Homomorphisms and Polynomial Invariants of Graphs [1]

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Abstract

This paper initiates a study of the connection between graph homomorphisms and the Tutte polynomial. This connection enables us to extend the study to other important polynomial invariants associated with graphs, and closely related to the Tutte polynomial. We then obtain applications of these relationships in several areas, including Abelian Groups and Statistical Physics. A new type of uniqueness of graphs, strongly related to chromatically-unique graphs and Tutte-unique graphs, is introduced in order to provide a new point of view of the conjectures about uniqueness of graphs stated by Bollobas, Peabody and Riordan.

Keywords: Homomorphism, Tutte polynomial, Polynomial invariant.

1 Introduction

Counting homomorphisms between graphs arise in many different areas including extremal graph theory, partition functions in statistical physics and property testing of large graphs. Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a *homomorphism of G to H*, written as $f : G \to H$, is a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. The number of homomorphisms of G to H is denoted by $\hom(G, H)$. This number, considered as a function of G with H fixed is a *graph parameter*, i.e, a function of graphs invariant under isomorphisms. A more broader class of parameters related to homomorphisms was recently intensively studied in the context of statistical physics, see for example [3].

The motivation of this work is to show the usefulness of the homomorphism perspective in the study of polynomial invariants of graphs. Thus, our main contribution is to prove that there exists a strong connection between counting graph homomorphisms and evaluating polynomials associated with graphs.

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One of the most studied polynomial invariants in combinatorics is the Tutte polynomial, a two-variable polynomial $T(G; x, y)$ associated with any graph G (see for instance [10]). It is well-known that homomorphisms of a graph G to the complete graph K_n are the $n-$colourings of G (see [5]). Since the Tutte polynomial can be regarded as an extension of the chromatic polynomial, a natural question arises: can we find a graph H such that $\hom(G, H)$ is given (up to a determined term) by an evaluation of the Tutte polynomial of G? In 1984, Joyce [7] showed that the number of homomorphisms of any graph G to a complete graph with loops, but not with multiple edges, could be deduced from the Tutte polynomial of G. We prove that every complete graph with p loops at each vertex and multiplicity q at each non-loop edge, being p different than q, can play the role of H. As well as the Tutte polynomial is an extension of the chromatic polynomial, this complete graph which we denote by $K_n^{p,q}$, is a natural extension of K_n (see Figure 1).

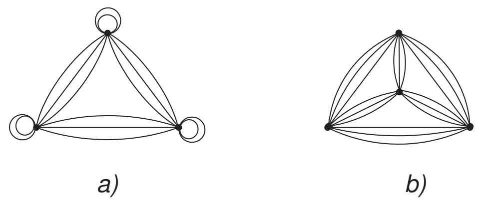

Fig. 1. a) $K_3^{2,3}$ b) $K_4^{0,3}$

We also prove that, by assuming a local condition, every graph H' such that the parameter $\hom(\_\_, H')$ can be recovered from the Tutte polynomial is necessarily isomorphic to some $K_n^{p,q}$. The characterization of those graphs H leads to important connections between homomorphisms and other polynomial invariants associated with graphs, such as, the *transition*, the *circuit partition*, the *boundary*, and the *coboundary polynomials*.

This work also provides applications of all the obtained relationships. We list several applications to: duality, homogeneous graphs, difference sets in abelian groups, the combinatorial analysis of the Potts model, and the Gibbs probability. We conclude the paper by introducing the concept of *colouring-unique graph* and by showing its connection with two well-known notions: the *Tutte-unique graphs* and the *chromatically-unique graphs*.

2 Connection between homomorphisms and polynomial invariants of graphs

Our first aim in this section is to define the concept of local function, and to determine those graphs H, such that the parameter $\hom(\_\_, H)$ is given by an evaluation of the Tutte polynomial, up to a local function. Throughout this section, we consider the family of graphs $K_n^{p,q}$ with $p, q \geq 0$, $n \geq 1$ and $p \neq q$.

Definition 2.1 Let H and G be two graphs. Denote by $G - e$ and G/e the result of both, deleting and contracting the edge e in G respectively. A function depending on G and H, denoted by $h(G, H)$, is said to be a local function if the fractions $h(G, H)/h(G - e, H)$ and $h(G, H)/h(G/e, H)$ depend only on H and independently of the choice of $e \in E(G)$.

Theorem 2.2 *For every graph G with λ vertices, m edges and c connected components, the following holds:*

(1) $\hom(G, K_n^{p,q}) = n^c(p-q)^{\lambda-c}q^{m-\lambda+c}T\left(G; \frac{p+q(n-1)}{p-q}, \frac{p}{q}\right)$ *with* $n > 1$.

(2) $\hom(G, K_1^{p,0}) = (p/2)^m T(G; 2, 2)$ *with* $p > 0$.

Theorem 2.3 *Let H be a graph. There exist two constants x_H and y_H such that, for every graph G there is a local function $h(G,H)$ verifying that* $\hom(G,H) = h(G,H)T(G;x_H,y_H)$ *if and only if there exist $p,q,n \in \mathbb{N}$ with $p \neq q$ and $n \geq 1$ so that H is isomorphic to $K_n^{p,q}$.*

There are many polynomial invariants of graphs that can be recovered from the Tutte polynomial (see [1,6,8,11] for more details). Such relationships and the two previous results lead to the characterization of the graphs H such that $\hom(\_\_, H)$ can be expressed in terms of the following polynomials: transition, circuit partition, boundary, and coboundary. Some of the relationships obtained in this work are summarized in the following table.

Homomorphisms-Transition polynomial $\hom(G, K_n^{p,q}) = n^{m-\lambda+1}(p-q)^m\delta^m Q(M(G), A, \sqrt{n})$ if $p-q \neq q\sqrt{n}$, $\lambda = \lvert V(G)\rvert$ and $m = \lvert E(G)\rvert$ $\hom(G, K_n^{p,q}) = (\sqrt{n})^{\lambda+1} q^m \delta^m Q(M(G), A, \sqrt{n})$ if $p-q = q\sqrt{n}$, $\lambda = \lvert V(G)\rvert$ and $m = \lvert E(G)\rvert$	G connected planar graph $M(G)$ medial graph
Homomorphisms-Circuit transition polynomial $\hom\left(G, K_n^{(1+\sqrt{n})q,q}\right) = (\sqrt{n})^\lambda q^m j(\overrightarrow{M(G)}; \sqrt{n})$ $q \geq 1$, $n > 1$, $\sqrt{n} \in \mathbb{N}$, $\lambda = \lvert V(G)\rvert$ and $m = \lvert E(G)\rvert$	G planar graph $\overrightarrow{M(G)}$ directed medial graph
Homomorphisms-Boundary and Coboundary polynomials $\hom(G, K_n^{p,q}) = n^{\lambda-m}(p-q)^m F\left(G; n, \frac{p+q(n-1)}{p-q}\right)$ $\hom(G, K_n^{p,q}) = q^m P(G; n, p/q)$	G any graph $\lambda = \lvert V(G)\rvert$ $m = \lvert E(G)\rvert$

The next result characterizes the graphs H such that for every connected planar graph G, $\hom(G,H)$ can be recovered from a transition polynomial of its medial graph, denoted by $M(G)$.

Theorem 2.4 *Let H be a graph and let $\delta_H \neq 0$, $\mu_H \neq 0$, and τ_H be three constants depending on H. Suppose that for every* *connected planar graph G with set of faces R, there is a weight function A for $M(G)$, which assigns the values α, β and γ to all black, white and crossing transitions respectively, satisfying the following equations:* $\tau_H\alpha+\beta+\gamma = \frac{1}{\delta_H}+\frac{\tau_H}{\mu_H}$; $\alpha+\tau_H\beta+\gamma = \frac{1}{\mu_H}+\frac{\tau_H}{\delta_H}$; $\alpha+\beta+\tau_H\gamma = \frac{1}{\delta_H}+\frac{1}{\mu_H}$. *Assume also that there exists a local function $h(G,H)$ so that* $\hom(G,H) = h(G,H)Q(M(G);A,\tau_H)$. *Then there exist $p,q,n \in \mathbb{N}$ with $p \neq q$ and $n \geq 1$ such that H is isomorphic to $K_n^{p,q}$.*

We now state a similar result for the circuit partition polynomial which generalizes by giving appropriate weights one of Jaeger's transition polynomials, defined on 4−regular graphs, to Eulerian digraphs with arbitrary even degrees. This polynomial is a simple transform of the Martin polynomial.

Theorem 2.5 *Let H be a graph. If there exists a constant x_H such that, for every planar graph G there is a local function $h(G,H)$ verifying that* $\hom(G,H) = h(G,H)j(\overrightarrow{M(G)};x_H)$, *then there exist $p,q,n \in \mathbb{N}$ with $p \neq q$ and $n \geq 1$ such that H is isomorphic to $K_n^{p,q}$.*

Finally, analogous results can be formulated for the boundary and coboundary polynomials which were introduced as a generalization of the flow and chromatic polynomials respectively (see [11]). In the case of the boundary polynomial, the connection with homomorphisms is stated as follows.

Theorem 2.6 *Let H be a graph. There exist a constant x_H and a positive integer number $r_H > 1$ such that, for every graph G there is a local function $h(G,H)$ verifying that* $\hom(G,H) = h(G,H)F(G;r_H,x_H)$ *if and only if there exist $p,q,n \in \mathbb{N}$ with $p \geq 0$, $p \neq q$ and $n \geq 1$ so that $H \cong K_n^{p,q}$.*

3 Applications

In this section we sketch several applications of the above-stated connections.

3.1 Duality

Proposition 3.1 *For every planar graph G with λ vertices, m edges, and c connected components, the following holds:*

(1) $\hom(G, K_n^{p,q}) = \left(\frac{p-q}{q}\right)^m n^{\lambda-m-1} \hom\left(G^*, K_n^{q+\frac{q^2 n}{p-q}, q}\right)$ *where G^* denotes the dual graph of G, $p \geq 0$, $q \geq 1$, $p \neq q$, and $q + \frac{q^2 n}{p-q} \in \mathbb{N}$.*

(2) $\hom(G, K_1^{p,0}) = \hom(G^*, K_1^{p,0})$ *with $p > 0$.*

3.2 Homogeneous Graphs

A finite graph G is said to be *homogeneous* if any isomorphism between induced subgraphs of G extends to an automorphism of G. In 1976, Sheehan and Gardiner [4] determined the finite homogeneous graphs. They fall into the following families: disjoint union of complete graphs of the same size, regular complete multipartite graphs, $5-$cycle C_5, and the line graph of $K_{3,3}$.

Proposition 3.2 *Let H be a finite homogeneous graph with n vertices. There exist two constants x_H and y_H such that, for every graph G there is a local function $h(G, H)$ verifying that $\hom(G, H) = h(G, H)T(G; x_H, y_H)$ if and only if H is isomorphic to K_n.*

3.3 Difference Sets in Abelian Groups

Let A be an abelian group of order r and $2 \leq k \leq r$. A $(r, k, l)-$*difference set* in A is a subset B of k elements of A

such that, for all $0 \neq a \in A$ there exist l pairs $(b_1, b_2) \in B \times B$ with $b_1 - b_2 = a$.

Proposition 3.3 *Let G be a graph, A an abelian group on r elements, $B \subseteq A$ an $(r,k,l)-$*difference set *in A, and let q be a positive integer number such that $(\frac{rl}{k-l}+1)q \in \mathbb{N}$. If two functions $f_1, f_2 : E(G) \to B$ are chosen uniformly at random, then the event that f_1 and f_2 have the same boundary has the following probability,*

$$Pr(d^* f_1 = d^* f_2) =$$
$$= k^{-2|E(G)|}(k-l)^{|E(G)|} r^{-|V(G)|} q^{-|E(G)|} \hom\left(G, K_r^{(\frac{rl}{k-l}+1)q,q}\right)$$

3.4 The Potts Model

For the combinatorial analysis of the Potts model on a finite graph G, it is assumed that the interaction energy, which measures the strength of the interaction between neighbourings pairs of vertices, is constant and equal to J. Consider that each atom can be in Q different states and $K = 2\beta J$, where β is a parameter determined by the temperature.

Proposition 3.4 *Let G be a finite graph, $K = 2\beta J$, and $q \in \mathbb{N}$ such that $e^K q \in \mathbb{N}$. Then, the partition function of the Potts model is given by the following formula,*

$$Z(G) = e^{-K|E(G)|} q^{-|E(G)|} \hom(G, K_Q^{e^K q,q}).$$

3.5 The Gibbs Probability

There are many different interpretations of the random cluster model summarized by A. Sokal [9], but one of the reasons for studying percolation in the random cluster model is its relation with phase transitions. In fact, this model can be regarded as the analytic continuation of the Potts model to non integer Q. Let G be a finite graph and A a subset of $E(G)$. The *Gibbs probability*, is a two parameter family of probability measures $\mu(t,Q)$ given by $\mu(A) = t^{|A|}(1-t)^{|E(G)-A|}Q^{k(A)}Z(G)^{-1}$ where $0 \leq t \leq 1$, $Q > 0$, and $k(A)$ denotes the number of connected components of the graph $(V(G), A)$.

Proposition 3.5 *For every $Q, s \in \mathbb{N}$ such that $(1-t)s$ is a positive integer number,*

$$\mu(A) = \frac{1}{\hom\left(G, K_Q^{s,(1-t)s}\right)} \left(\left(\frac{t}{1-t}\right)^{|A|} Q^{k(A)} (1-t)^{|E(G)|} s^{|E(G)|} \right)$$

4 Colouring Uniqueness

A graph G is said to be *Tutte-unique* if $T(G;x,y) = T(H;x,y)$ implies that H is isomorphic to G for every other graph H. In 2000, Bollobas, Peabody, and Riordan [2] conjectured that almost all graphs are Tutte-unique.

Definition 4.1 A graph G is colouring-unique if $\hom(G, K_n^{p,q}) = \hom(H, K_n^{p,q})$ for all $n \geq 1$, $p, q \geq 0$, $p \neq q$ implies $H \cong G$ for every other graph H.

Theorem 4.2 *Let G be a simple $2-$connected graph. If G is colouring-unique, then G is Tutte-unique.*

Consequently, if almost all graphs are colouring-unique, then also almost all graphs are Tutte-unique.

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