Additive and multiplicative tolerance in multiobjective linear programming

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Abstract

We consider a multiobjective linear program and the coefficients of the multiobjective function are subject to some uncertainties. Let x^* be an efficient point. We propose a procedure to compute an additive and multiplicative (percentage) tolerance in which all the objective function coefficients may simultaneously and independently vary while preserving the efficiency of x^*. Although the tolerances are not maximal in general, they are satisfactorily large. If x^* is a nondegenerate basic solution, then the procedure runs in a polynomial time.

Keywords: *Multiobjective linear programming, efficient point, sensitivity analysis, tolerance analysis, generalized fractional programming.*

1 Introduction

Sensitivity analysis as an important way to study uncertainties occurring in real life problems. It gives us a basic information on stability, robustness and impact of errors on a model. Tolerance approach was developed [11, 12] for dealing with simultaneous and independent changes of linear program coefficients. It provides a decision maker with an easy-to-use method to tackle perturbations of selected parameters. However, this approach is, in the essence, restricted to the considering only parameters in one line: the right-hand side, the objective function coefficients, or a row/column of the coefficient matrix.

Multiobjective linear programming (MOLP) problems are often solved using weighted sum of objective functions. This is why tolerance analysis in MOLP was mainly concerned with sensitivity of these weights [1, 2, 5, 6]. Considering all the objective functions coefficients seemed to be unrealistic. Nevertheless, we propose a simple procedure calculating quite large (but not maximal in general) tolerance in which *all* the objective functions coefficients may vary while retaining efficiency of a given point.

Let us consider the multiobjective linear program

$$\max_{x \in \mathcal{M}} \; Cx, \tag{1}$$

where the feasible set $\mathcal{M} := \{x \in \mathbb{R}^n : Ax \le b\}$, and $C \in \mathbb{R}^{s\times n}$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Let us introduce a general δ-neighborhood, of a matrix C as

$$\mathcal{O}_{\delta,G}(C) := \{\hat{C} : |\hat{C} - C| < \delta |G|\},$$

where G is a given matrix. Although in classical tolerance approach in linear programming there is naturally used closed neighborhood, here is much more usefull to use an open one because of noncompactness of efficiency.

For $G = C$, this concept corresponds to the most frequently used percentage deviation. When G consists of ones, then it represents an additive perturbation. For the purpose of this paper, additive δ-neighborhood of a certain matrix Z is denoted separately by

$$\begin{aligned}\mathcal{O}_\delta(Z) &:= \{\hat{Z} : \|\hat{Z} - Z\|_\infty < \delta\} \\ &= \{\hat{Z} : |\hat{Z}_{ij} - Z_{ij}| < \delta \; \forall i, j\}.\end{aligned}$$

A feasible solution x^* to (1) is called *efficient* if there is no $x \in \mathcal{M}$ such that $Cx \ge Cx^*$ and $Cx \ne Cx^*$.

Aan additive tolerance for an efficient solution x^* is any real $\delta > 0$ such that x^* remains efficient to

$$\max_{x \in \mathcal{M}} \; \hat{C}x \tag{2}$$

for every $\hat{C} \in \mathcal{O}_\delta(C)$. *A multiplicative tolerance* for an efficient x^* and given G is any $\delta > 0$ such that x^* remains efficient to (2) for every $\hat{C} \in \mathcal{O}_{\delta,G}(C)$.

From a practical viewpoint, multiplicative tolerance is much more useful than the additive deviation, since it takes into account diverse values of the coefficients. But the most common percentage case of the multiplicative approach has one drawback. Any percentage tolerance for zero values gives us no information—the zeros have to remain zeros.

Notation

$A_{i\bullet}$	the i-th row of a matrix A
A_B	submatrix of A consisting of the rows indexed by B
$sgn(z)$	a sign of a real z, i.e., $sgn(z) = 0$ if $z = 0$, $sgn(z) = 1$ if $z > 0$, and $sgn(z) = -1$ if $z < 0$
e	a vector of ones (with convenient dimension)
$\|x\|_\infty$	Chebyshev norm, i.e., $\|x\|_\infty = \max_i \|x_i\|$
$\|A\|_\infty$	Chebyshev norm for matrices, i.e., $\|x\|_\infty = \max_{i,j} \|A_{ij}\|$
$cl\, \mathcal{S}$	closure of a set $\mathcal{S}$
$int\, \mathcal{S}$	interior of a set $\mathcal{S}$

2 Additive tolerance

Let x^* be a vertex of $\mathcal{M}$ that is efficient to (1). First we compute the normal (polar) [8, 10] cone $\mathcal{N}(x^*)$ of $\mathcal{M}$ at the point x^* by means of inequalities

$$\mathcal{N}(x^*) = \{x \in \mathbb{R}^n : Dx \geq 0\}. \tag{3}$$

If x^* is a nondegenerate basic solution with the optimal basis B, then $D = (A_B^{-1})^T$. That is, the normal cone is described by $(A_B^{-1})^T x \geq 0$. Otherwise, let h_i, $i \in I$, be directions of all edges emerging from x^*. Then the normal cone $\mathcal{N}(x^*)$ is characterized by the system $-h_i^T x \geq 0\ \forall i \in I$.

Let us remind the well known [3] characterization of efficiency in multiobjective linear programming.

Theorem 1. *The feasible solution x^* is efficient if and only there exists $\lambda > 0$ such that $DC^T\lambda \geq 0$.*

Since x^* is efficient due to our assumption, there is some $\lambda > 0$ such that $DC^T\lambda \geq 0$. That is, x^* is an optimal solution to $\max_{x\in\mathcal{M}} c(\lambda)^T x$, where

$c(\lambda)^T = C^T\lambda$. The main idea is is to determine the maximum tolerance δ for the vector $c(\lambda)$ with fixed λ, and subsequently also such a vector λ for which the corresponding maximum tolerance of $c(\lambda)$ is as large as possible.

Lemma 1. *Let $\lambda \geq 0$ with $e^T\lambda = 1$, and $\delta > 0$ be given. If $\hat{c} \in int\,\mathcal{N}(x^*)$ for every $\hat{c} \in \mathcal{O}_\delta(C^T\lambda)$, then $\hat{C}^T\lambda \in int\,\mathcal{N}(x^*)$ for every $\hat{C} \in \mathcal{O}_\delta(C)$.*

Proof. Let $\hat{C}$ be an arbitrary matrix with $\|\hat{C} - C\|_\infty < \delta$. It is sufficient to prove that $\hat{C}^T\lambda \in \mathcal{O}_\delta(C^T\lambda)$, i.e. $\|\hat{C}^T\lambda - C^T\lambda\|_\infty < \delta$. We have

$$\|\hat{C}^T\lambda - C^T\lambda\|_\infty = \|(\hat{C}^T - C^T)\lambda\|_\infty \leq \|(\hat{C}^T - C^T)\|_\infty e^T\lambda < \delta,$$

which completes the proof. □

Lemma 2. *Let $\lambda \geq 0$ and suppose that $C^T\lambda \in int\,\mathcal{N}(x^*)$. Then x^* is an efficient solution to* (1).

Proof. If $\lambda > 0$, then the statement follows from Theorem 1. Otherwise, define $c_\varepsilon := C^T(\lambda + \varepsilon e)$, where $\varepsilon > 0$ is sufficiently small. Then $c_\varepsilon \in \mathcal{N}(x^*)$ and, hence, x^* is an efficient solution. □

The following theorem says how to effectively compute an additive tolerance δ^*, in which can independently vary all coefficients of C. As we see in Example 1, the tolerance δ^* is not maximal in general. It is a drawback when dealing with perturbation of all entries of C.

Theorem 2 (Additive tolerance). *Let (λ^*, δ^*) be an optimal solution to the linear program*

$$\max \delta \quad \textit{subject to} \quad DC^T\lambda - |D|e\delta \geq 0,\ \ \lambda, \delta \geq 0,\ \ e^T\lambda = 1. \tag{4}$$

Then for every $\hat{C} \in \mathcal{O}_{\delta^*}(C)$ the point x^* is an efficient solution to the multiobjective linear program (2).

Proof. Let $\lambda \geq 0$, $e^T\lambda = 1$. Suppose that $c := C^T\lambda \in \mathcal{N}(x^*)$ and that for certain $\delta > 0$ we have $cl\,\mathcal{O}_\delta(c) \subseteq \mathcal{N}(x^*)$, which is equivalent to $\mathcal{O}_\delta(c) \subseteq int\,\mathcal{N}(x^*)$. By Lemma 1, $\hat{C}^T\lambda \in int\,\mathcal{N}(x^*)$ for all $\hat{C} \in \mathcal{O}_\delta(C)$. By Lemma 2, x^* is an efficient solution of (2) for every $\hat{C} \in \mathcal{O}_\delta(C)$.

It remains to determine the maximum tolerance δ satisfying our assumptions. We use the optimization problem

$$\max \delta \quad \text{subject to} \quad cl\,\mathcal{O}_\delta(c) \subseteq \mathcal{N}(x^*),$$

which we rewrite as

$$\max \delta \ \text{ subject to } \ D\hat{c} \geq 0 \ \forall \hat{c} \in \mathcal{O}_\delta(c),$$

or

$$\max \delta \ \text{ subject to } \ Dc + D\hat{e}\delta \geq 0 \ \forall \hat{e} : |\hat{e}| \leq e.$$

The k-th inequality $D_{k\bullet}c + D_{k\bullet}\hat{e}\delta \geq 0$ is true for all $\hat{e}$ such that $|\hat{e}| \leq e$ if and only if $D_{k\bullet}c - |D|_{k\bullet}e\delta \geq 0$. Hence, the optimization problem

$$\max \delta \ \text{ subject to } \ Dc - |D|e\delta \geq 0$$

yields the same or smaller tolerance. So far, the weights λ were fixed. To compute the largest tolerance we introduce λ as new variables to the optimization problem. We get the linear program (4). □

Let us notice that the linear program (4) is infeasible if and only if x^* is not efficient solution of (1).

It is also easy to see that for each index $i \in \{1, \dots, s\}$ with $\lambda_i^* = 0$ the tolerance of i-th row of C is infinite. That is, for such indices the elements of $C_{i\bullet}$ can vary absolutely arbitrarily and independently while the other elements only in δ-neighborhood. This is common for large s, which is, however, not usual in real-life problems.

Example 1. Consider (inspired by [9])

$$C = \begin{pmatrix} 2.5 & 2 \\ 3.5 & 0.65 \end{pmatrix},$$

$$\mathcal{M} = \{x \in \mathbb{R}^2 : 3x_1 + 4x_2 \leq 42,\ 3x_1 + x_2 \leq 24,\ x_2 \leq 9,\ x \geq 0\},$$

and the (nondegenerate) basic solution $x^* = (6,6)^T$. Then $B = (1,2)$ and the normal cone at x^* is described as follows

$$\begin{aligned} \mathcal{N}(x^*) &= \{x \in \mathbb{R}^2 : (A_B^{-1})^T x \geq 0\} \\ &= \{x \in \mathbb{R}^2 : -x_1 + 3x_2 \geq 0,\ 4x_1 - 3x_2 \geq 0\}. \end{aligned}$$

Hence $D = \left(\begin{smallmatrix} -1 & 3 \\ 4 & -3 \end{smallmatrix}\right)$. According to Theorem 2 we solve the linear program

$$\begin{aligned} \max \delta \ \text{ subject to } \ 3.5\lambda_1 - 1.55\lambda_2 - 4\delta &\geq 0, \\ 4\lambda_1 + 12.05\lambda_2 - 7\delta &\geq 0, \\ \lambda_1 + \lambda_2 &= 1, \\ \lambda, \delta &\geq 0. \end{aligned}$$

The optimal solution is $\lambda \simeq (0.8742, 0.1258)^T$, $\delta^* \simeq 0.7161$. Therefore, we obtain the additive tolerance 0.7161. However, this tolerance is not maximal. In this simple example it is easy to find out that the maximum tolerance is $\frac{7}{8}$.

3 Multiplicative tolerance

In this section we generalize the additive tolerance approach to the multiplicative one. Although a multiplicative tolerance is no more calculated by a simple linear program, the computation can be done in a polynomial time under the same assumptions.

Let x^* be a vertex of $\mathcal{M}$ that is efficient to (1). First we proceed similarly as in the previous section and compute the normal cone $\mathcal{N}(x^*)$ of $\mathcal{M}$ at the point x^* by means of inequalities (3).

By Theorem 1, the efficiency of x^* implies that there is some $\lambda > 0$ such that x^* is an optimal solution to the linear program $\max_{x\in\mathcal{M}} c(\lambda)^T x$, where $c(\lambda)^T = C^T\lambda$. Again, we determine a maximum tolerance δ for the vector $c(\lambda)$ with fixed λ. Afterwards we relax λ and compute for which λ the corresponding maximum tolerance of $c(\lambda)$ is as large as possible.

Lemma 3. *Let $\lambda \geq 0$ with $e^T\lambda = 1$, and $\delta > 0$ be given. Suppose that $\hat{c} \in \operatorname{int}\mathcal{N}(x^*)$ for every $\hat{c}$ such that $|\hat{c} - C^T\lambda| < \delta|G|^T\lambda$. Then $\hat{C}^T\lambda \in \operatorname{int}\mathcal{N}(x^*)$ for every $\hat{C} \in \mathcal{O}_{\delta,G}(C)$.*

Proof. Let $\hat{C}$ be an arbitrary matrix with $|\hat{C} - C| < \delta|G|$. It is sufficient to prove that $|\hat{C}^T\lambda - C^T\lambda| < \delta|G|^T\lambda$. We have

$$|\hat{C}^T\lambda - C^T\lambda| = |(\hat{C}^T - C^T)\lambda| \leq |(\hat{C}^T - C^T)|\lambda < \delta|G|^T\lambda,$$

which completes the proof. □

The following theorem says how to compute a multiplicative tolerance δ^*. Such a tolerance δ^* may not be the best possible; see Example 2.

Theorem 3 (Multiplicative tolerance). *Let (λ^*, δ^*) be an optimal solution to the mathematical program*

$$\max \delta \quad \text{subject to} \quad DC^T\lambda - \delta|D||G|^T\lambda \geq 0,\ \lambda, \delta \geq 0,\ e^T\lambda = 1. \qquad (5)$$

Then for every $\hat{C} \in \mathcal{O}_{\delta^*,G}(C)$ the point x^* is an efficient solution to the multiobjective linear program (2).

Proof. Let $\lambda \geq 0,\ e^T\lambda = 1$ and denote $c := C^T\lambda$. Suppose that for certain $\delta > 0$ we have $\hat{c} \in int\,\mathcal{N}(x^*)$ for every $\hat{c}$ such that $|\hat{c} - c| < \delta|G|^T\lambda$. By Lemma 3, $\hat{C}^T\lambda \in int\,\mathcal{N}(x^*)$ for all $\hat{C} \in \mathcal{O}_{\delta,G}(C)$. By Lemma 2, x^* is an efficient solution of (2) for every $\hat{C} \in \mathcal{O}_{\delta,G}(C)$.

It remains to determine the maximum tolerance δ satisfying our assumption. Its equivalent formulation is that $\hat{c} \in \mathcal{N}(x^*)$ for every $\hat{c}$ such that $|\hat{c} - c| \leq \delta|G|^T\lambda$. To obtain the maximal δ we use the optimization problem

$$\max \delta \ \text{ subject to } \ D\hat{c} \geq 0\ \forall \hat{c} : |\hat{c} - c| \leq \delta|G|^T\lambda. \tag{6}$$

The k-th inequality $D_{k\bullet}\hat{c} \geq 0$ is true for all $\hat{c}$ such that $|\hat{c} - c| \leq \delta|G|^T\lambda$ if and only if $D_{k\bullet}c - \delta|D|_{k\bullet}|G|^T\lambda \geq 0$. It is because the worst case is when $\hat{c}_i = c_i - \delta sgn(D_{ki})|G|^T_{i\bullet}\lambda$, and then $D_{k\bullet}\hat{c} = D_{k\bullet}c - \delta|D|_{k\bullet}|G|^T\lambda$. Putting all inequalities togehter we get a sufficient condition. Hence, the optimization problem

$$\max \delta \ \text{ subject to } \ Dc - \delta|D||G|^T\lambda \geq 0$$

yields the optimal value that is equal or less than (6). So far, the weights λ were fixed. To compute the largest possible tolerance we introduce λ as new variables to the optimization problem. We get the optimization problem (5). $\square$

Remark 1. The nonlinear program (5) is a particular case of so called generalized linear fractional program

$$\max \min_i \frac{P_{i\bullet}x}{Q_{i\bullet}x} \ \text{ subject to } \ Qx > 0,\ Rx \geq r,$$

or,

$$\max \alpha \ \text{ subject to } \ Px - \alpha Qx \geq 0,\ Qx \geq 0,\ Rx \geq r,$$

which is polynomially solvable using an interior point method [4, 7].

Example 2. Reconsider the problem from Example 1 and set $G = C$. Recall that

$$C = \begin{pmatrix} 2.5 & 2 \\ 3.5 & 0.65 \end{pmatrix},\ D = \begin{pmatrix} -1 & 3 \\ 4 & -3 \end{pmatrix}.$$

According to Theorem 3 we solve the generalized linear fractional problem

$$\begin{aligned} \max \delta \quad \text{subject to} \quad \begin{pmatrix} 3.5 & -1.55 \\ 4 & 12.05 \end{pmatrix} \lambda - \delta \begin{pmatrix} 8.5 & 5.45 \\ 16 & 15.95 \end{pmatrix} \lambda \geq 0, \\ \lambda_1 + \lambda_2 = 1, \\ \lambda, \delta \geq 0. \end{aligned}$$

The optimal solution is $\lambda^* = \left(\frac{101}{121}, \frac{20}{121}\right)^T$, $\delta^* = \frac{1}{3}$. Therefore, we obtain the percentage tolerance 33.33%. However, this tolerance is not maximal. In this simple example it is easy to find out that the maximal tolerance is $\frac{7}{17} \simeq 41.18\%$.

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