A Finite Presentation of the rational Urysohn Space

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Abstract

We give a new construction of the rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$, which yields a finite presentation of $\mathbb{U}_{\mathbb{Q}}$. This may be viewed as an extension of the finite presentation of the generic partial order.

1 Introduction and background

There is a unique (up to isometry) separable Polish space $\mathbb{U}$ which is both universal (for all separable metric spaces) and ultrahomogeneous. (Recall, that a space is ultrahomogeneous if every isometry between finite subspaces extends to a total isometry.)

This remarkable result is due to P. Urysohn [27] and it is quoted as his last paper (written in 1925). The paper was almost neglected until 1986 when M. Katětov wrote (one of his last paper in his distinguished career) a paper [12] where he gave a new construction of the Urysohn space. The recent activity and importance of the Urysohn space, besides being a beautiful result in topology (it is the universal Polish space [27, 12, 28]), stems from several sources:

*Supported by a Grant 1M0021620808 of the Czech Ministry of Education

1.1 Early limit argument

The proof of Urysohn uses a construction of a countable metric space with rational distances $\mathbb{U}_{\mathbb{Q}}$ of which $\mathbb{U}$ is then the Cauchy completion. This $\mathbb{U}_{\mathbb{Q}}$ is a direct limit of the set of all finite rational metric spaces. This limit is a special case of a general model theoretic construction, now called *Fraïssé limit*, which holds for general structures. This is a key result of modern model theory. It appears that Urysohn anticipated this construction in a quite general (and complicated) case. (It also appears that Katětov was unaware of Fraïssé's work.)

1.2 Topological Dynamics

The Urysohn space is not only an important (and generic) space in the context of topological dynamics. The automorphism group $Aut(\mathbb{U})$ is extremely amenable which in turn is related to triviality of minimal flows. This important connections were discovered in [23, 22] and then on a very abstract level by [16], see the recent book [24].

1.3 Combinatorial connection I

The paper [16] relates the extreme amenability (of subgroups of S_ω) to purely combinatorial problems of *Ramsey classes*. Ramsey classes are (top of the line) generalizations of Ramsey Theorem [17, 19]. Several permutation groups were shown to be extremely amenable using combinatorial examples of Ramsey classes (such as the class of all finite graphs, the class of all finite posets or the class of Hales-Jewett cubes) and thus some further examples of extremely amenable groups were found [6, 16, 22, 23]. This also provoked combinatorial questions which led to new examples of Ramsey classes:

1. Particularly, the second author proved that all (ordered) finite metric spaces form a Ramsey class [20], see also writeup [25]. This gives [16] a simpler new proof that $Aut(\mathbb{U})$ is an extremely amenable group ([23]).

2. More recently Farah-Solecki [5] isolated in the context of extreme amenability a new "group valued" Hales-Jewett Theorem in the context of Lévy groups.

Other combinatorial aspects of Urysohn space are related to the concept of divisibility (see e.g. [4, 18, 8, 26]).

All those examples illustrate the broad context of the Urysohn space. We had to review it as our contribution in this paper draws from all parts of this context. This will be introduced now.

1.4 Combinatorial connection II – finite presentations

The Fraïssé limit of the class of all finite graphs is called the *Rado Graph* $\mathcal{R}$. This is both universal and ultrahomogeneous graph which can be defined in surprisingly different ways (see [1]). Here want to single out the following succinct definition:

i. the vertices $V(\mathcal{R})$ are all finite 0–1 sequences $(a_1, a_2, \ldots, a_t), t \in \mathbb{N}$,

ii. pair $\{(a_1, a_2, \ldots a_t), (b_1, b_2, \ldots, b_s)\}$ form an edge of $E(\mathcal{R})$ iff $b_a = 1$ where $a = \sum_{i=1}^{t} a_i 2^i$.

This definition means that both vertices and edges are finitely described – axiomatized; they are specified as all finite models of a certain finite sets of formulas. In [10] the authors called this phenomenon *finite presentation* of $\mathcal{R}$ and they proved that all ultrahomogeneous undirected graphs, tournaments and posets (characterized by Schmerl [15], Lachlan's [13, 14] and Cherlin [2]) were shown to possess a finite presentation. (Of course, there are only countably many finite presentations of ultrahomogeneous oriented graphs which leads to an interesting problem [10].) Out of these presentations stands the generic poset $\mathcal{P}$ for which the finite presentation is the most involved case. Even countable universality is nontrivial to achieve for finitely presented posets, see [7, 11, 9]. So it seemed that a finite presentation of the generic rational metric space was out of reach of finite presentations. This was also the conclusion of discussions held with P. Cameron, A. Vershik and others in St. Petersburg meeting in 2005.

The purpose of this paper is to prove the following which may be viewed as a contribution to Problem 12 of [24] (about a model of the Urysohn Space $\mathbb{U}$).

Theorem 1.1 *The rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$ has a finite presentation.*

This is proved in Section 2. This finite presentation is made possible by reproving (in a more concise way) the finite presentation of the generic poset $\mathbb{P}$ [10] which we include both for motivation and illustration in Section 3. In Section 4 we modify the finite presentation of $\mathbb{U}_{\mathbb{Q}}$ to other Urysohn-type spaces (with special values of its metrics).

2 The Urysohn space

A *finite presentation* is given by a set of formulas which define elements and a set of formulas which induce relations (and metrics).

We start to develop the theory for vertices as follows:

1. A *triplet* $\mathbf{A}$ is a triple $(A, \preceq_{\mathbf{A}}, d_{\mathbf{A}})$ where

i. A is finite set;

ii. $(A, \preceq_{\mathbf{A}})$ is partial order on A;

iii. $(A, d_{\mathbf{A}})$ is a rational metric space (i.e. $d_{\mathbf{A}} : A \times A \to \mathbb{Q}$ is a metric)

$\preceq_{\mathbf{A}}$ is called the *standard* order of $\mathbf{A}$.

Triplets $\mathbf{A} = (A, \preceq_{\mathbf{A}}, d_{\mathbf{A}})$ and $\mathbf{B} = (B, \preceq_{\mathbf{B}}, d_{\mathbf{B}})$ are said to be *isomorphic* if there exists a bijection $\varphi : A \to B$ which is both isomorphism of posets $(A, \preceq_{\mathbf{A}})$ and $(B, \preceq_{\mathbf{B}})$ and isometry of spaces $(A, d_{\mathbf{A}})$ and $(B, d_{\mathbf{B}})$.

Concerning posets we use the standard terminology. Particularly any element $a \in A$ determines *down set* $\downarrow a = \{b; b \preceq_{\mathbf{A}} a\}$, which induces by the restriction of $\preceq_{\mathbf{A}}$ and $d_{\mathbf{A}}$ the triplet $\downarrow a$. By abuse of the notation this triplet will be also denoted by $\downarrow a$. Let also $h(\mathbf{A})$ (*height of* $\mathbf{A}$) be the maximal size of a chain in $(A, \preceq_{\mathbf{A}})$.

2. The triplet $\mathbf{A}$ is said to be *proper* if all its down sets (as triplets) are non-isomorphic and if $(A, \preceq_{\mathbf{A}})$ has both a greatest element and a smallest element (denoted by $\max_{\mathbf{A}}$ and $\min_{\mathbf{A}}$).

3. A proper triplet $\mathbf{A}$ is said to be *path metric PM* if for every $a, a' \in A$ which are incomparable in $\preceq_{\mathbf{A}}$ there exist $a'' \in A, a'' \preceq_{\mathbf{A}} a, a'' \preceq_{\mathbf{A}} a'$ such that $d_{\mathbf{A}}(a, a') = d_{\mathbf{A}}(a, a'') + d_{\mathbf{A}}(a'', a')$. Such an a'' will be called the *witness of* $d_{\mathbf{A}}(a, a')$.

Proper path-metric triplet will be abbreviated as PPM-triplet. An example of PPM-triplet is on Fig 1.

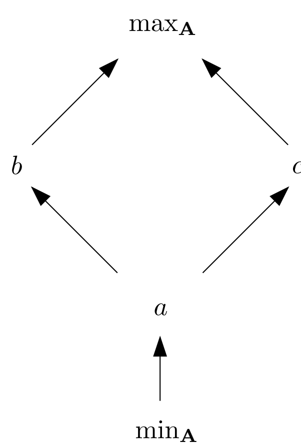
4. A PPM-triplet is said to be *complete* if the following holds for every $a \in A$:

$$a = \{(b, d_{\mathbf{A}}(a, b)); b \in \downarrow a, a \neq b\}.$$

Note that $\min_{\mathbf{A}} = \emptyset$.

An example of complete triplet isomorphic to the PPM triple of Fig 1 is shown at Fig 2.

Thus the structure of $max_{\mathbf{A}}$ encodes the whole complete triplet $\mathbf{A}$.



$d_{\mathbf{A}}$:

	$\min_{\mathbf{A}}$	a	b	c	$\max_{\mathbf{A}}$
$\min_{\mathbf{A}}$	0	1	3	2	6
a	1	0	2	1	5
b	3	2	0	3	3
c	2	1	3	0	5
$\max_{\mathbf{A}}$	6	5	3	5	0

Figure 1: PPM triplet $\mathbf{A}$

Observe also that every downset $\downarrow a$ is itself a complete triplet. This triplet will also be denoted shortly by $\downarrow a$. If $b \in A$ then we also say that $\downarrow b$ *is mentioned in* $\mathbf{A}$. By induction on the $h(\mathbf{A})$ we see easily the following fact (which is the reason why we introduced the complete triplets):

Fact 1 *Let* $\mathbf{A}$*,* $\mathbf{B}$ *be isomorphic complete triplets. Then* $\mathbf{A} = \mathbf{B}$*.*

Now we can state the basic construction of this paper, a finite presentation of $\mathbb{U}_{\mathbb{Q}}$:

Definition 2.1 (finite presentation of the Urysohn space $\mathbb{U}_{\mathbb{Q}}$) *Denote by* $\mathcal{U}$ *the set of all complete triplets. The metric* $d_{\mathcal{U}}$ *on* $\mathcal{U}$ *is defined as follows: Let* $\mathbf{A} = (A, \preceq_{\mathbf{A}}, d_{\mathbf{A}})$*,* $\mathbf{B} = (B, \preceq_{\mathbf{B}}, d_{\mathbf{B}})$ *be complete triplets. We put* $d_{\mathcal{U}}(\mathbf{A}, \mathbf{B}) = \min(d_{\mathbf{A}}(\max_{\mathbf{A}}, a) + d_{\mathbf{B}}(\max_{\mathbf{B}}, b))$ *where the minimum is taken over all* $a \in \mathbf{A}$*,* $b \in \mathbf{B}$ *such that* $a = b$*.*

If $\max_{\mathbf{B}} \in A$ (and thus also $d_{\mathcal{U}}(\mathbf{A}, \mathbf{B}) = d_{\mathbf{A}}(\max_{\mathbf{A}}, \max_{\mathbf{B}})$) we say that $\mathbf{B}$ *is mentioned in* $\mathbf{A}$.

If neither $\mathbf{A}$ is mentioned in $\mathbf{B}$ nor is $\mathbf{B}$ mentioned in $\mathbf{A}$ then for a, b reaching the minimum, we call the triplet $\downarrow a = \downarrow b$ a *witness of* $d_{\mathcal{U}}(\mathbf{A}, \mathbf{B})$.

We will show that this construction yields a finite presentation of $\mathbb{U}_{\mathbb{Q}}$. This will be done in the sequence of statements formed by Proposition 2.1, Proposition 2.2, and Theorem 2.1 which is the main result of the paper.

Proposition 2.1 $(\mathcal{U}, d_{\mathcal{U}})$ *is a metric space.*

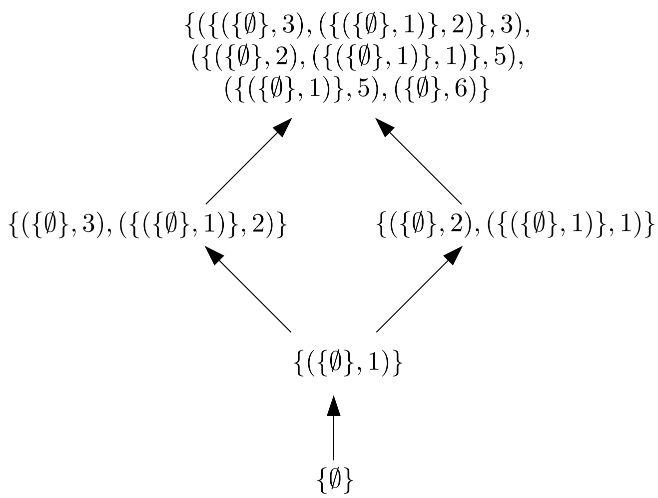


Figure 2: Complete triplet $\mathbf{A}$

Proof. Clearly $d_{\mathcal{U}} \geq 0$ and $d_{\mathcal{U}}(\mathbf{A},\mathbf{B}) = 0$ iff $\mathbf{A} = \mathbf{B}$

Assume that the triangle inequality does not hold. Take the triangle $\mathbf{A},\mathbf{B},\mathbf{C} \in \mathcal{U}$ such that $h(\mathbf{A}) + h(\mathbf{B}) + h(\mathbf{C})$ is minimal and the triangle inequality does not hold. Without loss of generality, assume that

$$d_{\mathcal{U}}(\mathbf{A},\mathbf{B}) > d_{\mathcal{U}}(\mathbf{B},\mathbf{C}) + d_{\mathcal{U}}(\mathbf{C},\mathbf{A}).$$

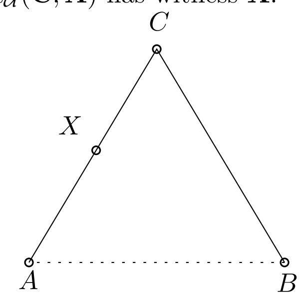
We distinguish several cases according to the existence of witness elements:

Case 1: The distances $d_{\mathcal{U}}(\mathbf{A},\mathbf{B})$, $d_{\mathcal{U}}(\mathbf{B},\mathbf{C})$ and $d_{\mathcal{U}}(\mathbf{C},\mathbf{A})$ do not have any witness:

1. If $\mathbf{A}$ and $\mathbf{B}$ are both mentioned in $\mathbf{C}$, then there exists $a \in C, b \in C$ such that $d_{\mathcal{U}}(\mathbf{B},\mathbf{C}) = d_{\mathbf{C}}(b, \max_{\mathbf{C}})$, $d_{\mathcal{U}}(\mathbf{A},\mathbf{C}) = d_{\mathbf{C}}(a, \max_{\mathbf{C}})$ and thus the triangle $a, b, \max_{\mathbf{C}}$ violates the triangular inequality in $d_{\mathbf{C}}$. Similarly we can proceed for any other vertex of the triangle and thus no vertex defines the distances to both remaining vertices.

2. If $\mathbf{A}$ is mentioned in $\mathbf{B}$ and $\mathbf{C}$ mentioned in $\mathbf{A}$, then there will be some $a \in \mathbf{B}$ such that $\downarrow a = \mathbf{A}$ and also there will be some $c \in \mathbf{A}$ such that $c \preceq_{\mathbf{B}} a \in \mathbf{B}$ such that $\downarrow c = \mathbf{C}$. Then the triangle $a, c, \max_{\mathbf{B}}$ would violates triangular inequality of $d_{\mathbf{B}}$.

Case 2: Assume that $d_{\mathcal{U}}(\mathbf{C}, \mathbf{A})$ has witness $\mathbf{X}$.



Since $\mathbf{X}$ is witness:

$$d_{\mathcal{U}}(\mathbf{A}, \mathbf{X}) = d_{\mathcal{U}}(\mathbf{A}, \mathbf{C}) - d_{\mathcal{U}}(\mathbf{X}, \mathbf{C})$$

The triangles $\mathbf{B}, \mathbf{C}, \mathbf{X}$ and $\mathbf{A}, \mathbf{B}, \mathbf{X}$ do not violate the triangular inequality (since $h(\mathbf{A}) + h(\mathbf{B}) + h(\mathbf{C})$ would not be minimal):

$$d_{\mathcal{U}}(\mathbf{X}, \mathbf{B}) \leq d_{\mathcal{U}}(\mathbf{X}, \mathbf{C}) + d_{\mathcal{U}}(\mathbf{C}, \mathbf{B})$$

$$d_{\mathcal{U}}(\mathbf{A}, \mathbf{B}) \leq d_{\mathcal{U}}(\mathbf{A}, \mathbf{X}) + d_{\mathcal{U}}(\mathbf{X}, \mathbf{B})$$

It follows that:

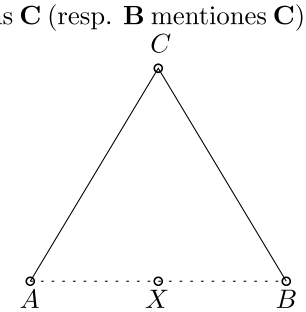
$$d_{\mathcal{U}}(\mathbf{A}, \mathbf{B}) \leq d_{\mathcal{U}}(\mathbf{A}, \mathbf{C}) - d_{\mathcal{U}}(\mathbf{X}, \mathbf{C}) + d_{\mathcal{U}}(\mathbf{X}, \mathbf{C}) + d_{\mathcal{U}}(\mathbf{C}, \mathbf{B})$$

$$d_{\mathcal{U}}(\mathbf{A}, \mathbf{B}) \leq d_{\mathcal{U}}(\mathbf{A}, \mathbf{C}) + d_{\mathcal{U}}(\mathbf{C}, \mathbf{B})$$

which is a contradiction.

Case 3: If $d_{\mathcal{U}}(\mathbf{C}, \mathbf{B})$ has a witness $\mathbf{X}$ then we proceed in complete analogy with case 2. (i.e. exchanging the roles of $\mathbf{A}$ and $\mathbf{B}$).

Case 4: Assume that $\mathbf{X}$ is a witness of $d_{\mathcal{U}}(\mathbf{A}, \mathbf{B})$ and that $d_{\mathcal{U}}(\mathbf{A}, \mathbf{C})$ and $d_{\mathcal{U}}(\mathbf{B}, \mathbf{C})$ have no witness. Thus $\mathbf{A}$ mentions $\mathbf{C}$ (resp. $\mathbf{B}$ mentiones $\mathbf{C}$) or the other way around.



Since **C** cannot mention both **A** and **B**, we can assume that **A** mentions **C**.

Now, if **B** mentionned **C** as well, then **C** would be a witness for $d_{\mathcal{U}}(\mathbf{A}, \mathbb{B})$. It would follow that

$$d_{\mathcal{U}}(\mathbf{A}, \mathbf{B}) = d_{\mathcal{U}}(\mathbf{A}, \mathbf{C}) + d_{\mathcal{U}}(\mathbf{C}, \mathbf{B}).$$

Contradiction.

Assume that **C** mentions **B**. Now again from transitivity property we have that **A** defines the distances to both **B** and **C** and thus for the triangle $\mathbf{A}, \mathbf{B}, \mathbf{C}$ the triangular inequality holds, a contradiction. □

Proposition 2.2 $(\mathcal{U}, d)$ *is a metric space which contains all finite metric spaces.*

Proof. We describe an algorithm for an isometric embedding of a given metric space (X, d') into $\mathcal{U}$.

We fix a linear ordering of the vertices $x \in X$ by assigning to each vertex a unique natural number $n(x) \in \{0, 1, \ldots, |X| - 1\}$.

For a vertex $x \in X$, the triplet $\mathbf{U}(x) = (U(x), \preceq_{\mathbf{U}(x)}, d_{\mathbf{U}(x)})$ representing it is defined recursively as follows:

1. Put:

   $\max(x) = \emptyset$ for $n(x) = 0$;

   $\max(x) = \{(U(y), d'(y, x)); y \in X, n(y) < n(x)\}$ for $n(x) > 0$;

   $U(x) = \{\max(y); y \in X, n(y) \leq n(x)\}$.

2. The ordering $\preceq_{\mathbf{U}(x)}$ is the linear order defined by:

   $$U(y) \preceq_{\mathbf{U}(x)} U(y') \text{ iff } n(y) \leq n(y').$$

3. The distance is defined by $d_{\mathbf{U}(x)}(U(y), U(y')) = d'(y, y')$.

We verify that $\mathbf{U}(x)$ is a complete triplet:

The finite linear order $\preceq_{\mathbf{U}(x)}$ has clearly smallest element 0 and greatest element $\max_{\mathbf{U}(x)} = \max(x)$ and no two downsets are isomorphic. Thus $\mathbf{U}(x)$ is a proper triplet.

In the linear order, each two elements are in relation, so trivially $d_{\mathbf{U}(x)}$ has path metric property. From construction of $U(x)$ it follows that $\mathbf{U}(x)$ is complete triplet and thus $\mathbf{U}(x) \in \mathcal{U}$.

Consider $x, y \in X$, $n(x) \leq n(y)$. As $\mathbf{U}(y)$ mentions $\mathbf{U}(x)$:

$$d'(x, y) = d_{\mathbf{U(y)}}(\mathbf{U}(x), \mathbf{U}(y)).$$

$\square$

Theorem 2.1 *$(\mathcal{U}, d)$ is the generic metric space.*

Proof. The set $\mathcal{U}$ is obviously countable, since all elements are finite. By Proposition 2.1 $(\mathcal{U}, d)$ is a metric space. By a construction similar to the construction performed in the proof of Proposition 2.2, we verify that $(\mathcal{U}, d)$ has the extension property. Clearly it suffices to verify the extersion property in the following form:

Fix $\mathcal{X}$ any finite subset of $\mathcal{U}$ together with a distance function $D : \mathcal{X} \to \mathbb{Q}$ defining a single vertex extension of the metric subspace induced by $\mathcal{X}$ (i.e. the desired distances to the new vertex such that D not violate triangular inequality property of $d_{\mathcal{U}}$ partialized to $\mathcal{X}$). (Remark, that Katětov axiomatized all possible functions D, now called Katětov functions, [21].) We find a finite triplet $\mathbf{M}(\mathcal{X}, D) = (M(\mathcal{X}, D), \preceq_{\mathbf{M}(\mathcal{X},D)}, d_{\mathbf{M}(\mathcal{X},D)}) \in \mathcal{U}$, such that $d_{\mathcal{U}}(\mathbf{M}(\mathcal{X}, D), \mathbf{A}) = D(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{X}$.

$\mathbf{M}(\mathcal{X}, D)$ is defined according to the following algorithm:

(1) The vertex set of $M(\mathcal{X}, D)$ is the union of all A such that there exists $\mathbf{A} = (A, \preceq_{\mathbf{A}}, d_{\mathbf{A}}) \in \mathcal{X}$, together with the single new vertex m which we describe later (in (4)).

(2) For a, b in $\mathbf{M}(\mathcal{X}, D)$ we set $a \preceq_{\mathbf{M}(\mathcal{X},D)} b$ iff $b = m$ or there exist $\mathbf{A} = (A, \preceq_{\mathbf{A}}, d_{\mathbf{A}}) \in \mathcal{X}$ such that $a, b \in A$ and $a \preceq_{\mathbf{A}} b$.

Observe that $m = \max_{\mathbf{M}(\mathcal{X},D)}$

(3) For $a, b \in M(\mathcal{X}, D)$ we set:

i. $d_{\mathbf{M}(\mathcal{X},D)}(a, b) = 0$ when $a = b$.

ii. $d_{\mathbf{M}(\mathcal{X},D)}(a, b) = d_{\mathcal{U}}(\downarrow a, \downarrow b)$, when $a, b \neq m$.

iii. $d_{\mathbf{M}(\mathcal{X},D)}(m, b) = \min_{\mathbf{C} \in \mathcal{X}} D(\mathbf{C}) + d_{\mathcal{U}}(\mathbf{C}, \downarrow b)$.
We call $\mathbf{C} \in \mathcal{X}$ such that $d_{\mathbf{M}(\mathcal{X},D)}(m, b) = D(\mathbf{C}) + d_{\mathcal{U}}(\mathbf{C}, \downarrow b)$ with $\mathbf{C} \neq \downarrow b$ a *witness* of $d_{\mathbf{M}(\mathcal{X},D)}(m, b)$. Observe that $d_{\mathbf{M}(\mathcal{X},D)}(m, b)$ has no witness iff $\downarrow b \in \mathcal{X}$ and in that case, $d_{\mathbf{M}(\mathcal{X},D)}(m, b) = D(\downarrow b)$.

iv. $d_{\mathbf{M}(\mathcal{X},D)}(a, m) = d_{\mathbf{M}(\mathcal{X},D)}(m, a)$ defined in *iii.*

(4) $m = \{(a, d_{\mathbf{M}(\mathcal{X},D)}(m,a)); a \in \bigcup_{\mathbf{B}\in\mathcal{X}} B\}$

We verify that $\mathbf{M}(\mathcal{X}, D)$ is a complete triplet by verifying conditions Definition 2.1. **1.–4.**

We first verify **1.** *ii.*:

$\preceq_{\mathbf{M}(\mathcal{X},D)}$ is partial order: for $a \preceq_{\mathbf{M}(\mathcal{X},D)} b \preceq_{\mathbf{M}(\mathcal{X},D)} c$ either $c = m$ and thus $a \preceq_{\mathbf{M}(\mathcal{X},D)} c$ holds trivially from definition or there exists $\mathbf{A} \in \mathcal{X}$ such that $a, b, c \in A$ and the fact that $a \preceq_{\mathbf{M}(\mathcal{X},D)} c$ follows from $a \preceq_{\mathbf{A}} c$.

There is single maximal element m and single minimal element $\emptyset$.

$\mathbf{M}(\mathcal{X}, D)$ is proper triplet:

Secondly, the downsets of every $a \in A$, $\mathbf{A} \in \mathcal{X}$ are preserved (i.e. downset of a in $\mathbf{M}(\mathcal{X}, D)$ is equivalent to the downset of a in $\mathbf{A}$). This follow from fact that $\preceq_{\mathbf{M}(\mathcal{X},D)}$ is inherited from $\preceq_{\mathbf{A}}$ and that the downset of $a \in A$ is identical to the donwset of a in any $\mathbf{B} \in \mathcal{U}$ such that $a \in B$. Since also all $\mathbf{A} \in \mathcal{X}$ are complete triplets, all the downsets are non-isomorphic. This verifies **2.**

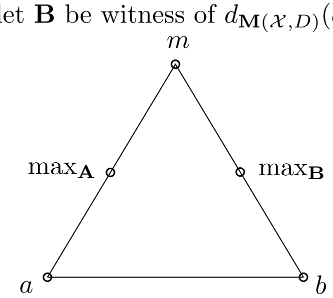
Next, we prove that $d_{\mathbf{M}(\mathcal{X},D)}$ is a rational metric (see Definition 2.1 **1.** *iii.*):

$d_{\mathbf{M}(\mathcal{X},D)}(a, b)$ is a positive rational number for each $a \preceq_{\mathbf{M}(\mathcal{X},D)} b$. Observe that for the last part of construction of $d_{\mathbf{M}(\mathcal{X},D)}$, the shortest path always exists: there is always a path from any element to the minimal element. The fact that that $d_{\mathbf{M}(\mathcal{X},D)}$ is symmetric directly follows from the construction.

We verify the triangular inequality property for $d_{\mathbf{M}(\mathcal{X},D)}$:

Since rule *ii.* merely translates metric $d_{\mathcal{U}}$ to $d_{\mathbf{M}(\mathcal{X},D)}$, any triplet violating triangular inequality property must have two distances defined by *iii.* or *iv.* Let $a, b \in \mathbf{M}(\mathcal{X}, D)$ and consider the triangle a, b, m.

First assume that $a, b \in \mathcal{X}$ and that $\downarrow a, \downarrow b \notin \mathcal{X}$. Thus let $\mathbf{A}$ be a witness of $d_{\mathbf{M}(\mathcal{X},D)}(a, m)$ and let $\mathbf{B}$ be witness of $d_{\mathbf{M}(\mathcal{X},D)}(b, m)$.



We first check that:

$$d_{\mathbf{M}(\mathcal{X},D)}(a,m) + d_{\mathbf{M}(\mathcal{X},D)}(b,m) \geq d_{\mathbf{M}(\mathcal{X},D)}(a,b).$$

By the definition of $d_{\mathbf{M}(\mathcal{X},D)}$ we have:

$$\begin{gathered} d_{\mathbf{M}(\mathcal{X},D)}(a,m) + d_{\mathbf{M}(\mathcal{X},D)}(b,m) = \\ = d_{\mathbf{M}(\mathcal{X},D)}(a,\max_{\mathbf{A}}) + d_{\mathbf{M}(\mathcal{X},D)}(\max_{\mathbf{A}},m) + d_{\mathbf{M}(\mathcal{X},D)}(b,\max_{\mathbf{B}}) + \\ + d_{\mathbf{M}(\mathcal{X},D)}(\max_{\mathbf{B}},m) = \\ = d_{\mathbf{M}(\mathcal{X},D)}(a,\max_{\mathbf{A}}) + D(\mathbf{A}) + d_{\mathbf{M}(\mathcal{X},D)}(b,\max_{\mathbf{B}}) + D(\mathbf{B}) \end{gathered}$$

By the property of D:

$$D(\mathbf{A}) + D(\mathbf{B}) \geq D_{\mathcal{U}}(\mathbf{A},\mathbf{B}) = \mathbf{M}(\mathcal{X},D)(\max_{\mathbf{A}},\max_{\mathbf{B}})$$

Thus:

$$\begin{gathered} d_{\mathbf{M}(\mathcal{X},D)}(a,m) + d_{\mathbf{M}(\mathcal{X},D)}(b,m) \geq \\ \geq d_{\mathbf{M}(\mathcal{X},D)}(a,\max_{\mathbf{A}}) + d_{\mathbf{M}(\mathcal{X},D)}(b,\max_{\mathbf{B}}) + d_{\mathbf{M}(\mathcal{X},D)}(\max_{\mathbf{A}},\max_{\mathbf{B}}) \geq \\ \geq d_{\mathbf{M}(\mathcal{X},D)}(a,b) \end{gathered}$$

(The last inequality follows from the triangular inequality of $d_{\mathbf{M}(\mathcal{X},D)}$ on $M(\mathcal{X},D) - \{m\}$.) This proves that

$$d_{\mathbf{M}(\mathcal{X},D)}(a,b) \leq d_{\mathbf{M}(\mathcal{X},D)}(a,m) + d_{\mathbf{M}(\mathcal{X},D)}(b,m).$$

Now we prove:

$$d_{\mathbf{M}(\mathcal{X},D)}(a,m) \leq d_{\mathbf{M}(\mathcal{X},D)}(a,b) + d_{\mathbf{M}(\mathcal{X},D)}(b,m)$$

(the case $d_{\mathbf{M}(\mathcal{X},D)}(b,m) \leq d_{\mathbf{M}(\mathcal{X},D)}(a,b) + d_{\mathbf{M}(\mathcal{X},D)}(a,m)$ can be handled similarly).

$$d_{\mathbf{M}(\mathcal{X},D)}(a,b) + d_{\mathbf{M}(\mathcal{X},D)}(b,m) \leq d_{\mathbf{M}(\mathcal{X},D)}(a,\max_{\mathbf{B}}) + d_{\mathbf{M}(\mathcal{X},D)}(\max_{\mathbf{B}},m)$$

Because witness $\mathbf{A}$ is minimal, we have

$$d_{\mathbf{M}(\mathcal{X},D)}(a,m) = d_{\mathbf{M}(\mathcal{X},D)}(a,\max_{\mathbf{A}}) + d_{\mathbf{M}(\mathcal{X},D)}(\max_{\mathbf{A}},m) \leq$$

$$\leq d_{\mathbf{M}(\mathcal{X},D)}(a,\max_{\mathbf{B}}) + d_{\mathbf{M}(\mathcal{X},D)}(\max_{\mathbf{B}},m).$$

The case when $\downarrow a$ belongs to $\mathcal{X}$ can be handled similarly if we put $\mathbf{A} = \downarrow a$.

Now we show that $d_{\mathbf{M}(\mathcal{X},D)}$ has the PM property.

Recall that we have to prove that for each a,b incomparable by $\leq_{\mathbf{M}(\mathcal{X},D)}$ then there exists a'' such that:

1. $a'' \preceq_{\mathbf{M}(\mathcal{X},D)} a$

2. $a'' \preceq_{\mathbf{M}(\mathcal{X},D} a'$

3. $d_{\mathcal{U}}(a,a') = d_{\mathcal{U}}(a,a'') + d_{\mathcal{U}}(a'',d')$.

The case $a, b \neq m$ follows directly from definition of $d_{M(\mathcal{X},D)}$. For $a = m$ we can put $a'' = \max_{\mathbf{A}}$ (where $\mathbf{A}$ is the witness of $d_{M(\mathcal{X},D)}(a,m)$.

The triplet $\mathbf{M}(\mathcal{X},D)$ is complete (**4.**) and thus $\mathbf{M}(\mathcal{X},D) \in \mathcal{U}$. From the construction it directly follows that $\mathbf{M}(\mathcal{X},D)$ mentions every $\mathbf{A} \in \mathcal{X}$ with the desired distance $d_{\mathcal{U}}(\mathbf{A},\mathbf{M}(\mathcal{X},D)) = d_{\mathbf{M}(\mathcal{X},D)}(\max_{\mathbf{A}},m) = D(\mathbf{A})$ (by *ii.*).

This proves Theorem 2.1 of finite presentation of $\mathbb{U}_{\mathbb{Q}}$.

□

3 Generic poset

Let $\mathbb{P}$ be the generic poset (i.e. countable ultrahomogenous and universal poset).

In [10] we found a finite presentation of $\mathbb{P}$. This presentation is related to (the extension of) surreal numbers [3] (and in fact surreal numbers with Conway ordering form a linear extension of the presentation [10]). Based on the presentation of $\mathbb{U}_{\mathbb{Q}}$ in Section 2 we outline here a new finite presentation of $\mathbb{P}$. As this presentation and its proof are closely related to Section 2 we present only the definitions and statements without the proof.

The points of our finite presentation of $\mathbb{P}$ will be triplets defined similarly as in Section 2:

1. Triple $\mathbf{A} = (A, \preceq_{\mathbf{A}}, \leq_{\mathbf{A}})$ is a *$\mathcal{P}$-triplet* iff

- A is finite set;
- Relation $\preceq_{\mathbf{A}}$ is partial order on A;
- Relation $\leq_{\mathbf{A}}$ is partial order on A;

Similarly to Section 2, we say that $\mathcal{P}$-triplets $\mathbf{A} = (A, \preceq_{\mathbf{A}}, \leq_{\mathbf{A}})$ and $\mathbf{B} = (B, \preceq_{\mathbf{B}}, \leq_{\mathbf{B}})$ are said to be *isomorphic* if there exists a bijection $\varphi : A \to B$ which is both isomorphism of posets $(A, \preceq_{\mathbf{A}})$ and $(B, \preceq_{\mathbf{B}})$ and posets $(A, \leq_{\mathbf{A}})$ and $(B, \leq_{\mathbf{B}})$. The downset $\{x; x \preceq_{\mathbf{A}} a\}$ will be denoted by $\downarrow a$. (There will be no downsets with respect to $\leq_{\mathbf{A}}$.)

2. $\mathcal{P}$-triplets are *proper* if no two downsets considered a $\mathcal{P}$-triplets are isomorphic and if $(A, \preceq_{\mathbf{A}})$ has both a greatest and a smallest elements (denoted by $\max_{\mathbf{A}}$ and $\min_{\mathbf{A}}$).

3. $\leq_A$ is said to be *induced by edges of* $\preceq_A$ if for every $a, a' \in A$, $a \leq_{\mathbf{A}} a'$ which are incomparable in $\preceq_{\mathbf{A}}$ there exists $a'' \in A, a'' \preceq_{\mathbf{A}} a, a'' \preceq_{\mathbf{A}} a'$ such that $a \leq_{\mathbf{A}} a'' \leq_{\mathbf{A}} a'$.

4. A proper $\mathcal{P}$-triplet $\mathbf{A} = (A, \preceq_{\mathbf{A}}, \leq_{\mathbf{A}})$ where $\leq_{\mathbf{A}}$ is induced by edges of $\preceq_{\mathbf{A}}$ is said to be complete if the following holds:

1. $\min_{\mathbf{A}} = \emptyset$.

2. For every $a \in A$ holds:

$$a = \{(b, -1); b \in\downarrow a, a \neq b, b \leq_{\mathbf{A}} a\} \cup \{(b, 1); b \in\downarrow a, a \neq b, a \leq_{\mathbf{A}} b\} \cup$$

$$\cup \{(b, 0); b \in\downarrow a, a \neq b, a \not\leq_{\mathbf{A}} b, b \not\leq_{\mathbf{A}} a\}.$$

Denote by $\mathcal{P}$ the set of all complete $\mathcal{P}$-triplets. Complete triplets induce a partial order denoted simply by $\leq$:

Definition 3.1 *For* $\mathbf{A} = (A, \preceq_{\mathbf{A}}, \leq_{\mathbf{A}})$, $\mathbf{B} = (B, \preceq_{\mathbf{B}}, \leq_{\mathbf{B}}) \in \mathcal{P}$ *we write* $\mathbf{A} \leq \mathbf{B}$ *iff there exists* $a \in A$ *such that* $a \in B$ *and* $\max_{\mathbf{A}} \leq_{\mathbf{A}} a$, $a \leq_{\mathbf{B}} \max_{\mathbf{B}}$.

Similarly as in Section 2 we can then prove:

Theorem 3.1 *1.* $(\mathcal{P}, \leq)$ *is partially ordered set.*

2. $(\mathcal{P}, \leq)$ *is isomorphic to the generic poset* $\mathbb{P}$.

4 Other metrics, other structures

It is obvious that the finite presentation given in Section 2 for $\mathbb{U}_{\mathbb{Q}}$ can be easily modified for Urysohn spaces with the rational metrics restricted to some interval (say $\mathbb{U} \cap [0, 1]$ or $\mathbb{U} \cap [0, a]$). Such presentations of the Urysohn spaces has been thoroughly investigated in [21, 16] where the Urysohn space with X-valued metric was denoted by $\mathbb{U}_X$. $\mathbb{U}_X$ need not exist as is demonstrated by the failure of the amalgamation property. However this is characterized in [4] by *4-value property.*

It is easy to see (by a cardinality argument) that the 4-value property does not suffice for the finite presentation of $\mathbb{U}_X$. However we have the following:

A class $\mathcal{K}$ of rational finite metric spaces is said to be *triangle axiomatized* if $A \in \mathcal{K}$ iff every 3-point subspace of A belongs to $\mathcal{K}$. An ultrahomogeneous metric space $\mathcal{X}$ is said to be triangle axiomatized if the class of all finite subspaces is triangle axiomatized. We can prove

Theorem 4.1 *Every ultrahomogeneous space $\mathcal{X}$ which is triangle axiomatized has a finite presentation.*

The details of this and other generalizations will appear elsewhere.

Triangle axiomatized classes include classes of ultrametric spaces thoroughly investigated recently in [21].

5 Acknowledgement

We thank to A. Louveau and to an anonymous referee for several remarks which improved the quality of this paper.

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