COMBINATORIAL PROOF THAT SUBPROJECTIVE CONSTRAINT SATISFACTION PROBLEMS ARE NP-COMPLETE

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Abstract. We introduce a new general polynomial-time construction- the *fibre construction*- which reduces any constraint satisfaction problem CSP($\mathcal{H}$) to the constraint satisfaction problem CSP($\mathcal{P}$), where $\mathcal{P}$ is any subprojective relational system. As a consequence we get a new proof (not using universal algebra) that CSP($\mathcal{P}$) is NP-complete for any subprojective (and thus also projective) relational system. The fibre construction allows us to prove the NP-completeness part of the conjectured Dichotomy Classification of CSPs, previously obtained by algebraic methods. We show that this conjectured Dichotomy Classification is equivalent to the dichotomy of whether or not the template is subprojective. This approach is flexible enough to yield NP-completeness of coloring problems with large girth and bounded degree restrictions thus reducing the Feder-Hell-Huang and Kostočka-Nešetřil-Smolíková problems to the Dichotomy Classification of coloring problems.

1. Introduction and Previous Work

Many combinatorial problems can be expressed as Constraint Satisfaction Problems (CSPs). This concept originated in the context of Artificial Intelligence (see e.g. [22]) and is very active in several areas of Computer Science. CSPs includes standard satisfiability problems and many combinatorial optimization problems, thus are also a very interesting class of problems from the theoretical point of view. The whole area was revitalized by Feder and Vardi [9], who reformulated CSPs as homomorphism problems (or H-coloring problems) for relational systems. Motivated by the results of [28] and [13], they formulated the following conjecture.

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Conjecture 1.1. (Dichotomy) *Every Constraint Satisfaction Problem is either P or NP-complete.*

Schaefer [28] established the dichotomy for CSPs with binary domains, and Hell-Nešetřil [13] established the dichotomy for undirected graphs; it follows from [9] that the dichotomy for CSPs can be reduced to the dichotomy problem for H-coloring for oriented graphs. This setting, and related problems, have motivated intensive research in descriptive complexity theory. This is surveyed, for example, in [7], [13] and [11].

Recently the whole area was put in yet another context by Peter Jeavons and his collaborators, in [15] and [5], when they recast the complexity of CSPs into properties of algebras and polymorphisms of relational systems. Particularly, they related the complexity of CSPs to a Galois correspondence between polymorphisms and definable relations (obtained by Bodnarčuk et al. [1] and by Gaiger [10]; see [26] and [27]). This greatly simplified elaborate and tedious reductions of particular problems and led to the solution of the dichotomy problem for ternary CSPs [2] and other results which are surveyed, for example, in [5] and [12]. This approach to studying CSPs via certain algebraic objects yields, in particular, that for every *projective* structure H the corresponding CSP(H) is an NP-complete problem [16], [15]. The success of these general algebraic methods gave motivation for some older results to be restated in this new context. For example, [4] treats H-coloring problems for undirected graphs in such a way that the dichotomy between the tractable and NP-complete cases of H-coloring problem agrees with the general conjecture stated in [6].

In this paper we propose a new approach to the dichotomy problem. We define a general construction- the *fibre construction*- which allows us to prove in a simple way that for every projective structure H, CSP(H) is NP-complete. In fact we define a *subprojective* structure and prove that for every subprojective relational structure H, CSP(H) is NP-complete. We then show (Theorem 2.2) that this dichotomy (to be or not to be a subprojective structure; Conjecture 2.4) coincides with the dichotomy for CSPs that is conjectured in [5], thus reproving in a combinatorial way the main result yielded by algebraic methods. This is stated as Corollary 2.3.

The fibre construction lends easily to restricted versions of CSPs, so allows us to address open problems from [8] and [17]. In particular, we reduce the Feder-Hell-Huang conjecture that NP-complete CSPs are NP-complete for instances of bounded degree, to the Dichotomy Classification Conjecture 2.4.

Our approach is motivated by the *Sparse Incomparability Lemma* [24] and *Müller's Extension Theorem* [23] (both these results are covered in [14]). These results are recalled and extended in Section 6. Strictly speaking, we do not need these results for the dichotomy results Theorem 2.2 and Corollary 2.3, but they provided an inspiration for early forms of the fibre construction in [29] and [30] and for the general case presented here. Moreover, we do need these results to address the Dichotomy Conjecture for instances of large girth, extending results of [17].

The fibre construction is simple, and is a refinement of gadgets, or indicator constructions [13, 14], using familiar extremal combinatorial results [23, 24, 25]). However, the simplicity becomes obscured by the notation when dealing with general relational structures. Thus we find it useful to prove, in Section 3, only a simple case of the fibre construction. The case we prove is simple, but contains all the essential ingredients of the general fibre construction.

In Section 2 we introduce all the definitions and state the main results. In Section 3 we prove a simple case of Theorem 2.2. In Section 4 we relate this to the Dichotomy Classification Conjecture. In Section 5 we consider further applications of the fibre construction. Section 6 contains an extension of the some of the motivating results of our construction.

2. Definitions and statement of results

We work with finite relational structures of a given type (or signature). A *type* is a vector $K = (k_i)_{i \in I}$ of positive integers, called *arities*. A *relational system* $\mathcal{H}$ of type K, consists of a finite vertex set $V = V(\mathcal{H})$, and a k_i-ary relation $R_i = R_i(\mathcal{H}) \subset V^{k_1}$ on V, for each $i \in I$. An element of R_i is called an k_i-tuple. Thus a (di)graph is just a relational system of type $K = (2)$. Its edges (arcs) are 2-tuples in the 2-ary relation R_1.

Given two relational systems $\mathcal{G}$ and $\mathcal{H}$ of the same type, an $\mathcal{H}$-*coloring of* $\mathcal{G}$ is a map $\phi : V(\mathcal{G}) \to V(\mathcal{H})$ such that for all $i \in I$ and every k_i-tuple $(v_1, \ldots, v_{k_i}) \in R_i(\mathcal{G})$, $(\phi(v_1), \ldots, \phi(v_{k_i}))$ is in $R_i(\mathcal{H})$. Fix a relational system $\mathcal{H}$ (sometimes called *template*). CSP($\mathcal{H}$) is the following problem decision problem:

Problem CSP($\mathcal{H}$)
Instance: A relational structure $\mathcal{G}$;
Question: Does there exists an $\mathcal{H}$-coloring of $\mathcal{G}$?

We write $\mathcal{G} \to \mathcal{H}$ to mean that $\mathcal{G}$ has an $\mathcal{H}$-coloring.

A relational system $\mathcal{H}$ is a *core* if its only $\mathcal{H}$-colorings are automorphisms.

It is well known, (see, for example, [14]) that $\mathcal{G} \to \mathcal{H}$ if and only if $\mathcal{G}' \to \mathcal{H}'$, where $\mathcal{G}'$ and $\mathcal{H}'$ are the cores of $\mathcal{G}$ and $\mathcal{H}$ respectively. Therefore, in the sequel, we only consider relational systems that are cores.

All relational systems of a given type form a category with nice properties. In particular, this category has products and powers which are defined explicitly as follows:

Given a relational system $\mathcal{H}$, and a positive integer d, the d-ary *power* $\mathcal{H}^d$ of $\mathcal{H}$ is the relational system of the same type as $\mathcal{H}$, defined as follows.

- $V(\mathcal{H}^d) = \{(v_1, \ldots, v_d) \mid v_1, \ldots, v_d \in V(\mathcal{H})\}$.
- For $i \in I$, $((v_{1,1}, v_{1,2}, \ldots, v_{1,d}), \ldots, (v_{k_i,1}, \ldots, v_{k_i,d}))$ is in $R_i(\mathcal{H}^d)$ if and only if all of $(v_{1,1}, v_{2,1}, \ldots, v_{k_i,1}), \ldots, (v_{1,d}, \ldots, v_{k_i,d})$ are in $R_i(\mathcal{H})$.

An $\mathcal{H}$-coloring of $\mathcal{H}^d$ (i.e. a homomorphism $\mathcal{H}^d \to \mathcal{H}$) is called a *$d$-ary polymorphism* of $\mathcal{H}$. A d-ary polymorphism ϕ is called a *projection* if there exists some $i \in 1, \ldots, d$ such that $\phi((v_1, \ldots, v_d)) = v_i$ for any $v_1, \ldots, v_d \in V(\mathcal{H})$. Let $\mathrm{Pol}(\mathcal{H})$, $\mathrm{Aut}(\mathcal{H})$ and $\mathrm{Proj}(\mathcal{H})$ be the sets of polymorphisms, automorphisms and projections (of all arities) of $\mathcal{H}$. A relational system $\mathcal{H}$ is *projective* if for every $\phi \in \mathrm{Pol}(\mathcal{H})$, $\phi = \sigma \circ \pi$ for some $\sigma \in \mathrm{Aut}(\mathcal{H})$ and some $\pi \in \mathrm{Proj}(\mathcal{H})$. (It is shown in [21] that almost all relational structures are projective.)

The following definition of graphs that are, in a sense, locally projective, is our principal definition.

Definition 2.1. A subset S of $V(\mathcal{H})$ is called *projective* if for every $\phi \in \mathrm{Pol}(\mathcal{H})$, ϕ restricts on S to the same function as does $\sigma \circ \pi$ for some $\sigma \in \mathrm{Aut}(\mathcal{H})$ and some $\pi \in \mathrm{Proj}(\mathcal{H})$. S is called *non-trivial* if $|S| > 1$. A relational structure $\mathcal{H}$ is called *subprojective* if it is a core and it contains a non-trivial projective subset.

Note that any subset of a projective set is again projective. A structure is projective if and only if the set of all its vertices is projective. It is easy to find subprojective structures which fail to be projective.

The main tool of the paper is the following general indicator construction which we call the *fibre construction.* This construction extends a construction first used in a in a Ramsey theory setting in [29], and then proved in [30] in the present form, for $\mathcal{H} = K_3$ and $\mathcal{P}$ being projective. A special case of it is proved in Section 3, the full proof is relegated to the full version of the paper.

Theorem 2.2. *Let $\mathcal{H}$ be any relational system, and let $\mathcal{P}$ be any subprojective relational system. Then there exists a polynomial time construction, the fibre construction, $\mathcal{M}_{\mathcal{H}}^{\mathcal{P}}$ which provides for any instance $\mathcal{G}$ of* CSP($\mathcal{H}$), *an instance $\mathcal{M}_{\mathcal{H}}^{\mathcal{P}}(\mathcal{G})$ of $\mathcal{P}$ such that*

$$\mathcal{G} \to \mathcal{H} \iff \mathcal{M}_{\mathcal{H}}^{\mathcal{P}}(\mathcal{G}) \to \mathcal{P}.$$

Note that $\mathcal{H}$ and $\mathcal{P}$ need not be of the same type. Since CSP(K_3) is NP-complete, taking $\mathcal{H}$ in to be K_3 gives the following result.

Corollary 2.3. *For any subprojective relational system $\mathcal{P}$, the problem* CSP($\mathcal{P}$) *is NP-complete.*

In [15] and [16], algebraic techniques were used to show that CSP($\mathcal{P}$) is NP-complete for any projective relational system $\mathcal{P}$. In [5] and [6], the techniques of [15] were extended to show that CSP($\mathcal{H}$) is NP-complete for a certain class of relational systems whose description requires some algebraic definitions that we present in Section 4. It turns out that all of these relational systems are subprojective (see Proposition 4.4). It is conjectured in [5] (see Conjecture 4.5) that CSP($\mathcal{H}$) is polynomial time solvable for any other relational system $\mathcal{H}$. We use a reformulation of Conjecture 4.5 by Larose and Zádori [18] to show that it is equivalent to the following formulation.

Conjecture 2.4. (Dichotomy Classification Conjecture) *The problem* CSP($\mathcal{H}$) *is NP-complete if and only if $\mathcal{H}$ is a subprojective.*

The fibre construction also has immediate applications to restricted versions of CSP complexity.

The *degree* of a vertex v in a relational system $\mathcal{G}$ is the number of tuples it occurs in in $\bigcup R_i$, and the maximum degree, over all vertices in $\mathcal{H}$, is denoted by $\Delta(\mathcal{G})$. $\mathcal{G}$ is called b-*bounded* if $\Delta(\mathcal{G}) \leq b$.

It is conjectured in [8] that for any relational system $\mathcal{H}$, if CSP($\mathcal{H}$) is NP-complete, then there is some finite b such that CSP($\mathcal{H}$) is NP-complete when restricted to b-bounded instances.

In [30], this was shown to be true in the case of graphs and projective relational systems $\mathcal{H}$. Further explicit bounds were given on $b(\mathcal{H})$, which is the minimum b such that CSP($\mathcal{H}$) is NP-complete when restriced to b-bounded instances.

In Section 5, we observe the following corollary of the proof of Theorem 2.2.

Corollary 2.5. *For any subprojective relational system* $\mathcal{P}$,

$$b(\mathcal{P}) < (4 \cdot \Delta(\mathcal{P})^6).$$

This greatly improves the bound on $b(H)$ from [30] in the case of graphs H. More importantly, this shows that if the Dichotomy Classification Conjecture 2.4 (and, equivalently, Conjecture 4.5 of [5]) is true, then so is the conjecture from [8].

Degrees and short cycles are classical restrictions for coloring problems. Recall that *girth* $g(G)$ of a graph G is the length of the shortest cycle in G. We then observe that the following result about sparse graphs follows from our extension (from Section 6) of the *Sparse Incomparability Lemma* [24], [25].

Theorem 2.6. *Let H be a subprojective graph, and ℓ a positive integer. Then the problem* CSP(H) *is NP-complete when restricted to graphs with girth* $\geq \ell$.

This solves a problem of [17] where the question of CSPs when restricted to instances with large girth was studied. This result can be generalized futher to relational structures but we decided to stop here.

3. The Fibre Construction

So called indicator constructions are often used relate the conplexity of different CSPs. The basic idea is that one can reduce (encode) CSP(H) to (in) CSP(H') by constructing in polynomial time, for any instance G of CSP(H), an instance G' of CSP(H'), such that

$$G \to H \iff G' \to H'.$$

If CSP(H) is NP-complete, then CSP(H') must also be NP-complete. See the proof of the H-coloring dichotomy in [13] for an intricate use of such constructions.

One of the difficulties with indicator constructions is that one uses many ad hoc tricks to find a constrution for a particular graphs H' or H. The fibre construction, Theorem 2.2, is an indicator construction that will suffice for all reductions.

In this section, we prove the following simple case of the fibre construction.

Proposition 3.1. *There exists a polynomial time construction which provides for and graph G, a graph $M(G)$ such that*

$$G \to C_5 \iff M(G) \to K_3.$$

Ours is not the most elegant known reduction of C_5-coloring to K_3-coloring, but it has the advantage that it can be easily generalized. After the proof, we discuss a couple of issues that we must deal with in the general case, Theorem 2.2. The proof of the general case can be found in the full version of the paper.

3.1. **Notation.** Given an indexed set $W^* = [w_1^*, \dots, w_d^*]$ of vertices, *a copy W^a of the set W^** will mean the indexed set $W^a = [w_1^a, \dots, w_d^a]$. Given two copies W^a and W^b of the same set W^* we say that we *identify W^a and W^b index-wise* to mean we identify the vertices w_i^a and w_i^b for $i = 1, \dots, d$. When we define a function f on W^*, we will assume it to be defined on any copy W^a of W^* by $f(w_\alpha^a) = f(w_\alpha^*)$ for all $\alpha = 1, \dots, d$. We will often refer to a function f on an indexed set W^* as a *pattern* of W^*. In the case that the image of f is contained in the vertex set of some graph H we speak about *H-pattern* of W^*.

3.2. **The Fibre Gadget.** The construction consists of two parts. In the first part we build a fibre gadget M which depends only on C_5 and K_3. To build the fibre gadget M we need the following simple lemma which is motivated by a result of Müller, [23].

Lemma 3.2. *Let $\mathcal{P}$ be a subprojective relational system with non-trivial projective subset S. Let W be an indexed set, and let $\Gamma = \{\gamma_1, \dots, \gamma_d\}$ be a set of S patterns of W satisfying the following condition (*).*

> *For any pair $w \neq w' \in W$, there exists some $\gamma \in \Gamma$ for which $\gamma(w) \neq \gamma(w')$.*

Then there exists a relational system $\mathcal{M}$, isomorphic to $\mathcal{P}^d$, with $W \subset V(\mathcal{M})$, such that the set of $\mathcal{P}$-colorings of $\mathcal{M}$, when restricted to W, is exactly

$$\{\alpha \circ \gamma \mid \alpha \in \mathrm{Aut}(\mathcal{P}), \gamma \in \Gamma\}.$$

Proof of Lemma 3.2. Put $\mathcal{M} = \mathcal{P}^d$ and for each $w \in W$, identify w with the vertex $(\gamma_1(w), \dots, \gamma_d(w))$ of $\mathcal{M}$. By condition (*), these are distinct elements of $V(\mathcal{M})$.

Since S is a projective subset of $\mathcal{P}$, the only $\mathcal{P}$-colorings of $\mathcal{M} = \mathcal{P}^d$ restrict on S^d, which contains W, to $\alpha \circ \pi$ where α is a automorphism of $\mathcal{P}$ and π is a projection. But the projections restrict on W to exactly the maps of Γ, so the lemma follows. □

The following lemma provides the fibre gadget M.

Lemma 3.3. *There exists a graph M containing two copies W^a and W^b of an indexed set W^*, and a set $F = \{f_x \mid x \in V(C_5)\}$ of distinct K_3-patterns of W^*, such that the following conditions are true upto some permutation of $V(K_3)$.*

(i) *Any K_3-coloring of M, restricted to W^a, (or to W^b) is in F.*
(ii) *For any K_3-coloring ϕ of M, ϕ restricts on W^a to f_x and on W^b to f_y for some edge xy of C_5.*
(iii) *For any edge xy (or yx) of C_5, there is a K_3-coloring ϕ of M that restricts on W^a to f_x and on W^b to f_y.*

Moveover, $M \cong (K_3)^{10}$.

Proof. Let $V(K_3) = \{0, 1, 2\}$. Let $W^* = [w^*_x \mid x \in V(C_5)]$, and let $F = \{f_x \mid x \in V(C_5)]$, where f_x is the $\{0, 1\}$-pattern (K_3-pattern) defined by

$$f_x(w^*_y) = \begin{cases} 1 & \text{x = y} \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = W^a \cup W^b$, where W^a and W^b are disjoint copies of W^* and let $\Gamma = \{\gamma_{xy} \mid xy \in E(C_5)\}$ where γ_{xy} is the $\{0, 1\}$-pattern of W defined by

γ_{xy} restricted to W^a is f_x and restricted to W^b is f_y.

Observe that γ_{xy} and γ_{yx} are distinct elements of Γ for every edge xy of C_5, so $|\Gamma| = 10$.

Apply Lemma 3.2 to W and Γ. The instance $\mathcal{M}$ of $\mathrm{CSP}(K_3)$ that it returns is clearly the graph M that we are looking for. □

The name 'fibre gadget' comes from the relation of the vertices of W^* to the set of K_3-patterns F. We view $w \in W^*$ as a fibre in $V(K_3)^{|F|}$, whose i^{th} postition corresponds to its image under the i^{th} pattern f_{x_i} of F.

3.3. **The Fibre Construction.**

The fibre gadgets are put together with the following construction, which call the fibre construction.

Construction 3.4. *Let W^*, F, and M be as in Lemma 3.3. Let G be an instance of $\mathrm{CSP}(K_3)$, and construct $M(G)$ as follows. (See Figure 1.)*

(i) *For each vertex v of G let W^v be a copy of W^*.*
(ii) *For each edge uv of G let M_{uv} be a copy of M. Index-wise, identify W^u and W^v with the copies of W^a and W^b, respectively, in M_{uv}.*

Thus $M(G)$ consists of $|V(G)|$ copies of W^* and $|E(G)|$ copies of M. All vertices are distinct unless identified above.

We can now prove Proposition 3.1.

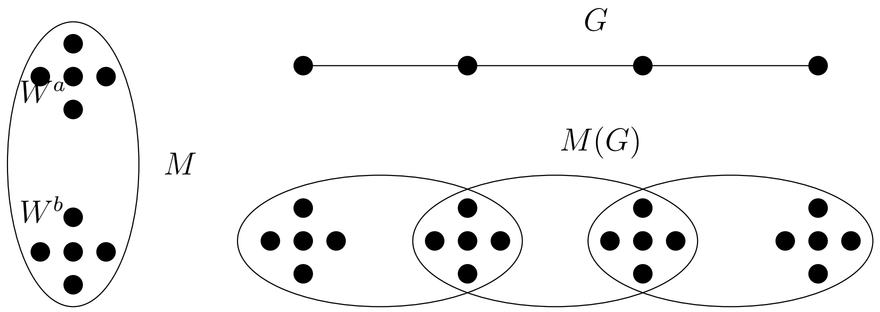

FIGURE 1. Fibre Construction

Proof. For any graph G let $M(G)$ be the graph defined by the fibre construction, Construction 3.4. As $M(G)$ is made of $|E(G)|$ copies of M, which is independent of G, this is a polynomial time construction.

Let ϕ be a K_3-coloring of $M(G)$. We show that this defines a C_5 colouring ϕ' of G. It is enough to show this for a compontent of $M(G)$. Now ϕ restricts on W^v, for each vertex v of G, to $\sigma \circ f$ for some permutation σ of $V(K_3)$ and some pattern f in F. Since the number of vertices of each color is constant over all patterns of F, and this property is not preserved under any permutaion of $V(K_3)$, the permutaion σ must be constant for all W^v. We assume that it is the identity permutation, so ϕ restricts on each W^v to some pattern f in F. Thus $\phi' : V(G) \to V(C_5)$ is well defined by letting $\phi'(v) = x$ where ϕ restricts on W^v to the pattern f_x. Moreover, by property (ii) of Lemma 3.3, ϕ' is a C_5-coloring of G.

On the other hand, given a C_5-coloring ϕ' of G we define a K_3-coloring ϕ of $M(G)$ as follows. For all vertices v of G, let ϕ be $f_{\phi'(v)}$ on the set W^v. For every edge uv of G, the sets W^u and W^v are already colored by ϕ, and we must extend this coloring to M^{uv}. Now ϕ restricts on W^u to $f_{\phi'(u)}$ and on W^v to $f_{\phi'(v)}$, and $\phi'(u)\phi'(v)$ is an edge of C_5, so by property (iii) of Lemma 3.3 ϕ can be extended to M^{uv}. Thus ϕ can be extended to a K_3-coloring of $M(G)$. □

3.4. **Remark.** This outline gives only the idea of the general proof. There are several obstacles. For example, in the general case of relational systems, we will need a different fibre gadget for each kind of relation. And, of course, our relations need not be symmetric. In the general case, the set F from Lemma 3.3 will be S-patterns, instead of K_3-patterns. Also, for general

$\mathcal{H}$, it may be more difficult to to ensure that Γ in the proof of Lemma 3.3 satisfies property (*) of Lemma 3.2. We thus use a more general version on Lemma 3.2 in which Γ need not satisfy (*), but which returns a $\mathcal{M}$ that is not necessarily isomorphic to $\mathcal{P}^d$.

These are just technicalities which can be handled with care.

4. CSP Dichotomy Conjecture - algebraic approach

In [5], the universal algebra approach of [15] is extended to to show that CSP($\mathcal{H}$) is NP-complete for a large class of CSPs. A conjecture is made that CSP($\mathcal{H}$) is polynomial time solvable for all other CSPs $\mathcal{H}$. In [19], this conjecture is then transported to the language of posets.

An algebra $\mathcal{A} = (A, F)$ consists of a non-empty set A, and a set F of finitary operations on A. It is *finite* if A is finite. Given a relational system $\mathcal{H}$, recall that Pol($\mathcal{H}$) is the set of polymorphisms of $\mathcal{H}$. This defines an algebra $\mathcal{A}_\mathcal{H} = (V(\mathcal{H}), \mathrm{Pol}(\mathcal{H}))$. We say that $\mathcal{A}_\mathcal{H}$ is NP-complete if CSP($\mathcal{H}$) is.

The following two definitions are borrowed directly from [5].

Definition 4.1. Let $\mathcal{A} = (A, F)$ be an algebra and B a subset of A such that, for any $f \in F$ and for any $b_1, \ldots, b_d \in B$, where d is the arity of f, we have $f(b_1, \ldots, b_d) \in B$. Then the algebra $\mathcal{B} = (B, F|_B)$ is called a subalgebra of $\mathcal{A}$, where $F|_B$ consists of the restrictions of all operations in F to B.

Definition 4.2. Let $\mathcal{B} = (B, F_1)$ and $\mathcal{C} = (C, F_2)$ be such that $F_1 = \{f_i^1 \mid i \in I\}$ and $F_2 = \{f_i^2 \mid i \in I\}$, where both f_i^1 and f_i^2 are d_i-ary, for all $i \in I$. Then $\mathcal{C}$ is a homomorphic image of $\mathcal{B}$ if there exists a surjection $\psi : B \to C$ such that the following identity holds for all $i \in I$, and all $b_1, \ldots, b_{d_i} \in B$.

$$\psi \circ f_i^1(b_1, \ldots, b_{d_i}) = f_i^2(\psi(b_1), \ldots, \psi(b_{d_i})).$$

Given an algebra $\mathcal{C} = (C, F)$, the *term operators* of $\mathcal{C}$ refer to the set of finitary operators of C that preserve the same relations on C as F does. Thus all operators in F are term operators. A d-ary operator f of F is *essentially unary* if $f = f' \circ \pi$ for some projection $\pi : C^d \to C$ and some non-constant function $f' : C \to C$. Because this f' is non-constant, if F has any essentially unary operators, then $|C| \geq 2$.

The following result is Corollary 7.3 in [5].

Theorem 4.3. *A finite algebra $\mathcal{A}$ is NP-complete if it has a subalgebra $\mathcal{B}$ with a homomorphic image $\mathcal{C}$, all of whose term operators are essentially unary.*

Using our main theorem, the following proposition reproves Theorem 4.3 in the case that $\mathcal{A} = \mathcal{A}_{\mathcal{H}}$ for some relational system $\mathcal{H}$.

Proposition 4.4. *Let $\mathcal{H}$ be a relational system such that $\mathcal{A}_{\mathcal{H}}$ has a subalgebra $\mathcal{B} = \mathcal{A}(B, F_1 = \mathrm{Pol}(\mathcal{H})|_B)$ with a homomorphic image $\mathcal{C} = \mathcal{A}(C, F_2)$, all of whose term operators are essenially unary. Then $\mathcal{H}$ is subprojective.*

The proof of this is not difficult, but in the interest of space, we omit it.

Bulatov, Jeavons, and Krokhin make the following conjecture.

Conjecture 4.5. ([5]) *For a relational system $\mathcal{H}$,* CSP($\mathcal{H}$) *is NP-complete if and only if $\mathcal{A}_{\mathcal{H}}$ has a subalgebra $\mathcal{B} = \mathcal{A}(B, F_1 = \mathrm{Pol}(\mathcal{H})|_B)$ with a homomorphic image $\mathcal{C} = \mathcal{A}(C, F_2)$, all of whose term operators are essenially unary.*

In [19], Larose and Zádori restate this conjecture in terms of Taylor operations.

An idempotent d-ary operation ϕ on an algebra $\mathcal{A} = (A, F)$ is called a Taylor operation if for every $a \neq b \in A$ and every $i \in 1, \dots, d$, there is some way to fill in the slots in the following equation, with some choice of a and b, such that it is true. (Here, the shown a and b are both in the i^{th} slot of ϕ.)

$$\phi(-, -, \dots, -, a, -, \dots, -, -) = \phi(-, -, \dots, -, b, -, \dots, -, -)$$

The following result immediately follows from a result of Taylor [31] that characterises all algebras without any Taylor operations (see Theorem 4 in [19],) and Proposition 5 of [19].

Theorem 4.6 ([19])**.** *For a relational system $\mathcal{H}$, $\mathcal{A}_{\mathcal{H}}$ has a subalgebra with a homomorphic image all of whose term operators are essentially unary if and only if there are no Taylor operations among the term operators of $\mathcal{A}_{\mathcal{H}}$.*

With this theorem, Larose and Zádori get that the following equivalent statement of Conjecture 4.5.

Conjecture 4.7. *The problem* CSP($\mathcal{H}$) *is NP-complete if and only if there are no Taylor operations among the term operators of $\mathcal{A}_{\mathcal{H}}$.*

The following is now straight forward, and again we omit the proof.

Proposition 4.8. *Let $\mathcal{H}$ be a subprojective relational system. Then there are no Taylor operations among the term operations of the algebra $\mathcal{A}_{\mathcal{H}}$.*

By Propositions 4.4 and 4.8, and Theorem 4.6 we get the following.

Corollary 4.9. *Conjectures 2.4, 4.5, and 4.7 are equivalent.*

5. Applications

5.1. **Degree Bounded** CSP**s.** We mentioned in the introduction, that because CSP(K_3) is NP-complete, taking $\mathcal{H} = K_3$, Corollary 2.3 follows from Theorem 2.2. In fact, CSP(K_3) is NP-complete for 4-bounded instances G.

For a 4-bounded graph G, the fibre construction would yield an instance $\mathcal{M}^{\mathcal{P}}_{K_3}(G)$ of CSP($\mathcal{P}$) with maximum degree $(4 \cdot \Delta(\mathcal{P}))^6$. Thus Corollary 2.5, follows from the proof of Theorem 2.2.

5.2. **Girth Restricted H-coloring.** The results in this subsection are for graphs.

The following lemma is proved in [23] in the case that P is a complete graph, and is proved in [25] without item (iii) in the case that P is projective. In both of these cases, $S = V(P)$.

Lemma 5.1. *Let P be a subprojective graph with projective subset S, and let $\ell \geq 3$ be an integer. Let W be an indexed set, and let $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ be a set of S patterns of W. Then there exists a relational system M with $W \subset V(M)$, such that the following are true:*

(i) *The set of P-colorings of M, when restricted to W, is exactly*

$$\{\alpha \circ \gamma \mid \alpha \in \mathrm{Aut}(P), \gamma \in \Gamma\}.$$

(ii) *M has girth at least ℓ.*

(iii) *The distance, in M, between any two vertices of W is at least g.*

Proof. This lemma follows from Theorem 6.2 which is a local form of the main result of [25]. The result will be stated in Section 6. □

Using this in place of Lemma 3.2 in the fibre construction, we can ensure that the graph $\mathcal{M}^{P}_{H}(G)$ that is returned has girth ℓ. Thus Theorem 2.6 follows.

5.3. **Conservative CSPs.** A constraint satisfaction problem CSP($\mathcal{H}$) is *conservative* if $\mathcal{H}$ contains all possible unary relations. Such a CSP is also known as List $\mathcal{H}$-colouring.

In [3], Bulatov proves the following dichotomy for conservative CSPs.

Theorem 5.2. [3] *A conservative constraint satisfaction problem* CSP($\mathcal{H}$) *is NP-complete if there there is a set $B \subset V(\mathcal{H})$ of size at least 2 such that for any polymophism ϕ of $\mathcal{H}$, ϕ restricted to B is essentially unary. Otherwise,* CSP($\mathcal{H}$) *is polynomial time solvable.*

The difficult part of Bulatov's paper is the polynomial time solvable part of this result. The NP-complete part follows quickly from the algebraic approach of [15] and [5]. We observe that the NP-complete part is also immedieate from Corollary 2.3. Indeed, since we only consider cores $\mathcal{H}$, any essentially unary operation on B is $\alpha \circ \pi$ where π is a projection of B^d to B and α is an automorphism of $\mathcal{H}$. Thus B is a projective subset of $\mathcal{H}$.

6. Coloring Theorems - Combinatorial Background

The main motivation for our construction is a result of Müller [23], which is a special graph case of Lemma 3.2, except that it returns a graph M of arbitrary girth. The difficult part of the lemma is, of course, ensuring that M has arbitrary girth. He did this with a special case of Lemma 6.2. Müller's lemma was extended in [25], and the form here, is a localisation of their version.

Localising the notion of H-pointed graphs H, from [25], we get the following definition.

Definition 6.1. Let H, H' be graphs. Subsets S of $V(H)$ and S' of $V(H')$ are said to be (H, H')-*pointed subsets* if for any two homomorphisms $g, g' : H \to H'$ which satisfy $g(x) = g'(x) \in S'$, whenever $x \neq x_0$ and $x \in S$ (for some fixed vertex $x_0 \in S$), then $g(x_0) = g'(x_0) \in S'$.

Theorem 6.2. *For every graph H and every choice of positive integers k and l there exists a graph G together with a surjective homomorphism $c : G \to H$ with the following properties.*

(i) *$g(G) > l$;*
(ii) *For every graph H' with at most k vertices and there exists a homomorphism $g : G \to H'$ if and only if there exists a homomorphism $f : H \to H'$.*
(iii) *For every (H, H')-pointed subsets $S \subset V(H), S' \subset V(H')$ with at most k vertices and for every homomorphism $g : G \to H'$ holds: if homomorphisms $f, f' : H \to H'$ satisfy $g = f \circ c$, then $f(x) = f'(x)$ for every $x \in S$.*

The proof of Theorem 6.2 is along the same lines as less general version proved in [25]. We essentially repeat their proof replacing the notion of H-pointed with its localization, (H, H')-pointed. This is routine but lengthy, so we omit the proof.

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