x_k) = \lambda(H; x_1, \ldots, x_k)$, but this fact itself does not imply the existence a universal subgraph H for all choices of $x_1, \ldots, x_k$ which is claimed in Theorem 3.

Griggs and Jin [9] studied the function $\lambda(G; x, y)$ for G being a λ-graph obtained from the infinite triangular lattice Γ_Δ by connecting neighbors with edges of the first type and vertices at distance exactly two with edges of the second type. For the sake of simplicity, only the edges of the first type will be drawn in our figures. Griggs and Jin [9] determined $\lambda(\Gamma_\Delta; x, 1)$ for $x \in [0, 1/3] \cup [4/5, \infty)$.

Theorem 4 (Griggs and Jin [9]). *For $k \geq 0$ the minimum span of any L(x, 1)-labelling of triangular lattice is given by:*

$$\lambda(\Gamma_\Delta; x, 1) \begin{cases} = 2x + 3, & \textit{if } 0 \leq x \leq 1/3, \\ \in [2x + 3, 11x], & \textit{if } 1/3 \leq x \leq 9/22, \\ \in [2x + 3, \frac{9}{2}], & \textit{if } 9/22 \leq x \leq 3/7, \\ \in [9x, \frac{9}{2}], & \textit{if } 3/7 \leq x \leq 1/2, \\ \in [\frac{9}{2}, \frac{16}{3}], & \textit{if } 1/2 \leq x \leq 2/3, \\ \in [\frac{16}{3}, \frac{23}{4}], & \textit{if } 2/3 \leq x \leq 3/4, \\ \in [\frac{23}{4}, 6], & \textit{if } 3/4 \leq x \leq 4/5, \\ = 6, & \textit{if } 4/5 \leq x \leq 1, \\ = 6x, & \textit{if } 1 \leq x \leq 4/3, \\ = 8, & \textit{if } 4/3 \leq x \leq 2, \\ = 4x, & \textit{if } 2 \leq x \leq 11/4, \\ = 11, & \textit{if } 11/4 \leq x \leq 3, \\ = 3x + 2, & \textit{if } 3 \leq x \leq 4, \\ = 2x + 6, & \textit{if } x \geq 4. \end{cases}$$

We complete the behavior of the function $\lambda(\Gamma_\Delta; x, 1)$ for $x \in [1/3, 4/5]$.

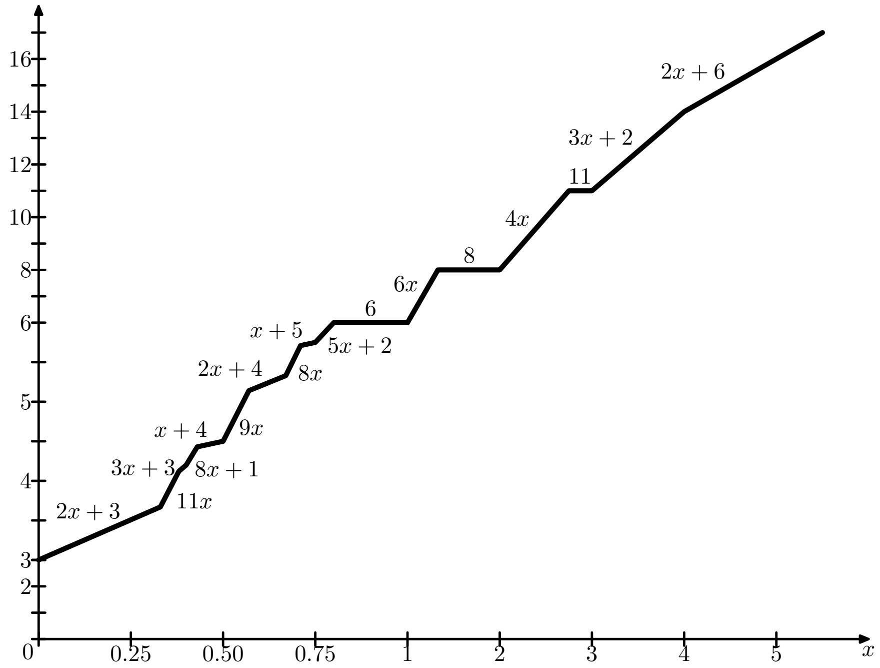

Figure 1: The optimal value of $\lambda(\Gamma_\Delta; x, 1)$. Note that some parts of the graph are enlarged to reflect the behavior $\lambda(\Gamma_\Delta; x, 1)$ in more detail.

3 General bounds

In this section, we prove several bounds on $\lambda(G;x,1)$ of the following type: given the values of $\lambda(G;x,1)$ at two points x_1 and x_2, we estimate the values of $\lambda(G;x,1)$ in the whole interval (x_1,x_2).

Since the λ-function $\lambda(G;x,1)$ is a piecewise linear function of x, it is easy to complete the function inside the interval if there is only one acceptable linear function connecting the two known points of the function. One such case is described in the next lemma ($\mathbb{Z}^+$ stands for the set of non-negative integers).

Lemma 5. *Let G be a λ-graph and assume that $\lambda(G;x,1) = \alpha + \beta x$ for $x = x_1, x_2 \in [0,\infty)$, $x_1 < x_2$. If the slope of every linear function $\alpha' + \beta' x$, $\alpha', \beta' \in \mathbb{Z}^+$, that is equal to $\alpha + \beta x$ for some $x \in [x_1, x_2]$, is greater than or equal to the slope of $\alpha + \beta x$, i.e. $\beta' \geq \beta$ for any such function $\alpha' + \beta' x$, then $\lambda(G;x,1) = \alpha + \beta x$ for all $x \in [x_1, x_2]$.*

Proof. By Theorem 1, the function $\lambda(G;x,1)$ is a continuous piecewise linear function of x. It follows from Theorem 2 that the value of $\lambda(G;x,1)$ is equal to an element of $D(x,1)$.

Suppose that $\lambda(G;x,1)$ is strictly greater than $\alpha + \beta x_0$ for some $x_0 \in (x_1,x_2)$. Since the function $\lambda(G;x,1)$ is equal to $\alpha+\beta x$ for x_2, there exists a linear function $\alpha' + \beta' x$ and $y_1, y_2 \in [x_1,x_2]$, $y_1 < y_2$, such that

- $\lambda(G;x,1) = \alpha' + \beta' x$ for $x \in [y_1, y_2]$,
- $\alpha + \beta y_1 < \alpha' + \beta' y_1$, and
- $\alpha + \beta y_2 = \alpha' + \beta' y_2$.

Hence, $\beta'(y_2 - y_1)$ is strictly smaller than $\beta(y_2 - y_1)$ and thus $\beta' < \beta$ which is excluded by the assumptions of the lemma. We conclude that there is no x_0 such that $\lambda(G;x_0,1) > \alpha + \beta x_0$.

An analogous argument yields that $\lambda(G;x,1)$ is not smaller than $\alpha+\beta x$ for any $x \in (x_1,x_2)$. We infer that $\lambda(G;x,1)$ is equal to $\alpha + \beta x$ for all $x \in [x_1,x_2]$. $\square$

There is a variant of Lemma 5 where all the linear functions has smaller slope than the segment of the two known points. The proof is analogous to that of Lemma 5.

Lemma 6. *Let G be a λ-graph and $\lambda(G;x,1) = \alpha + \beta x$ for $x = x_1, x_2 \in [0,\infty)$. If the slope of every linear function $\alpha' + \beta' x$, $\alpha', \beta' \in \mathbb{Z}^+$, that is equal to $\alpha + \beta x$ for some $x \in [x_1, x_2]$ is* smaller than or equal to *the slope of $\alpha + \beta x$, then $\lambda(G;x,1) = \alpha + \beta x$ for all $x \in [x_1, x_2]$.*

For every linear function $f(x) = \alpha + \beta x$, $\alpha, \beta \in \mathbb{Z}^+$, there exist intervals I where no linear function with a non-negative integer slope smaller than β is equal to $f(x)$ for any $x \in I$. We utilize this fact in our further considerations. For an interval $I \subseteq [0,\infty)$, $\xi(I)$ denotes the smallest positive integer q such that $p/q \in I$ for $p \in \mathbb{Z}$, i.e., $\xi(I)$ is the smallest denominator of an integral fraction contained in I.

Proposition 7. *Let $I \subseteq [0,\infty)$ be an interval and α, β, α', and β' are non-negative integers. If $\beta' < \beta < \xi(I)$, then $\alpha + \beta x \neq \alpha' + \beta' x$ for every $x \in I$.*

Proof. Assume the opposite, i.e. there exist non-negative integers α, β, α' and β' such that $\alpha + \beta x_0 = \alpha' + \beta' x_0$ for $x_0 \in I$ and $\beta' < \beta < \xi(I)$. A simple computation yields that

$$x_0 = -\frac{\alpha - \alpha'}{\beta - \beta'} = \frac{|\alpha - \alpha'|}{|\beta - \beta'|} \in I \ .$$

Hence, x_0 is a rational number with the denominator $|\beta - \beta'| \leq \beta$ which is impossible since $\beta < \xi(I)$. □

Proposition 7 immediately yields the next simple Lemma.

Lemma 8. *Let $\lambda(G;x,1) = \alpha + \beta x$ for $x = x_1, x_2 \in [0,\infty)$, $x_1 < x_2$. If $\xi([x_1, x_2]) > \beta$, then $\lambda(G;x,1) = \alpha + \beta x$ for all $x \in [x_1, x_2]$.*

Proof. By Proposition 7, there is no linear function with non-negative integer coefficients and slope smaller than β that is equal to $\alpha + \beta x$ for $x \in [x_1, x_2]$. Therefore, the statement of the corollary follows from the Lemma 5. □

Lemma 5 handles the case when the interval contains no function with a smaller slope. The counterpart, Lemma 6, can be applied, too. It is enough to assume that the left end-point of the interval is sufficiently large as stated in the next lemma.

Lemma 9. *Let $\lambda(G;x,1) = \alpha + \beta x$ for $x = x_1, x_2 \in [0,\infty)$, $x_1 < x_2$. If x_1 is greater than $\frac{\alpha}{\xi([x_1,x_2])}$, then $\lambda(G;x,1) = \alpha + \beta x$ for all $x \in [x_1, x_2]$.*

Proof. We want to apply Lemma 6. Assume that there is a linear function $\alpha' + \beta' x$, $\alpha', \beta' \in \mathbb{Z}^+$, that is equal to $\alpha + \beta x$ for $x_0 \in [x_1, x_2]$ and such that the slope β' is greater than β. A simple computation yields that

$$x_0 = -\frac{\alpha' - \alpha}{\beta' - \beta} = \frac{|\alpha' - \alpha|}{|\beta' - \beta|} \ .$$

Since $|\beta' - \beta| \geq \xi([x_1, x_2])$, it holds that

$$x_0 \leq \frac{|\alpha - \alpha'|}{\xi([x_1, x_2])} \leq \alpha\xi([x_1, x_2]) < x_1 \ .$$

This implies that $x_0 < x_1$. Therefore, all linear functions $\alpha' + \beta' x$ equal to $\alpha + \beta x$ for some $x \in [x_1, x_2]$ have slope smaller or equal than β. The statement of the lemma now follows from Lemma 6. □

Consider an optimal labelling f of a λ-graph G for for $x_1 = x_0$ and $x_2 = 1$. As we have explained in Section 2, we can assume that $f(v) \in D(x_0, 1)$ for every vertex v. It is often the case that the labelling defined by considering the same integer combinations that appears in f is optimal for some $x_1 < x_0$ (and $x_2 = 1$). In the next lemma, we find an estimate of the interval to which the labelling f can be extended.

Lemma 10. *Let x_0 be a point equal to p_0/q_0 for some positive integers p_0 and q_0 that are mutually prime. If $\lambda(G; x_0, 1) = \alpha + \beta x_0$ and $\beta < q_0$ then*

$$\lambda(G; x, 1) \leq \alpha + \beta x \text{ for all } x \in \left[\frac{\tilde{p}}{\tilde{q}}, x_0\right] ,$$

where $\tilde{p}$ and $\tilde{q}$ are integers such that $\frac{\tilde{p}}{\tilde{q}} < \frac{p_0}{q_0}$, $1 \leq \tilde{q} \leq q_0$ and $\frac{\tilde{p}}{\tilde{q}}$ is maximal among all fractions formed by such a pair of integers $\tilde{p}$ and $\tilde{q}$.

Proof. Let $\tilde{x} = \tilde{p}/\tilde{q}$. By the choice of $\tilde{p}$ and $\tilde{q}$, $\xi((\tilde{x}, x_0])$ is greater or equal to q_0. By Proposition 7, there is no linear function with non-negative integer coefficients and slope smaller than β that were equal to $\alpha + \beta x$ for $x \in (\tilde{x}, x_0]$. Therefore, the value of $\lambda(G; x, 1)$ is smaller or equal to $\alpha + \beta x$ for all $x \in (\tilde{x}, x_0]$ (there is simply no linear function that would be greater than $f(x)\alpha + \beta x$ and meet $f(x)$ for $x \in (\tilde{x}, x_0]$). Since $\lambda(G; x, 1)$ is a continuous function of x, it also holds that $\lambda(G; \tilde{x}, 1) \leq \alpha + \beta\tilde{x}$. □

In the rest of the paper, we will apply the bounds obtained in this section to complete the behavior of the function $\lambda(\Gamma_\Delta; x, 1)$.

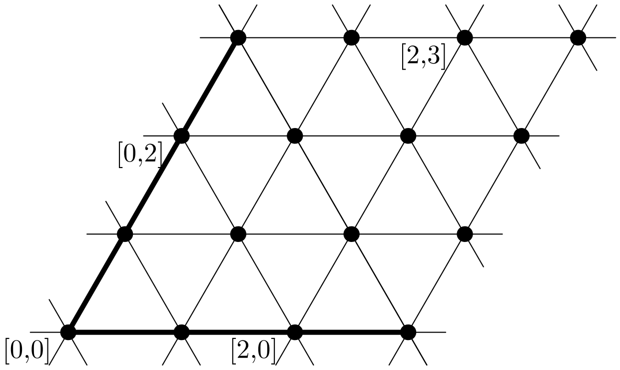

Figure 2: The infinite triangular lattice Γ_Δ with an origin and two axes.

4 Upper bounds for triangular lattice

The simplest way of obtaining an upper bound is a construction of a labelling. Let us start with introducing some additional notation related to the description of the lattice Γ_Δ. First, choose an arbitrary vertex of the lattice Γ_Δ to be the origin. The lattice contains two naturally defined axes (see Figure 2). This gives every vertex unique coordinates as shown in the figure.

Our construction of labellings of Γ_Δ is similar to the technique of matrix labelling used in [9]. However, in addition, the patterns used to label the lattice can be "shifted" in the way described further.

A matrix M of size $n \times m$ specifies the labelling of nm vertices of Γ_Δ. If M is *placed* at coordinates $[a, b]$ then a vertex with coordinates $[a+i, b+j]$, $i \in \{0, \ldots, m-1\}$, $j \in \{0, \ldots, n-1\}$, is labelled with $M_{n-j,i+1}$. The labelling is produced with "horizontal offset" o if the matrix M is placed at all coordinates $[c \cdot m, d \cdot n + c \cdot o]$, for all integers c and d. The constructed labelling could also be created by the original matrix labelling technique but our method allows using smaller matrices to describe the labelling.

The next lemma uses shifted matrix labelling to construct proper labellings of Γ_Δ for $x \in [3/8, 2/5]$.

Lemma 11. *The value of* $\lambda(\Gamma_\Delta; x, 1)$ *for* $x \in [3/8, 2/5]$ *is at most* $3 + 3x$.

Proof. The labelling of the infinite triangular lattice Γ_Δ will be obtained

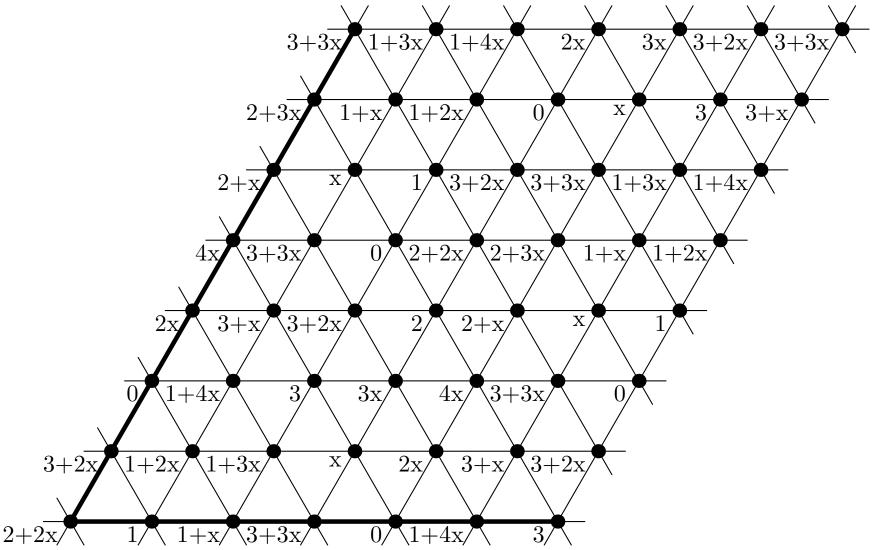

Figure 3: A part of the labelling of the infinite triangular lattice.

through the matrix labelling technique with horizontal offsets.

The matrix M we use is the following:

$$\begin{bmatrix} 2 & 0 & x & 2+3x \\ 3x & 3+2x & 3+3x & 2+x \\ x & 3 & 3+x & 4x \\ 3+3x & 1+3x & 1+4x & 2x \\ 2+3x & 1+x & 1+2x & 0 \\ 2+x & x & 1 & 3+2x \\ 4x & 3+3x & 0 & 2+2x \\ 2x & 3+x & 3+2x & 2 \\ 0 & 1+4x & 3 & 3x \\ 3+2x & 1+2x & 1+3x & x \\ 2+2x & 1 & 1+x & 3+3x \end{bmatrix}.$$

The horizontal offset of the matrix labelling is 9 (see Figure 3 for a part of the obtained labelling of Γ_Δ). The reader is kindly asked to verify that the produced labelling assign neighboring vertices labels that differ by at least x and those at distance two labels that differ by at least one. Since the span of the constructed labelling of Γ_Δ is equal to $3+3x$, the statement of the lemma follows. □

5 Lower bounds

Our lower bound proofs are partly computer assisted—note that this is also the case in some of the previous bounds on $\lambda(\Gamma_\Delta; x, 1)$. By Compactness Principle, if Γ_Δ does not have a labelling with a certain span, then there exists a finite subgraph of Γ_Δ that does not have a labelling of that span either. We have independently prepared two programs that test all labellings of finite parts of the triangular lattice Γ_Δ by brute force to reveal specific values of x; one of the programs can be downloaded at `http://kam.mff.cuni.cz/~kral/lattice.c`. Both the programs use the same coordinate system as we define in the proof of Lemma 11. Finally, for an integer n, $\Gamma_{\Delta,n}$ denotes the subgraph of Γ_Δ induced by the vertices with coordinates $[x, y]$ from set $\{0 \le x \le n, 0 \le y \le n\}$.

In order to decrease the running time, our program does not try to assign all values of the set $D(x, 1)$ to the points of the lattice but it only assigns labels contained in a set $\widetilde{D}_m(x) \subset D(x, 1)$. The set $\widetilde{D}_m(x)$ is defined recursively. We start with $\widetilde{D}^0_m(x)$ equal to the set of the labels contained in $D(x,1)$ that are smaller than or equal to m. If $\widetilde{D}^i_m(x)$ contains two numbers a and b, $a < b$, such that for every label $c \in \widetilde{D}^i_m(x)$ the following holds:

- if $c \geq a + x$, then $c \geq b + x$ and
- if $c \geq a + 1$, then $c \geq b + 1$,

then we set $\widetilde{D}^{i+1}_m(x) = \widetilde{D}^i_m(x) \setminus \{a\}$. We proceed in this way while $\widetilde{D}_m(x)$ contains a pair a and b of such labels. The final set $\widetilde{D}^i_m(x)$ is denoted by $\widetilde{D}_m(x)$.

The input of our programs is the size of the lattice n, the value of x and the maximal label m. The programs return whether $\Gamma_{\Delta,n}$ can be labelled by $\widetilde{D}_m(x)$. However, it remains to verify that restricting the set of possible labels to $\widetilde{D}_m(x)$ does not affect the correctness of our algorithm. We do so in the next lemma.

Lemma 12. *Let G be a λ-graph. If G has a proper labelling for $x_1 = x$ and $x_2 = 1$ of span at most m, it has a proper labelling of span at most m that uses only the labels contained in the set $\widetilde{D}_m(x)$.*

Proof. Assume that G has a proper labelling of span at most m. By Theorem 2, there also exists a proper labelling f that uses only the labels contained in the set $D(x,1)$.

We now prove by induction on i that for every $i \geq 0$ there exists a proper labelling f_i that uses only labels from $\widetilde{D}^i_m(x)$. Consider the pair of the numbers a and b such that b was removed from $\widetilde{D}^i_m(x)$ at the i-th step. We claim that changing the labels of all the vertices labelled by f_i with a to b yields a proper labelling of G. Let f_{i+1} be the resulting labelling. If f_{i+1} is not proper, then there exists a vertex v such that $f_i(v) = a$ (and thus $f_{i+1}(v) = b$) and a vertex u adjacent to v such that the labels $f_{i+1}(v)$ and $f_{i+1}(u)$ are too close. Let $c = f_i(u) = f_i(u+1)$.

If $c < a$, then the difference between the labels of u and v increased after relabelling. Hence, we can assume $c > a$. If u and v are joined by an x_1-edge, then $c \geq a + x$ (recall that $x_1 = x$). Consequently, by the choice of a and b, $c \geq b + x$. An analogous argument applies when u and v are joined by an x_2-edge (recall that $x_2 = 1$). We infer from the fact that f_i is a proper labelling of G that f_{i+1} is also a proper labelling of G.

We eventually conclude that there is a proper labelling of G that uses only the labels contained in $\widetilde{D}_m(x)$. □

The following four lemmas have been established using our computer assisted technique.

Lemma 13. *The optimal span* $\lambda(\Gamma_\Delta; x, 1)$ *of the infinite triangular lattice* Γ_Δ *is at least* $4 + x$ *for* $x = 3/7$.

Proof. We have verified by a computer that $\lambda(\Gamma_{\Delta,9}; x, 1) > 3 + 3x$ for $x = 3/7$, i.e. $\Gamma_{\Delta,9}$ has no labellings of span $30/7$ for $x = 3/7$. Therefore, $\lambda(\Gamma_\Delta; 3/7, 1) > 4 + 2/7$. By Theorem 2, the value of $\lambda(\Gamma_\Delta; 3/7, 1)$ is contained in the set $D(3/7, 1)$. The smallest element of $D(3/7, 1)$ greater than $4 + 2/7$ is $4 + 3/7$. This yields the statement of the lemma. □

Lemma 14. *The optimal span* $\lambda(\Gamma_\Delta; x, 1)$ *of the infinite triangular lattice* Γ_Δ *is at least* $3 + 3x$ *for* $x = 3/8$ *and* $x = 2/5$.

Proof. We proceed similarly as in Lemma 13. Again, we have verified using a computer that $\lambda(\Gamma_{\Delta,10}; x, 1) > 4$ for $x = 3/8$. Hence, $\lambda(\Gamma_\Delta; 3/8, 1) > 4$. Since the optimal span is a non-decreasing function of x it holds that $\lambda(\Gamma_\Delta; 2/5, 1) > 4$. Again, by Theorem 2, $\lambda(\Gamma_\Delta; 3/8, 1) \in D(3/8, 1)$ and $\lambda(\Gamma_\Delta; 2/5, 1) \in D(2/5, 1)$. The smallest elements greater than 4 is $3+3\cdot3/8$ for $x = 3/8$ and $3+3\cdot2/5$ for $x = 2/5$. We conclude that $\lambda(\Gamma_\Delta; x, 1) \geq 3+3x$ for $x = 3/8$ and $x = 2/5$ □

Lemma 15. *The optimal span* $\lambda(\Gamma_\Delta; x, 1)$ *of the infinite triangular lattice* Γ_Δ *is at least* $4 + 2x$ *for* $x = 4/7$ *and* $x = 3/5$.

Proof. Using a computer, we have found out that $\lambda(\Gamma_{\Delta,13}; x, 1) > 5$ for $x = 4/7$. Therefore, $\lambda(\Gamma_\Delta; 4/7, 1) > 5$. It follows from the fact that the optimal span is a non-decreasing function that also $\lambda(\Gamma_\Delta; 3/5, 1) > 5$. By Theorem 2, the value of $\lambda(\Gamma_\Delta; 4/7, 1)$ is contained in the set $D(4/7, 1)$. The smallest elements greater than 5 is $4 + 2x$ for both $x = 4/7$ and $x = 3/5$. The assertion of the lemma follows. □

Lemma 16. *The optimal span* $\lambda(\Gamma_\Delta; x, 1)$ *of the infinite triangular lattice* Γ_Δ *is at least* $5 + x$ *for* $x = 5/7$.

Proof. Again, we have verified that $\lambda(\Gamma_{\Delta,15}; x, 1) > 2 + 5x$ for $x = 5/7$. Therefore, the value of $\lambda(\Gamma_\Delta; 5/7, 1)$ is greater than $5+4/7$. By Theorem 2, the value of $\lambda(\Gamma_\Delta; 5/7, 1)$ is contained in $D(5/7, 1)$. The smallest element of $D(5/7, 1)$ greater than $5+4/7$ is $5+5/7 = 5+x$ which yields the lemma. □

6 Optimal labellings of triangular lattice

In this section we prove our main theorem, filling in the values of $\lambda(\Gamma_\Delta; x, 1)$ for $x \in [1/3, 4/5]$.

Theorem 17. *The following values are the spans of optimal labelling of the infinite triangular lattice Γ_Δ:*

$$\lambda(\Gamma_\Delta; x, 1) = \begin{cases} 3+2x, & \text{if } 0 \le x \le 1/3, \\ 11x, & \text{if } 1/3 \le x \le 3/8, \\ 3+3x, & \text{if } 3/8 \le x \le 2/5, \\ 1+8x, & \text{if } 2/5 \le x \le 3/7, \\ 4+x, & \text{if } 3/7 \le x \le 1/2, \\ 9x, & \text{if } 1/2 \le x \le 4/7, \\ 4+2x, & \text{if } 4/7 \le x \le 2/3, \\ 8x, & \text{if } 2/3 \le x \le 5/7, \\ 5+x, & \text{if } 5/7 \le x \le 3/4, \\ 2+5x, & \text{if } 3/4 \le x \le 4/5, \\ 6, & \text{if } 4/5 \le x \le 1, \\ 6x, & \text{if } 1 \le x \le 4/3, \\ 8, & \text{if } 4/3 \le x \le 2, \\ 4x, & \text{if } 2 \le x \le 11/4, \\ 11, & \text{if } 11/4 \le x \le 3, \\ 3x+2, & \text{if } 3 \le x \le 4, \\ 2x+6, & \text{if } x \ge 4. \end{cases}$$

Proof. The values of $\lambda(\Gamma_\Delta; x, 1)$ for $x \in [0, 1/3] \cup [4/5, \infty)$ were determined by Griggs and Jin [9] (see Theorem 4). For $x \in [1/3, 4/5]$, we split the proof into several parts dealing with subintervals of $[1/3, 4/5]$ separately. We do not consider the subintervals of $[1/3, 4/5]$ from left to right since some of the values we compute are used in our further considerations for neighboring subintervals.

- **the interval** $[3/8, 2/5]$
  It follows from Lemmas 14 and 11 that the value of $\lambda(\Gamma_\Delta; x, 1)$ is equal to $3+3x$ for $x = 3/8$ and $x = 2/5$. Since $\xi([3/8, 2/5]) = 5$, Lemma 8 implies that the value of $\lambda(\Gamma_\Delta; x, 1)$ is equal to $3+3x$ for all $x \in [3/8, 2/5]$.

- **the interval** $[1/3, 3/8]$
  By Theorem 4, the value of $\lambda(\Gamma_\Delta; 1/3, 1)$ is equal to $3+2/3$. Therefore, $\lambda(\Gamma_\Delta; x, 1)$ is equal to $11x$ for $x = 1/3$ and $x = 3/8$. Since $1/3$ is greater than 0, Lemma 9 asserts the value of $\lambda(\Gamma_\Delta; x, 1)$ being equal to $11x$ all for $x \in [1/3, 3/8]$.

- **the interval** $[3/7, 1/2]$
  By Theorem 4, $\lambda(\Gamma_\Delta; x, 1) = 4 + x$ for $x = 1/2$. It follows from Lemma 10 that the value of $\lambda(\Gamma_\Delta; x, 1)$ is smaller than or equal to $4 + x$ for $x \in [0, 1/2]$. On the other hand, by Lemma 13, the value of $\lambda(\Gamma_\Delta; x, 1)$ is greater or equal to $4 + x$ for $x = 3/7$. Therefore, $\lambda(\Gamma_\Delta; 3/7, 1) = 4 + 3/7$. Since $\xi([3/7, 1/2]) = 2$, Lemma 8 yields that $\lambda(\Gamma_\Delta; x, 1) = 4 + x$ for all $x \in [3/7, 1/2]$.

- **the interval** $[2/5, 3/7]$
  We have shown above that $\lambda(\Gamma_\Delta; x, 1)$ is equal to $1 + 8x$ for $x = 2/5$ and $x = 3/7$. Since $\xi([2/5, 3/7]) = 5$ and $2/5$ is greater than $1/5$, Lemma 9 implies that the value of $\lambda(\Gamma_\Delta; x, 1)$ is equal to $1 + 8x$ for all $x \in [2/5, 3/7]$.

- **the interval** $[4/7, 2/3]$
  By Theorem 4, the value of $\lambda(\Gamma_\Delta; x, 1)$ is equal to $4 + 2x$ for $x = 2/3$. We infer from Lemma 10 that the value of $\lambda(\Gamma_\Delta; x, 1)$ is smaller than or equal to $4 + 2x$ for $x \in [1/2, 2/3]$. On the other hand, by Lemma 15, the value of $\lambda(\Gamma_\Delta; x, 1)$ is greater or equal to $4 + 2x$ for $x = 4/7$. Therefore, $\lambda(\Gamma_\Delta; 4/7, 1)$ is equal to $4 + 2 \cdot 4/7$. Since $\xi([4/7, 2/3]) = 3$, we derive from Lemma 8 that $\lambda(\Gamma_\Delta; x, 1) = 4+2x$ for all $x \in [4/7, 2/3]$.

- **the interval** $[1/2, 4/7]$
  We have established that $\lambda(\Gamma_\Delta; x, 1) = 9x$ for $x = 1/2$ and $x = 4/7$. Since $1/2$ is greater than 0, Lemma 9 asserts that the value of $\lambda(\Gamma_\Delta; x, 1)$ is equal to $9x$ for all $x \in [1/2, 4/7]$.

- **the interval** $[5/7, 3/4]$
  By Theorem 4, $\lambda(\Gamma_\Delta; x, 1)$ is equal to $5 + x$ for $x = 3/4$. We infer from Lemma 10 that $\lambda(\Gamma_\Delta; x, 1)$ is smaller than or equal to $5 + x$ for $x \in [2/3, 3/4]$. On the other hand, by Lemma 16, the value of $\lambda(\Gamma_\Delta; x, 1)$ is greater than or equal to $5 + x$ for $x = 5/7$. Therefore, $\lambda(\Gamma_\Delta; 5/7, 1)$ equals $5+5/7$. Since $\xi([5/7, 3/4]) = 4$, Lemma 8 implies that $\lambda(\Gamma_\Delta; x, 1)$ is equal to $5 + x$ for all $x \in [5/7, 3/4]$.

- **the interval** $[2/3, 5/7]$
  We have shown that $\lambda(\Gamma_\Delta; x, 1)$ is equal to $8x$ for $x = 2/3$ and $x = 5/7$. Since $2/3$ is greater than 0, Lemma 9 yields that the value of $\lambda(\Gamma_\Delta; x, 1)$ is equal to $8x$ for all $x \in [2/3, 5/7]$.

- **the interval** $[3/4, 4/5]$
  By Theorem 4, $\lambda(\Gamma_\Delta; x, 1)$ is equal to $2+5x$ for $x = 3/4$ and $x = 4/5$. Since $\xi([3/4, 4/5]) = 4$ and $3/4$ is greater than $2/4$, we conclude using the Lemma 9 that the value of $\lambda(\Gamma_\Delta; x, 1)$ is equal to $2 + 5x$ for all $x \in [3/4, 4/5]$.

Since we have analyzed all subintervals of $[1/3, 4/5]$, the proof of the theorem is finished. □

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