Optimal value range in interval linear programming

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Abstract

We deal with the linear programming problem in which input data can vary in some given real compact intervals. The aim is to compute exact range of the optimal value function. We present a general approach to the situation the feasible set is described by an arbitrary linear interval system. Moreover, there can be certain dependencies between the constraint matrix coefficients. As long as we are able to characterize the primal and dual solution set (the set of all possible primal and dual feasible solutions, respectively), the bounds of the objective function result from two nonlinear programming problems. We demonstrate our approach on various cases of the interval linear programming problem (with and without dependencies).

Keywords: *Linear interval systems, linear programming, optimal value range, interval matrix, dependence.*

1 Introduction

Interval computing [1] is popular way for treating uncertainties in data measurements. Instead of exact values we compute with real intervals which are supposed to involve all gauging errors. An interval matrix is defined as

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A} \leq \overline{A}$ are fixed matrices; n-dimensional interval vectors can be regarded as interval matrices n-by-1. By

$$A_c \equiv \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta \equiv \frac{1}{2}(\overline{A} - \underline{A})$$

we denote the midpoint and radius of $\boldsymbol{A}$, respectively.

Let an interval matrix $\boldsymbol{A} \subset \mathbb{R}^{m \times n}$ and interval vectors $\boldsymbol{b} \subset \mathbb{R}^m$, $\boldsymbol{c} \subset \mathbb{R}^n$ be given. By a (standard) interval linear program [5, 6, 7, 10, 11, 13] we understand the family of problems

$$\min c^T x \ \text{ subject to } \ Ax = b, x \geq 0,$$

where $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$. Let us denote by

$$f(A, b, c) \equiv \inf c^T x \ \text{ subject to } \ Ax = b, x \geq 0$$

the optimal value of the linear program; also infinite values are allowed. The optimal value range problem [6, 7, 13] consists of computing the lower and upper bound of the optimal value

$$\underline{f} \equiv \inf f(A, b, c) \ \text{ subject to } \ A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c},$$
$$\overline{f} \equiv \sup f(A, b, c) \ \text{ subject to } \ A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}.$$

These bounds can be obtained by optimization problems as presented in Theorem 1. For the proof see Rohn's third chapter [7]. While the lower bound $\underline{f}$ is polynomially computable from a simple linear program, computing the upper bound $\overline{f}$ is an NP-hard problem [7].

Theorem 1. *We have*

$$\underline{f} = \inf \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \leq \overline{b}, \ \overline{A}x \geq \underline{b}, \ x \geq 0.$$

Let $\underline{f}$ be finite or let the right-hand side of the equation (1) *be positively infinite. Then*

$$\overline{f} = \sup b_c^T y + b_\Delta^T |y| \ \textit{ subject to } \ A_c^T y - A_\Delta^T |y| \leq \overline{c}. \tag{1}$$

The lower bound $\underline{f}$ can be computed by a standard linear program, whereas the upper bound $\overline{f}$ is solvable either by a nonlinear program, or by exponentially many linear programs applied on each orthant (cf. [6]).

In the next section, we generalize Theorem 1 for other types of linear programs. In addition, we would be able to solve interval linear programs involving some dependences. Dependences in linear interval systems have been studies in recent years e.g. in [2, 9, 12, 14], symmetric matrices as a particular case were dealt with in [3, 4].

2 Unified approach to optimal value range

Let an interval matrix $\boldsymbol{A} \subset \mathbb{R}^{m\times n}$ and interval vectors $\boldsymbol{b} \subset \mathbb{R}^m$, $\boldsymbol{c} \subset \mathbb{R}^n$ be given. The (general) interval linear program to be considered is

$$\min c^T x \quad \text{subject to} \quad x \in M(A,b), \tag{2}$$

where $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$. The set $M(A,b)$ is described by a linear system with the constraint matrix A and the right-hand side b; the system may consists of equations or inequalities or both of them, each variable may be (independently on the others) sign-restricted or free. Dependences in the matrix A are permitted. Let us generalize the notation of optimal value

$$f(A,b,c) \equiv \inf c^T x \quad \text{subject to} \quad x \in M(A,b)$$

and the notation of the lower and upper bounds of the optimal value

$$\begin{aligned} \underline{f} &\equiv \inf f(A,b,c) \quad \text{subject to} \quad A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}, \\ \overline{f} &\equiv \sup f(A,b,c) \quad \text{subject to} \quad A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}. \end{aligned}$$

For fixed A, c, b let us consider the dual problem to (2)

$$\max b^T y \quad \text{subject to} \quad y \in N(A,c), \tag{3}$$

where $N(A,c)$ is the appropriate linear system dual to $M(A,b)$. The solution sets of $M(A,b)$ and $N(A,c)$ are defined traditionally as

$$\begin{aligned} M &\equiv \{x \in M(A,b) \mid A \in \boldsymbol{A}, b \in \boldsymbol{b}\}, \\ N &\equiv \{y \in N(A,c) \mid A \in \boldsymbol{A}, c \in \boldsymbol{c}\}. \end{aligned}$$

More easily we can characterize the solution sets M and N, more easily we are able to compute the bounds $\underline{f}$ and $\overline{f}$. This follows from our main result formulated in Theorem 2. As long as there are no dependences in the description of $M(A,b)$, then one know how to characterize the solution set N; all the standard linear systems (equations or inequalities with non-negative or free variables) are surveyed in Rohn's second chapter [7]. But dependences are much more difficult to deal with [9].

Lemma 1. *Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Then one has*

$$c_c^T x - c_\Delta^T |x| = \min c^T x \quad \textit{subject to} \quad c \in \boldsymbol{c} \tag{4}$$

$$b_c^T y + b_\Delta^T |y| = \max b^T y \quad \textit{subject to} \quad b \in \boldsymbol{b}. \tag{5}$$

Proof. Define the vector $c^* \in \boldsymbol{c}$ componentwise as

$$c_i^* \equiv \begin{cases} \underline{c}_i & x_i \geq 0, \\ \overline{c}_i & x_i < 0. \end{cases}$$

We show that c^* is the witness of (4). For every $c \in \boldsymbol{c}$ we have

$$c^T x = \sum_{i:\, x_i \geq 0} c_i x_i + \sum_{i:\, x_i < 0} c_i x_i \geq \sum_{i:\, x_i \geq 0} \underline{c}_i x_i + \sum_{i:\, x_i < 0} \overline{c}_i x_i = c^{*T} x.$$

Furthermore

$$\begin{aligned} c^{*T} x &= \sum_{i:\, x_i \geq 0} \underline{c}_i x_i + \sum_{i:\, x_i < 0} \overline{c}_i x_i = \sum_{i\, :x_i \geq 0} (c_c - c_\Delta)_i x_i + \sum_{i:\, x_i < 0} (c_c + c_\Delta)_i x_i \\ &= c_c^T x - \sum_{i\, :x_i \geq 0} (c_\Delta)_i |x_i| - \sum_{i:\, x_i < 0} (c_\Delta)_i |x_i| \\ &= c_c^T x - c_\Delta^T |x|. \end{aligned}$$

Hence the formula (4) is valid, and the formula (5) can be proven accordingly. □

Theorem 2. *We have*

$$\underline{f} = \inf c_c^T x - c_\Delta^T |x| \quad \textit{subject to} \quad x \in M. \tag{6}$$

Let $\overline{f} < \infty$ or let the right-hand side of the equation (7) be positively infinite. Then

$$\overline{f} = \sup b_c^T y + b_\Delta^T |y| \quad \textit{subject to} \quad y \in N. \tag{7}$$

Otherwise we have only

$$\overline{f} \geq \sup b_c^T y + b_\Delta^T |y| \quad \textit{subject to} \quad y \in N. \tag{8}$$

Proof. 1. (lower bound) From the definition of $\underline{f}$ we have that

$$\underline{f} = \inf_{A \in \boldsymbol{A},\, c \in \boldsymbol{c},\, b \in \boldsymbol{b}} \; \inf_{x \in M(A,b)} c^T x = \inf_{A \in \boldsymbol{A},\, b \in \boldsymbol{b}} \; \inf_{x \in M(A,b)} \; \inf_{c \in \boldsymbol{c}} c^T x.$$

Lemma 1 asserts that $\min_{c \in \boldsymbol{c}} c^T x = c_c^T x - c_\Delta^T |x|$. Hence

$$\underline{f} = \inf_{A \in \boldsymbol{A},\, b \in \boldsymbol{b}} \; \inf_{x \in M(A,b)} c_c^T x - c_\Delta^T |x| = \inf_{x \in M} c_c^T x - c_\Delta^T |x|.$$

2. (upper bound) Let us suppose that $M(A, b) \neq \emptyset$ for all $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$. For every $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$ we have due to duality in linear programming

$$\inf_{x \in M(A,b)} c^T x = \sup_{y \in N(A,c)} b^T y.$$

We maximalize both sides of the equation over $b \in \boldsymbol{b}$,

$$\sup_{b \in \boldsymbol{b}} \inf_{x \in M(A,b)} c^T x = \sup_{b \in \boldsymbol{b}} \sup_{y \in N(A,c)} b^T y.$$

Since (by Lemma 1)

$$\sup_{b \in \boldsymbol{b}} \sup_{y \in N(A,c)} b^T y = \sup_{y \in N(A,c)} \sup_{b \in \boldsymbol{b}} b^T y = \sup_{y \in N(A,c)} b_c^T y + b_\Delta^T |y|,$$

we derive

$$\sup_{b \in \boldsymbol{b}} \inf_{x \in M(A,b)} c^T x = \sup_{y \in N(A,c)} b_c^T y + b_\Delta^T |y|.$$

If we maximalize both sides of this equation over $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$, we obtain

$$\sup_{A \in \boldsymbol{A},\, c \in \boldsymbol{c},\, b \in \boldsymbol{b}} \inf_{x \in M(A,b)} c^T x = \sup_{y \in N(A,c),\, A \in \boldsymbol{A},\, c \in \boldsymbol{c}} b_c^T y + b_\Delta^T |y|,$$

or

$$\overline{f} = \sup_{y \in N} b_c^T y + b_\Delta^T |y|.$$

Therefore the equation (7) holds.

If the supposition is not valid, i.e., there is some $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ such that $M(A, b) = \emptyset$, then $\overline{f} = \infty$. From the assumption of the theorem it follows that

$$\sup_{y \in N} b_c^T y + b_\Delta^T |y| = \infty.$$

In other words, the equation (7) is true, since both sides are positively infinite.

3. (inequality) The inequality (8) follows from the fact that for every $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$ we have from duality theory in linear programming

$$\inf_{x \in M(A,b)} c^T x \geq \sup_{y \in N(A,c)} b^T y.$$

Next proceed like in the part 2. $\square$

Observe that if the vector of variables x is known to be nonnegative, then the objective function of the optimization problem (6) is linear, namely $\underline{c}^T x$ (and likewise for the problem (7)). Generally, the more variables x_i's are nonnegative the more number of absolute values in the objective function of (6) can be omitted and the problem becomes more simpler (fewer orthants have to be taken into account when branching the optimization problem according to the sign of x_i's).

Statement of Theorem 2 is nice, but rather theoretical, as finiteness of $\overline{f}$ is not known. This must be checked in other way: We have $\overline{f} < \infty$ if and only if for all $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ the problem (2) is feasible (i.e., $M(A,b) \neq \emptyset$). This leads to the notion of strong solvability adopted from Rohn's second chapter [7]. Characterization of strong solvability for all standard linear systems is known; see survey in Rohn's second chapter [7] again.

Definition 1. The system from description of $M(A,b)$, $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, is *strongly solvable* if $M(A,b)$ is nonempty for all $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$.

Now we are able to fully algortithmically specify optimal value bounds.

Algorithm 1 (Optimal value range $[\underline{f}, \overline{f}]$).

1. Compute

$$\underline{f} := \inf c_c^T x - c_\Delta^T |x| \ \text{ subject to } \ x \in M.$$

2. If $\underline{f} = \infty$, then set $\overline{f} := \infty$ and stop.
3. Compute

$$\overline{\varphi} := \sup b_c^T y + b_\Delta^T |y| \ \text{ subject to } \ y \in N.$$

4. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.
5. If $M(A,b)$, $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, is strongly solvable, then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

The following example demonstrates the situation when $\overline{f} \neq \overline{\varphi}$. In other words, the equation (7) is not true as the assumptions of Theorem 2 are not satisfied.

Example 1. Consider the interval linear program

$$\inf -x \ \text{ subject to } \ [0,1]x = 1, \ x \geq 0.$$

Its dual is

$$\sup y \ \text{ subject to } \ [0,1]y \le -1$$

Compute $\underline{f} = -\infty$, $\overline{\varphi} = -1$. However, one can easily see that $\overline{f} = \infty$. The reason for the inequation $\overline{f} \neq \overline{\varphi}$ is that for one realization of the interval coefficient (when it is zero) there is a duality gap between the problems caused by infeasibility both of them.

The assumption of Theorem 1 on finiteness of $\underline{f}$ suggests that analogous sufficient condition may old for our unified approach as well. Unfortunately, such a generalization does not hold, particularly when dependences between constraint matrix coefficients are encountered; see Example 2. Nevertheless, a slight modification of formula (7), as presented in Theorem 3, helps.

Example 2. Consider the interval linear program

$$\inf x_1 - x_2 \ \text{ subject to } \ a_1x_1 - x_2 = 1,\ x_1 - a_2x_2 = 1,\ x_1, x_2 \ge 0,$$

where $a_1 \in [1,2]$, $a_2 \in [0.5,1]$ with the dependence $a_1a_2 = 1$. Its dual for fixed a_1 and a_2 is

$$\sup y_1 + y_2 \ \text{ subject to } \ a_1y_1 + y_2 \le 1,\ -y_1 - a_2y_2 \le -1.$$

For $a_1 = a_2 = 1$, both problems are optimal and the optimal value is 1. Oherwise, both problems are infeasible. Thus $\underline{f} = 1$, $\overline{f} = \infty$ and $\overline{\varphi} = 1$.

Theorem 3. *Let $\underline{f}$ from* (6) *be finite. Then*

$$\overline{f} = \sup b_c^T y + b_\Delta^T |y| \ \textit{ subject to } \ y \in N',$$

where $N' \equiv \{y \in N'(A,c) \mid A \in \boldsymbol{A},\ c \in \boldsymbol{c}\}$ and $N'(A,c)$ is the dual feasible set to

$$\inf c^T x \ \textit{ subject to } \ x \in M(A,b) \cup \{x \mid c^T x \ge \underline{f}\}.$$

Proof. Since $\underline{f}$ is finite, problems

$$\inf c^T x \ \text{ subject to } \ x \in M(A,b),$$

and

$$\inf c^T x \ \text{ subject to } \ x \in M(A,b) \cup \{x \mid c^T x \ge \underline{f}\}, \tag{9}$$

have the same optimal value for every $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Let us show that $N'(A,c)$ is feasible for every $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$. Suppose that the problem (9) have the form

$$\inf c_1^T x_1 + c_2^T x_2 \ \text{ subject to } \ x \in M(A,b) \cup \{x \mid c^T x \geq \underline{f}\},$$

where

$$M(A,b) = \{(x_1, x_2) \mid A_1 x_1 + A_2 x_2 = b_1,\ A_3 x_1 + A_4 x_2 \geq b_2,\ x_1 \geq 0\}.$$

Every linear program (even interval) can be expressed in this way. Its dual is

$$\sup b_1^T y_1 + b_2^T y_2 + \underline{f} y_3 \ \text{ subject to } \ x \in N'(A,b),$$

where

$$\begin{aligned} N'(A,c) &= \{(y_1,y_2,y_3) \mid \ A_1^T y_1 + A_3^T y_2 + c_1 y_3 \leq c_1, \\ &\qquad\qquad\qquad\quad A_2^T y_1 + A_4^T y_2 + c_2 y_3 = c_2,\ y_2, y_3 \geq 0\} \\ &= \{(y_1,y_2,y_3) \mid \ A_1^T y_1 + A_3^T y_2 + c_1(y_3 - 1) \leq 0, \\ &\qquad\qquad\qquad\quad A_2^T y_1 + A_4^T y_2 + c_2(y_3 - 1) = 0,\ y_2, y_3 \geq 0\}. \end{aligned}$$

Hence, $(y_1,y_2,y_3) = (0,0,1) \in N'(A,c)$ for every $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$. Therefore there is no duality gap between (9) and its dual and

$$\begin{aligned} \overline{f} &= \sup_{A \in \boldsymbol{A},\, c \in \boldsymbol{c},\, b \in \boldsymbol{b}} \ \inf_{x \in M(A,b)} c^T x = \sup_{A \in \boldsymbol{A},\, c \in \boldsymbol{c},\, b \in \boldsymbol{b}} \ \sup_{y \in N'(A,c)} b^T y \\ &= \sup_{A \in \boldsymbol{A},\, c \in \boldsymbol{c}} \ \sup_{y \in N'(A,c)} b_c^T y + b_\Delta^T |y| = \sup_{y \in N'} b_c^T y + b_\Delta^T |y|. \end{aligned}$$

$\square$

Note that the statement of Theorem 3 is true even if there are dependences between the coefficients of the constraint matrix A (but not in the objective function or in the right-hand side).

3 Special cases

Example 3 (introductory case). Consider the introductory case; the set $M(A,b)$ be described as

$$M(A,b) = \{x \mid Ax = b, x \geq 0\}.$$

Then the solution set M is characterized by the well-known Oettli–Prager condition [7, 15]

$$\begin{aligned} M &= \{x \mid A_\Delta|x| + b_\Delta \geq |A_c x - b_c|,\ x \geq 0\} \\ &= \{x \mid \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0\}. \end{aligned}$$

The dual feasible set $N(A, c)$ is described as

$$N(A, c) = \{y \mid A^T y \leq c\}$$

and we use the Gerlach condition [7, 8] to characterize the dual solution set N

$$N = \{y \mid A_c^T y - A_\Delta^T|y| \leq \overline{c}\}.$$

Since

$$c_c^T x - c_\Delta^T|x| = c_c^T x - c_\Delta^T x = \underline{c}^T x,$$

it follows from Algorithm 1 that

$$\begin{aligned} \underline{f} &= \inf \underline{c}^T x \ \text{ subject to } \ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0, \\ \overline{f} &= \sup b_c^T y + b_\Delta^T|y| \ \text{ subject to } \ A_c^T y - A_\Delta^T|y| \leq \overline{c}. \end{aligned}$$

The second formula is true only under strong solvability of $M(A, b)$, $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, which is equivalent [7] to solvability of

$$(A_c - diag(z)A_\Delta)x = b_c + diag(z)b_\Delta,\ x \geq 0$$

for all $z \in \{\pm 1\}^m$, where $diag(z)$ denotes the diagonal matrix with entries $z_1, \ldots, z_m$.

Example 4 (simple case). Let the feasible set $M(A, b)$ be described as follows

$$M(A, b) = \{x \mid Ax \leq b,\ x \geq 0\}.$$

The solution set is characterized by [7]

$$M = \{x \mid \underline{A}x \leq \overline{b},\ x \geq 0\}.$$

The dual feasible set $N(A, c)$ is

$$N(A, c) = \{y \mid A^T y \geq c,\ y \geq 0\}$$

and its solution set is accordingly

$$N = \{y \mid \overline{A}^T y \geq \underline{c},\ y \geq 0\}.$$

Strong solvability of $M(A, b)$, $A \in \boldsymbol{A},\ b \in \boldsymbol{b}$ is equivalent [7] to solvability of the system of inequalities

$$\overline{A}x \leq \underline{b}.$$

Finally, we compute

$$\underline{f} = \inf \underline{c}^T x \quad \text{subject to} \quad x \in M,$$
$$\overline{\varphi} = \sup \overline{b}^T y \quad \text{subject to} \quad y \in N.$$

Using Algorithm 1, the optimal value bounds are computable in polynomial time.

Example 5 (case with dependences). Now we consider an example with dependences. Let $M(A, (b^1, b^2))$ be described as follows

$$M(A, (b^1, b^2)) = \{x, y \mid Ax = b^1,\ Ay = b^2,\ x, y \geq 0\}.$$

Such dependences (double appearance of a matrix) occur for instance when dealing with complex numbers. The solution set M is described due to [9] by nonlinear inequalities

$$\begin{aligned} M = \Big\{x, y \mid\ & \underline{A}x \leq \overline{b}^1,\ \overline{A}x \geq \underline{b}^1,\ \underline{A}y \leq \overline{b}^2,\ \overline{A}y \geq \underline{b}^2,\ x, y \geq 0, \\ & b^1_\Delta y^T + b^2_\Delta x^T + A_\Delta |xy^T - yx^T| \geq \\ & \geq |(A_c x - b^1_c) y^T - (A_c y - b^2_c) x^T| \Big\}. \end{aligned}$$

The dual feasible set $N(A, (c^1, c^2))$ is described as

$$N(A, (c^1, c^2)) = \{u, v \mid A^T u \leq c^1,\ A^T v \leq c^2\}$$

and the dual solution set N is characterized by [9]

$$\begin{aligned} N = \Big\{u, v \mid\ & A_c^T u - A_\Delta^T |u| \leq \overline{c}^1,\ A_c^T v - A_\Delta^T |v| \leq \overline{c}^2, \\ & (\overline{c}^1 - A_c^T u)|v_k| + (\overline{c}^2 - A_c^T v)|u_k| + A_\Delta^T |v_k u - u_k v| \geq \\ & \geq 0\ \forall k : u_k v_k < 0 \Big\}. \end{aligned}$$

According to Theorem 2 or Algorithm 1 we compute optimal value bounds as

$$\underline{f} = \inf(c_c^1)^T x - (c_\Delta^1)^T|x| + (c_c^2)^T y - (c_\Delta^2)^T|y| \ \text{ subject to } \ (x,y) \in M,$$
$$\overline{f} = \sup(b_c^1)^T u + (b_\Delta^1)^T|u| + (b_c^2)^T v + (b_\Delta^2)^T|v| \ \text{ subject to } \ (u,v) \in N.$$

But how to check assumption of Theorem 2 or strong solvability of $M(A,b)$, $A \in \boldsymbol{A},\ b \in \boldsymbol{b}$, is not known.

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