# $k$-chromatic number of graphs on surfaces*

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**Abstract**

A well-known result (Heawood [6], Ringel [11], Ringel and Youngs [10]) states that the maximum chromatic number of a graph embedded in a given surface $S$ coincides with the size of the largest clique that can be embedded in $S$, and that this number can be expressed as a simple formula in the Eulerian genus of $S$. We study maximum chromatic number of $k$ edge-disjoint graphs embedded in a surface. We improve the previously known upper bounds, and show that in many cases, the new upper bound coincides with the lower bound obtained from embedding disjoint cliques in the surface. In the proof of this result, we derive a variant of Euler's Formula for union of several graphs that might be interesting independently.

## 1 Introduction and Definitions

We consider simple undirected graphs with no loops and parallel edges. Let $e(G)$ and $n(G)$ denote the number of edges and the number of vertices of a graph $G$, respectively. When the graph $G$ is clear from the context, we simply use $e$ and $n$. A *proper coloring* of a graph $G$ by $k$ colors is assignment of colors 1, 2,..., $k$ to vertices of $G$ such that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ of graph $G$ is the minimum $k$ such that $G$ has a proper coloring by $k$ colors.

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Let $\Sigma_h$ denote the orientable surface obtained from the sphere by attaching $h$ handles, and let $\Pi_h$ be the nonorientable surface obtained from the sphere by inserting $h$ crosscaps. We define *Eulerian genus* $g(S)$ of a surface $S$ by $g(\Sigma_h) = 2h$ and $g(\Pi_h) = h$. Let $g(G)$ denote the *Eulerian genus* of the graph $G$, i.e., the minimal Eulerian genus of a surface into which $G$ is embeddable.

Colorings of graphs on surfaces have been studied extensively. The fundamental result in this area is the well-known Four Color Theorem, that was proved by Appel and Haken [1] in 1977, and a shorter proof was later found by Robertson, Sanders, Seymour and Thomas [12]. Regarding the graphs on surfaces of genus $g \geq 1$, Heawood [6] showed that each graph embedded in such a surface has chromatic number at most

$$H(g) = \left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor .$$

Later, Ringel [11] and Ringel and Youngs [10] found the corresponding lower bounds, by showing that the complete graph on $H(g)$ vertices can be embedded into any surface of Eulerian genus $g$, with the exception of the Klein bottle, where the correct bound on the chromatic number is 6 (established by Franklin [4]).

We consider the properties (especially regarding the chromatic number) of partitions of a graph into several subgraphs. The *partition* of a graph $G$ to $k$ parts consists of $k$ edge-disjoint subgraphs $G_1, \ldots, G_k$ such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_k)$. Note that we do not require that the subgraphs $G_i$ are spanning, i.e., possibly $n(G_i) < n(G)$ for some $i$. We always assume that the graphs $G_i$ do not contain isolated vertices. We call the subgraphs $G_i$ *parts* of the partition.

The *$k$-chromatic number* $\chi_k(G)$ is the maximum of $\sum_{i=1}^{k} \chi(G_i)$ over all partitions $G_1, G_2, \ldots, G_k$ of $G$ into $k$ parts. The parameter $\chi_k$ has been studied for general graphs as well as for graphs of bounded genus. The fact that for a graph $G$ with $n$ vertices, $\chi_2(G) \leq n + 1$ follows from the well-known theorem of Nordhaus and Gaddum [8]. Plesník [9] proved that $n + \binom{k}{2} \leq \chi_k(K_n) \leq n + 2^{\binom{k+1}{2}}$ and conjectured that $\chi_k(K_n) = n + \binom{k}{2}$. Watkinson [15] has improved the upper bound to $\chi_k(K_n) \leq \frac{k!}{2}$ and Füredi et al. [5] to $\chi_k(K_n) \leq n + 7^k$.

Regarding the graphs with bounded genus $g$, let us define $\chi_k(S)$ to be maximum of $\chi_k(G)$ over all graphs $G$ that can be embedded in $S$. Stiebitz and Škrekovski [14] has determined the exact values of $\chi_2$ for all surfaces.

Füredi et al. [5] have shown that

$$\chi_k(S) \leq \left\lfloor \frac{7k + \sqrt{24kg + 49k^2 - 48k}}{2} \right\rfloor,$$

and found a lower bound of order

$$\frac{7k + \sqrt{24kg + k^2}}{2}.$$

In this paper, we decrease the upper bound; this way, we obtain exact values for many surfaces and values of $k$.

An embedding of a graph in a surface is called *cellular* if the interior of each face is homeomorphic to an open disk. In particular, the boundary walk of each face in a cellular embedding is connected. For a face $f$ of such an embedding, let $\ell(f)$ be the length of its boundary walk. If $G$ is a simple connected graph with at least three vertices, then $\ell(f) \geq 3$ for each face $f$. A *block* of a graph $G$ is a maximum 2-connected induced subgraph of $G$. Let us recall some fundamental facts about graph embeddings and surfaces that can be found e.g. in [7].

**Theorem 1** *Let $G$ be a connected graph of Eulerian genus $g$. Then, any embedding of $G$ in a surface with Eulerian genus $g$ is cellular.*

**Theorem 2 (Battle et al. [2], Stahl and Beineke [13])** *If $G_1, G_2, \ldots, G_n$ are the blocks of a graph $G$, then*

$$g(G) = \sum_{i=1}^{n} g(G_i).$$

**Theorem 3 (Franklin [4], Ringel [11], Ringel and Youngs [10])** *Eulerian genus of the complete graph $K_n$ is $g = \lceil \frac{1}{6}(n-3)(n-4) \rceil$. $K_n$ can be embedded into any surface with Eulerian genus $g$, with the exception of $K_7$, that cannot be embedded in $\Pi_2$, i.e. the Klein bottle.*

**Theorem 4 (Euler's Formula)** *If $f$ is the number of faces of a cellular embedding of a graph $G$ into a surface of Eulerian genus $g$, then $e(G) = n(G) + f + g - 2$.*

In the following section, we derive a version of the Euler's Formula that provides more information about a graph split into several parts (Theorem 7).

A graph $G$ is *critical* if for every edge $e$ of $G$, $\chi(G - e) < \chi(G)$. If $G$ is a critical graph and $\chi(G) = k$, we say that $G$ is *$k$-critical*. Obviously, if $G$ is $k$-critical, then $\delta(G) \geq k - 1$. For non-complete graphs, the following stronger result known as Dirac's inequality was shown in [3]:

**Theorem 5 (Dirac)** *If $G$ is a $k$-critical graph with $k \geq 4$ and $G$ is not a clique, then $2e(G) \geq (k - 1)n(G) + k - 3$.*

## 2 Generalized Euler's Formula

Let $F$ be the set of the faces of a cellular embedding of a simple connected graph $G$ with at least 3 vertices. Then, $\triangle = \sum_{f \in F}(\ell(f) - 3) \geq 0$ is the number of edges that must be added to $G$ to make it a triangulation (possibly introducing parallel edges and loops during the construction). One of the well-know consequences of Euler's Formula is the following lemma:

**Proposition 6** *If $G$ is a simple connected graph with $n \geq 3$ vertices and $e$ edges embedded cellularly to a surface of Eulerian genus $g$, then $e + \triangle = 3n + 3g - 6$. In particular, $e \leq 3n + 3g(G) - 6$.*

We include the proof for the sake of completeness.

**Proof.** Let $F$ be the set of faces of $G$. Since each edge of $G$ appears exactly twice in the facial walks, we have $2e = \sum_{f \in F} \ell(f)$, and consequently $2e - \triangle = 3|F|$. Using Theorem 4, we infer $3e = 3n + 3|F| + 3g - 6 = 3n + 2e - \triangle + 3g - 6$, from which the desired formula immediately follows. Also, by Theorem 1, the embedding of $G$ into a surface of Eulerian genus $g(G)$ is cellular, and since $\triangle \geq 0$, we have $e \leq 3n + 3g(G) - 6$. $\square$

To prove our upper bound, we need to generalize this inequality for union of several graphs:

**Theorem 7 (Generalized Euler's Formula)** *Let $G$ be a simple graph and let $G_1, \ldots, G_k$ be a partition of $G$ to $k$ parts. Let $n_i = n(G_i) \geq 3$ for each $1 \leq i \leq k$. If every component of each $G_i$ has at least three vertices, then*

$$e \leq 3g(G) + 3\sum_{i=1}^{k}(n_i - 2).$$

**Proof.** Suppose that the claim is false, and let $G$ together with its partition to graphs $G_1, \dots, G_k$ be a counterexample that is "smallest" in the following sense:

1. $\sum_{i=1}^{k}(n_i - 2)$ is the smallest possible, and
2. among all graphs that satisfy the first condition, $n$ is the largest possible.

By Proposition 6, we know that $k > 1$. Let us now describe some of the properties of $G$ and its partition:

(i) *Each $G_i$ is connected.* Otherwise, we may assume without loss of generality that $G_1$ is not connected, i.e., $G_1 = G_1^a \cup G_1^b$, where $G_1^a$ and $G_1^b$ are vertex-disjoint. By the minimality, the partition $G = G_1^a \cup G_1^b \cup G_2 \cup \ldots \cup G_k$ satisfies $e \leq 3g(G) + 3n(G_a) - 6 + 3n(G_b) - 6 + 3\sum_{i=2}^{k}(n_i - 2) < 3g(G) + 3\sum_{i=1}^{k}(n_i - 2)$, which is a contradiction with the fact that $G$ is a counterexample.

(ii) *$G$ is connected.* Otherwise, $G$ is a vertex-disjoint union of two smaller graphs $G_a$ and $G_b$, and we may assume that $G_a = G_1 \cup \ldots \cup G_t$ and $G_b = G_{t+1} \cup \ldots \cup G_k$ (the graphs $G_i$ are connected, thus they must be subgraphs of one of these two graphs). By Theorem 2, $g(G) = g(G_a) + g(G_b)$, and since $G$ is a minimal counterexample, we have $e(G_a) \leq 3g(G_a) + 3\sum_{i=1}^{t}(n_i - 2)$ and $e(G_b) \leq 3g(G_b) + 3\sum_{i=t+1}^{k}(n_i - 2)$. Summing these two inequalities brings a contradiction with the fact that $G$ is a counterexample.

(iii) *Each $n_i$ is at least 4.* Otherwise, we may assume that $n_1 = 3$ and let $G'$ be the union of graphs $G_2, \ldots, G_k$. Since $g(G') \leq g(G)$ and $G$ is a minimal counterexample, it follows that $e(G') = e - e(G_1) \leq 3g(G) + 3\sum_{i=2}^{k}(n_i - 2)$. However, $e(G_1) \leq 3 = 3(n_1 - 2)$, and hence $e \leq 3g(G) + 3\sum_{i=1}^{k}(n_i - 2)$, which is a contradiction.

(iv) *Minimum degree of each $G_i$ is at least three.* Otherwise, we may assume that $v$ is a vertex of $G_1$ with degree $d \leq 2$. Let $G_1' = G_1 - v$ and let $G'$ be the union of graphs $G_1', G_2, G_3, \ldots, G_k$. Suppose that $G_1'$ satisfies the assumptions of the theorem. Since $g(G') \leq g(G)$ and $G$ is a minimal counterexample, we get $e(G') = e(G) - d \leq 3g(G) + 3\sum_{i=1}^{k}(n_i - 2) - 3$, which is again a contradiction.

We need to verify that $G_1$ satisfies the assumptions of the theorem. This is trivial if $v$ is not a cut-vertex of $G_1$, since $n_1 \geq 4$ by the previous item. Therefore, if $d = 1$ then the assumptions are satisfied, and we may assume that $G_1$ does not contain a vertex of degree 1. Let us consider the case that $d = 2$ and $v$ is a cut-vertex. Since $\delta(G_1) \geq 2$, both components of $G_1'$ have at least three vertices, hence in this case $G_1'$ satisfies the assumptions of the theorem as well.

(v) *$G$ is 2-connected.* Otherwise, suppose that $G = G_a \cup G_b$, where $G_a$ and $G_b$ share just a single vertex $v$. By Theorem 2, $g(G) = g(G_a) + g(G_b)$. Suppose that the graphs $G_1, \ldots, G_t$ are subgraphs of $G_a$, the graphs $G_{t+1}, \ldots, G_r$ are subgraphs of $G_b$, and for $r < i \leq k$, $G_i = G_i^a \cup G_i^b$, where $G_i^a$ is a subgraph of $G_a$ and $G_i^b$ is a subgraph of $G_b$. Since the minimum degree of $G_i$ is at least three, both $G_i^a$ and $G_i^b$ have at least three vertices. Again, by summing the inequalities $e(G_a) \leq 3g(G_a) + 3\sum_{i=1}^{t}(n_i - 2) + 3\sum_{i=r+1}^{k}(n(G_i^a) - 2)$ and $e(G_b) \leq 3g(G_b) + 3\sum_{i=t+1}^{r}(n_i - 2) + 3\sum_{i=r+1}^{k}(n(G_i^b) - 2)$, we obtain a contradiction with the minimality of $G$.

(vi) *Each two graphs $G_i$ and $G_j$ share at most one vertex.* Otherwise, if $G_{k-1}$ and $G_k$ share $t \geq 2$ vertices, then let $G_{k-1}' = G_{k-1} \cup G_k$, and apply the theorem on $G$ split into graphs $G_1, \ldots, G_{k-2}, G_{k-1}'$. We obtain $e \leq 3g(G) + 3\sum_{i=1}^{k}(n_i - 2) + 6 - 3t$, which is a contradiction since $6 - 3t \leq 0$.

Let us now fix an embedding of $G$ on a surface of Eulerian genus $g(G)$. Recall that this embeding is cellular by Theorem 1. Given a vertex $v$ of degree $d$ in $G$, let $e_0, \ldots, e_{d-1}$ be the edges of $G$ in a cyclic ordering around $v$. A *segment* is a maximum interval $[a, b]$ such that all the edges $e_a$, $e_{a+1}$, $\ldots$, $e_b$ (with the indices taken modulo $d$) belong to a single graph $G_i$. The edges $e_a$ and $e_b$ are called *boundary edges* of the segment. The *length* of the segment is the number of its edges. The embedding of $G$ has the following properties:

- *If a vertex $v$ belongs to at least two parts, then there are at least two segments of edges at $v$ for each of these parts.* Otherwise, suppose that all the edges of $G_1$ at $v$ form just a single segment. In this case, we may split $v$ into two vertices $v_1$ and $v_2$ such that all edges of $G_1$ at $v$ are incident to $v_1$ and all the remaining edges at $v$ are incident to $v_2$. The created graph $G'$ is a counterexample embedded in the same surface with $e(G') = e(G)$ and $n(G') > n(G)$, which is a contradiction with the choice of $G$.

- *The following configuration* $(\star)$ *of edges cannot appear:* $e_1 = vw$ *belongs to* $G_i$*, all the remaining edges of* $G_i$ *at* $v$ *belong to one segment* $[a, b]$*, and the vertex* $w$ *appears at a face* $f$ *incident to* $e_a$ *or* $e_b$*.* If this were the case, we might redraw $G$ in such a way that $e_1$ is adjacent to $e_a$ or $e_b$ in the list of edges at $v$, by drawing it through the face $f$. We could then again split the vertex $v$, and obtain a contradiction.

We now plug the equality for $\triangle$ from Proposition 6 in the formula that we want to prove, thus obtaining the following equivalent inequality:

$$\triangle - 3n + 3\sum_{i=1}^{k} n_i \geq 6k - 6.$$

Therefore, we need to show that either $G$ has long faces, or the vertex sets of the graphs $G_i$ have a big overlap. In fact, we prove that if the embedding of the graph $G$ and its partition satisfies all the conditions described above, then the following stronger claim holds:

$$\triangle - 3n + 3\sum_{i=1}^{k} n_i \geq 6k.$$

We proceed by the discharging method. We assign an initial charge to each vertex and each face in the following way: a vertex $v$ that belongs to $x$ of the graphs $G_i$ has initial charge $3(x - 1)$. A face of length $\ell$ has initial charge $\ell - 3$. The sum of these charges is equal to $\triangle - 3n + 3\sum_{i=1}^{k} n_i$.

Next, we move some of this charge to the graphs $G_i$ in such a way that the final charge of each vertex and each face is nonnegative, and the final charge of each $G_i$ is at least 6. Since no charge is lost in the process, the required inequality follows.

We use the following rules to redistribute the charge:

(R1) Each vertex $v$ that belongs to $x \geq 2$ graphs $G_i$ sends charge $3/2$ to each of these graphs.

(R2) Let $f$ be a $\geq 4$-face and let $v_1v_2v_3v_4v_5$ be a subwalk of the facial walk of $f$ such that edges $v_2v_3$ and $v_3v_4$ belong to the same graph $G_i$, and neither $v_1v_2$ nor $v_4v_5$ belongs to $G_i$. Then, $f$ sends $1/2$ to $G_i$ through each of $v_2$ and $v_4$ (one unit of charge in total).

(R3) Let $f = w\,v_1 v_2 w\,v_4 v_5$ be a 6-face such that the edges $v_1v_2$, $v_2w$ and $v_1w$ belong to a graph $G_i$ and the edges $w\,v_4$, $v_4v_5$ and $v_5w$ belong to a different graph $G_j$. Then, $f$ sends $3/2$ to each of $G_i$ and $G_j$ through the vertex $w$.

(R4) Let $f$ be a face of length at least $t-1$ (where $t > 5$) for that Rule R3 does not apply, and let $v_1v_2\dots v_t$ be a subwalk of the facial walk of $f$ such that the edges $v_2v_3$, $v_3v_4$, $\dots$, and $v_{t-2}v_{t-1}$ belong to the same graph $G_i$, and neither $v_1v_2$ nor $v_{t-1}v_t$ belongs to $G_i$. Then, $f$ sends 1 to $G_i$ through each of $v_2$ and $v_{t-1}$ (two units of charge in total).

Let us first show that after the rules are applied, the final charge of each vertex and each face is nonnegative. If $v$ is a vertex that belongs to $x$ graphs $G_i$, then its final charge is zero if $x=1$ and it is $3(x-1)-3x/2 = 3x/2-3 \geq 0$ if $x \geq 2$, by Rule R1. Now, consider the charge of the faces. Let $f$ be an arbitrary face of $G$:

(a) If Rule R3 is applied to $f$, then its final charge is zero.

(b) If $f$ is a 3-face, then either all of its edges belong to the same graph, or each of them belongs to a different graph, as otherwise two of the graphs $G_i$ would intersect in at least two vertices. Therefore, no rule applies to $f$, and the final charge of $f$ is zero.

(c) Finally, suppose that Rule R2 applies $a$ times and Rule R4 applies $b$ times on an $\ell$-face $f$. The final charge of $f$ is $\ell - 3 - a - 2b$; therefore, it suffices to consider the case that $a + 2b + 2 \geq \ell \geq 4$. On the other hand, $\ell \geq 2a + 3b$, hence the final charge is at least $a + b - 3$, and we may assume that $a + b \leq 2$. It follows that $\ell \leq 6$ and exactly two of the graphs $G_i$ contain edges of the face $f$. Since these two graphs may share only one vertex and the graph is simple, $f$ must be a 6-face consisting of two triangles, $a = 0$ and $b = 2$. But then we obtain case (a), covered by Rule R3.

Now, let us consider the charge of the parts. We need to prove that the final charge of each of the parts is at least six. Let $G_i$ be one of the parts, and let $Y$ be the set of vertices that $G_i$ shares with the rest of the graph $G$. Since $G$ is 2-connected, $|Y| \geq 2$. By Rule R1, the subgraph $G_i$ receives $3|Y|/2$ units of charge, which is at least six if $|Y| \geq 4$. Therefore, it suffices to consider the cases $|Y| = 2$ and $|Y| = 3$.

We call a boundary edge $e$ of a segment of $G_i$ at a vertex $v \in Y$ *rich* if $e$ does not connect $v$ with another vertex of $Y$. Let $e = vw$ be a rich edge and let $f_e$ be a face that contains $e$ and an edge incident to $v$ that does not belong to $G_i$. Since $w \notin Y$, all the edges incident to $w$ must belong to $G_i$, hence one of Rules R2, R3 or R4 applies and $f_e$ sends at least $1/2$ units of charge through $v$ to $G_i$.

Suppose first that $|Y| = 3$. Let $v$ be an arbitrary vertex in $Y$. The edges of $G_i$ at $v$ form at least two segments. By the property (iv), the degree of $v$ in $G_i$ is at least 3, hence there are at least three boundary edges incident with $v$. Since $|Y \setminus \{v\}| = 2$, at least one of these edges is rich, hence $G_i$ receives at least $1/2$ units of charge through $v$. Therefore, $G_i$ receives $9/2$ units of charge by Rule R1, and at least $1/2$ units of charge by Rules R2–R4 through each vertex of $Y$, which sums to at least six units of charge.

Suppose now that $|Y| = 2$. The graph $G_i$ receives three units of charge by Rule R1. We prove that at least $3/2$ units of charge are sent to $G_i$ through each vertex of $Y$ by Rules R2–R4, thus showing that $G_i$ receives at least six units of charge. Suppose for contradiction that less than $3/2$ units of charge are sent to $G_i$ through a vertex $v \in Y$. Then, there are at most two rich edges incident with $v$. On the other hand, $G_i$ has at least two segments at $v$, the degree of $v$ is at least three by the property (iv), and $Y \setminus \{v\}$ consists of only one vertex $w$, thus at least two rich edges are incident with $v$. Hence, we conclude that there are exactly two rich edges at $v$. This is only possible in the following cases:

- *The degree of $v$ in $G_i$ is three, and each of the edges of $G_i$ incident with $v$ forms a segment of length one.* However, note that in this case, each of the four (not necessarily distinct) faces incident with the rich edges sends $1/2$ units of charge through $v$, for total of two units of charge.

- *There are exactly two segments of $G_i$ at $v$ and one of them is of length one.* Let $e_0 = vu_0$ be the edge of the segment of length one, and $e_1 = vu_1$ and $e_2 = vu_2$ the boundary edges of the other segment. Note that $u_1 \neq u_2$, as the degree of $v$ is at least three. If $w \neq u_0$ (say $w = e_1$), then each of the faces incident with $e_0$ send $1/2$ units of charge through $v$ and the face $f_{e_2}$ sends $1/2$ units of charge, for total of $3/2$ units.

  Let us now consider the case that $w = u_0$. The graph $G_i$ receives $1/2$ units of charge through $v$ for each of $e_1$ and $e_2$. If Rules R3 or R4 applied at $v$ at least once, $G_i$ would receive additional 1/2 units of charge, contradicting the choice of $v$. Let us assume that this is not the case. Let $w_1$ and $w_2$ be the vertices following $u_1$ and $u_2$ in the facial walks of $f_{e_1}$ and $f_{e_2}$, respectively. For $i = 1, 2$, the vertices $w_i$ and $v$ are both neighbours of $u_i$, hence $w_1 \neq v \neq w_2$. The edges following $w_1$ and $w_2$ in the facial walks do not belong to $G_i$, since otherwise one of Rules R3 or R4 applies. This means that $w_1, w_2 \in Y$, and hence $w_1 = w_2 = w$. This is the forbidden configuration $(\star)$, hence we obtain a contradiction with the assumption that less than 3/2 is sent through the vertex $v$.

It follows that the final charge of each of the graphs $G_i$ is at least six, thus we conclude that $\triangle + 3(\sum_{i=1}^{k} n_i - n) \geq 6k$, which finishes the proof.$\square$

Theorem 7 is tight – for example, the equality is obtained for disjoint union of $k$ triangulations, or graphs obtained from this graph by identifying the vertices in such a way that all edges of each graph form one segment at each vertex. Also, it is not possible to relax the condition on the number of vertices in $G_i$, as the claim is false if each $G_i$ is just an edge.

# 3 Upper Bound

We are now ready to prove the upper bound on the $k$-chromatic number $\chi_k(G)$ of a graph $G$ of Eulerian genus $g$. Our method is similar to the one used by Füredi et al. [5], except that we use a better estimate on the number of edges of $G$ obtained from Theorem 7.

**Theorem 8** *Let $G$ be a simple graph $G$ of Eulerian genus $g$. If $k \leq g$, then*

$$\chi_k(G) \leq \left\lfloor \frac{7k + \sqrt{24kg + k^2}}{2} \right\rfloor .$$

**Proof.** Let us embed $G$ in a surface of Eulerian genus $g$. Let $G_1, \ldots, G_k$ be a partition of $G$ into $k$ parts. For each $i$, let $G'_i \subseteq G_i$ be a critical subgraph of $G_i$ such that $\chi(G'_i) = \chi(G_i) = c_i$. We may assume that $c_1 \geq c_2 \geq \ldots \geq c_k$. Let $t$ be the largest number such that $c_t \geq 7$. Thus, $t = 0$ if $c_1 \leq 6$. We bound the sum of chromatic numbers of the graphs $G_1, \ldots, G_t$. Let $n_i = n(G'_i)$.

Let $G' = G'_1 \cup \ldots \cup G'_t$ and $e' = e(G')$. Using Theorem 7, we get

$$2e' \leq 6g + 6\sum_{i=1}^{t}(n_i - 2).$$

On the other hand, minimum degree of each $G'_i$ is at least $c_i - 1$, hence $(c_i - 1)n_i \leq 2e(G'_i)$. This implies that

$$\sum_{i=1}^{t}(c_i - 1)n_i \leq 6g + 6\sum_{i=1}^{t}(n_i - 2).$$

Using the fact that $c_i \geq 7$ and $n_i \geq c_i$, we obtain

$$\sum_{i=1}^{t}\left[(c_i - 7/2)^2 - 49/4\right] = \sum_{i=1}^{t}(c_i - 7)c_i \leq \sum_{i=1}^{t}(c_i - 7)n_i \leq 6g - 12t.$$

By the inequality between the arithmetic and quadratic mean,

$$\frac{1}{t}\left[\sum_{i=1}^{t}(c_i - 7/2)\right]^2 \leq 6g + t/4,$$

from which we infer

$$\sum_{i=1}^{t} c_i \leq \frac{7t + \sqrt{24tg + t^2}}{2}.$$

Taking into account the graphs $G_{t+1}, \ldots, G_k$, we get

$$\sum_{i=1}^{k} c_i \leq \frac{7t + \sqrt{24tg + t^2}}{2} + 6(k - t).$$

If $t \leq g$, this expression is increasing in $t$, thus we obtain

$$\sum_{i=1}^{k} c_i \leq \frac{7k + \sqrt{24kg + k^2}}{2}.$$

Since the expression on the left-hand side is integer, we may round the expression on the right-hand side down, thus finishing the proof of this theorem. □

# 4 Lower Bound

The proof of the upper bound hints at how the lower bound examples should look like. For each of the graphs in the partition, we should have $c_i = n_i$, hence all the graphs $G_i$ should be complete. Also, since we used the inequality between arithmetic and quadratic means, their sizes should be the same. This is only possible for special values of $g$ and $k$. For example, consider the case $g = \frac{1}{6}k(t-3)(t-4)$ for some $t \geq 4$, $t \equiv 0, 1 \pmod 3$. Then, $K_t$ can be embedded in a surface of genus $g/k$ ($K_7$ cannot be embedded in the Klein bottle, but it can be embedded in the torus), according to Theorem 3. By Theorem 2, the disjoint union of $k$ complete graphs on $t$ vertices can be embedded in a surface $S$ of genus $g$, hence

$$\chi_k(S) \geq k\,t = \frac{7k + \sqrt{24kg + k^2}}{2}.$$

For general $g$ and $k$, we cannot hope for a nice formula like the one in Theorem 8, thus we would be satisfied with some description of the best possible example. A natural guess is that this example is a disjoint union of cliques. We were not able to prove that this is the case – the best result that we obtained in this direction is the following proposition:

**Proposition 9** *Let $G_1, \ldots, G_k$ be a partition of a graph $G$ of Eulerian genus $g$ to $k$ parts, and let $c_i = \chi(G_i) \geq 7$ for each $i$. Let $G'_i$ be a $c_i$-critical subgraph of $G_i$. Suppose that $c_i \equiv 0, 1 \pmod 3$ whenever $G'_i$ is a clique. Then, the disjoint union of the cliques $K_{c_1}, \ldots, K_{c_k}$ has Eulerian genus at most $g$.*

**Proof.** Let $e' = e(G'_1 \cup \ldots \cup G'_k)$, $n_i = n(G'_i)$, and let $\delta_i = 0$ if $G'_i$ is a clique and $\delta_i = c_i - 3$ otherwise. By Theorem 7,

$$2e' \leq 6g + 6\sum_{i=1}^{k}(n_i - 2).$$

On the other hand, using Theorem 5, we get

$$2e' \geq \sum_{i=1}^{k}(c_i - 1)n_i + \delta_i.$$

Therefore, we obtain

$$
\begin{aligned}
g &\geq \frac{1}{6}\sum_{i=1}^{k}(c_i-7)n_i+12+\delta_i \\
&\geq \sum_{i=1}^{k}\frac{1}{6}\left((c_i-7)c_i+12+\delta_i\right) \\
&\geq \sum_{i=1}^{k}\left\lceil\frac{1}{6}(c_i-3)(c_i-4)\right\rceil=\sum_{i=1}^{k}g\left(K_{c_i}\right),
\end{aligned}
$$

where the last inequality holds because $\delta_i \geq 4$ whenever $c_i \equiv 2 \pmod 3$, by the assumptions of the lemma. The statement of the lemma follows from Theorem 2. □

# 5 Conclusions

Let us call the complete graph $K_n$ *bad* if it does not triangulate the minimal surface in which it can be embedded, i.e., $n \equiv 2 \pmod 3$. Proposition 9 shows that the best values of $\chi_k$ are achieved for disjoint unions of cliques, unless bad cliques appear in the partition. It is natural to ask whether the restriction on the appearance of the bad cliques is necessary, or whether it is always possible to "disentangle" cliques:

**Problem 1** *Let $G_1, \ldots, G_k$ be a partition of a graph $G$ to $k$ parts such that each subgraph $G_i$ is a clique. Is it true that the vertex-disjoint union of the cliques $G_i$ can be embedded in a surface of Eulerian genus $g(G)$?*

For $k = 2$, this follows from Theorem 2. The proof of Theorem 7 shows that unless the graphs in the partition can be trivially disentangled, we may decrease the bound by 6, which implies that the answer to Problem 1 is positive for $k = 3$.

One way to answer the question in Problem 1 positively for $k \geq 4$ would be to improve Theorem 7, by decreasing the right hand side of the inequality by 2 for each bad clique in the partition. Another way is suggested by the following conjecture of Stiebitz and Škrekovski [14]:

**Conjecture 1** *Let $G$ be an edge-disjoint union of a clique $K$ and an arbitrary graph $H$. Let $H'$ be the graph obtained from $H$ by contracting the set $V(K)$ to a single vertex. Then, $g(H') + g(K) \leq g(G)$.*

Because two complete graphs in a partition of a graph to $k$ parts cannot share more than one vertex, it is easy to show by induction that Conjecture 1 implies positive answer to Problem 1.

In our considerations, we do not distinguish between orientable and non-orientable surfaces – we only focus on their Eulerian genus. While assymptotically there does not seem to be much difference, for some values $k$ and $g$ the results may differ.

We have provided (almost) matching upper and lower bounds for $k$-chromatic number of graphs with bounded genus $g$, assuming that the genus is large enough relatively to $k$. The reason why our techniques cannot be directly applied in the case $k$ is larger than $g$ is that we would need to consider critical graphs with chromatic number $\leq 6$. Graphs with chromatic number $\leq 4$ are easy to handle – we may assume that they appear only as $K_4$ disjoint with the rest of the graph, since they are planar and hence do not affect genus of the graph. However, graphs with chromatic number 5 and 6 are difficult to deal with. For chromatic number 6, the list of critical graphs is known only for surfaces with $g \leq 2$, and for the chromatic number 5, there even are infinitely many of them on each surface with $g \geq 1$. Nevertheless, it might be interesting to determine the exact values of $\chi_k$ for some special cases, e.g., for graphs embedded in the torus or in the projective plane.

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