# List colorings with measurable sets

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**Abstract**

The measurable list chromatic number of a graph $G$ is the smallest number $\xi$ such that if each vertex $v$ of $G$ is assigned a set $L(v)$ of measure $\xi$ in a fixed atomless measure space, then there exist sets $c(v) \subseteq L(v)$ such that each $c(v)$ has measure one and $c(v) \cap c(v') = \emptyset$ for every pair of adjacent vertices $v$ and $v'$. We show that the measurable list chromatic number of a finite graph $G$ is equal to its fractional chromatic number. We also apply our method to obtain an alternative proof of a measurable generalization of Hall's theorem due to Hilton and Johnson [J. Graph Theory **54** (2007), 179–193].

## 1 Introduction

In this paper, we study colorings and list colorings of graphs with measurable sets. This concept extends the standard notion of fractional colorings, which

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we recall now. A *q-coloring* of a graph $G$ with $p$ colors is a mapping that assigns to each vertex a subset of $\{1, 2, \ldots, p\}$ of size $q$ such that the sets assigned to adjacent vertices are disjoint. The *fractional chromatic number* $\chi_f(G)$ of $G$ is the infimum of $p/q$ taken over all $p$ and $q$ for which $G$ admits a $q$-coloring with $p$ colors. It is well-known [9] that if $G$ is finite, then the infimum is always attained and thus $\chi_f(G)$ is rational. Fractional colorings form a prominent topic of graph theory, to which a whole book [8] is devoted.

We now consider an extension of fractional colorings introduced by Hilton and Johnson [6]. Let $(X, \mathcal{M}, \mu)$ be an *atomless measure space*, i.e., $\mathcal{M}$ is a $\sigma$-algebra on a set $X$, and $\mu$ a measure defined on $\mathcal{M}$ such that, for each $A \in \mathcal{M}$ with $\mu(A) > 0$, there exists $B \subset A$ such that $B \in \mathcal{M}$ and $0 < \mu(B) < \mu(A)$. A *measurable coloring* of a graph $G$ over the space $(X, \mathcal{M}, \mu)$ is a mapping $c : V(G) \to \mathcal{M}$ such that $\mu(c(v)) \geq 1$ for every $v \in V(G)$ and $c(v) \cap c(v') = \emptyset$ for every pair of adjacent vertices $v$ and $v'$. The *measurable chromatic number* $\chi_m(G)$ of $G$ is the infimum of $\mu(X)$ over all atomless measurable spaces $(X, \mathcal{M}, \mu)$ over which $G$ admits a measurable coloring. This notion is indeed a "continuous" extension of fractional colorings as $\chi_f(G) = \chi_m(G)$. We note that a graph $G$ has a measurable coloring over $(X, \mathcal{M}, \mu)$ for every atomless measurable space $(X, \mathcal{M}, \mu)$ with $\mu(X) \geq \chi_m(G)$.

A list variant of fractional colorings was studied by Alon, Tuza and Voigt [2]. Fix a graph $G$. For each positive integer $q$, let $p_q$ be the smallest integer such that if every vertex $v$ is assigned a list $L(v)$ of $p_q$ colors, then there exists a $q$-coloring $c$ of $G$ with $c(v) \subseteq L(v)$. The *list fractional chromatic number* $\chi_{f,\ell}(G)$ of a graph $G$ is the limit of the ratio $p_q/q$ when $q$ tends to infinity. Alon et al. [2] showed that if $G$ is finite then $\chi_{f,\ell}(G)$ exists and is equal to $\chi_f(G)$. It seems unavoidable to consider the limit when $q$ tends to infinity in this definition. The reader can consult the book by Scheinerman and Ullman [8] for further details.

We believe that measurable colorings are a natural extension of fractional colorings, in particular when related to list colorings since this notion allows us to avoid the limit in the definition of the list fractional chromatic number. Let us define measurable list colorings formally. The *measurable list chromatic number* $\chi_{m,\ell}(G)$ of a graph $G$ is the infimum of all positive reals $\alpha$ such that if each vertex $v$ of $G$ is assigned a set $L(v) \in \mathcal{M}$ with $\mu(L(v)) = \alpha$, then there exists a measurable coloring $c : V(G) \to \mathcal{M}$ such that $c(v) \subseteq L(v)$. A justification that this notion is a "continuous" list variant of fractional colorings is the following result, which we prove in Section 4: $\chi_{m,\ell}(G) = \chi_f(G)$ for every finite graph $G$. Hence, for every finite graph $G$, all the four quantities $\chi_f(G)$, $\chi_{f,\ell}(G)$, $\chi_m(G)$ and $\chi_{m,\ell}(G)$ are equal. As we note at the end of the paper, the equality does not hold in general for infinite graphs.

Our main tool to prove that $\chi_f(G) = \chi_{m,\ell}(G)$ is a limit theorem established in Section 3, which asserts that if for every $\varepsilon > 0$ there exists a coloring $c_\varepsilon(v) \subseteq L(v)$ with $\mu(c_\varepsilon(v)) \geq 1 - \varepsilon$, then there also exists a measurable coloring $c$, i.e., a coloring with $\mu(c(v)) = 1$. We also apply the same technique to give a new proof of a result of Hilton and Johnson [6], which generalizes an extension of Hall's theorem by Cropper et al. [4] as we outline in the next subsection. We believe that our alternative proof is conceptionally simpler since we restrict ourselves to using only elementary results from the measure theory and avoid using the Krein-Milman theorem. Our proof also suggests a method of translating other results on fractional colorings to measurable colorings. We do not apply our method to such results on fractional colorings since we believe that it would not provide any new insights.

## 1.1 Measurable extension of Hall's theorem

In this subsection, we introduce a theorem of Cropper, Gyarfás and Lehel [4], which is a generalization of Hall's theorem, and we discuss its extension to measurable colorings.

Cropper et al. [4] characterized the class of graphs which have the property that a Hall-type condition is sufficient for the existence of a multicoloring. Formally, if $\kappa : V(G) \to \mathbb{N}$ is a function that specifies the numbers of colors each vertex should be assigned and $L : V(G) \to 2^X$ is a list assignment, we say that the *Hall condition* is fulfilled if

$$\sum_{v \in V(H)} \kappa(v) \leq \sum_{x \in X} \alpha(x, L, H) \text{ for every subgraph } H \subseteq G, \tag{1}$$

where $\alpha(x, L, H)$ is the independence number of the subgraph of $H$ induced by the vertices $v$ with $x \in L(v)$. A color $x$ can be assigned to at most $\alpha(x, L, H)$ vertices of $H$ and thus the condition (1) is necessary for the existence of a coloring $c$. Observe that it is actually enough to verify it for induced subgraphs $H$ of $G$. The following theorem provides a characterization of graphs for which the Hall condition is also sufficient for the existence of a coloring $c$. We let $K_4^-$ be the graph obtained from $K_4$ by removing an edge.

**Theorem 1 (CGL Theorem [4]).** *The following statements are equivalent for a finite graph $G$.*

- *For every $\kappa : V(G) \to \mathbb{N}$ and $L : V(G) \to 2^X$, the Hall condition* (1) *is sufficient and necessary for the existence of a coloring $c$ such that for every $v \in V(G)$, $c(v)$ is a subset of $L(v)$ of size $\kappa(v)$, and $c(v) \cap c(v') = \emptyset$ for every pair of adjacent vertices $v$ and $v'$;*
- *$G$ is the line graph of a forest;*
- *every block of $G$ is a clique and each cut-vertex belongs exactly to two blocks;*
- *$G$ does not contain $K_4^-$, $K_{1,3}$ or a cycle of length more than* 3 *as an induced subgraph.*

Note that Theorem 1 applied to $G = K_n$ and $\kappa \equiv 1$ translates to Hall's theorem for a system of $n$ distinct representatives.

Bollobás and Varopoulos [3] generalized Hall's theorem to measurable sets. In particular, they proved the following.

**Theorem 2 (Bollobás and Varopoulos [3]).** *Let $(X, \mathcal{M}, \mu)$ be an atomless measure space, let $(X_\alpha)_{\alpha \in A}$ be a family of measurable sets of $\mathcal{M}$ with (finite or infinite) index set $A$, and let $\kappa : A \to \mathbb{R}^+$. If for every finite subset $B$ of $A$,*

$$\sum_{\alpha \in B} \kappa(\alpha) \leq \mu\left(\bigcup_{\alpha \in B} X_\alpha\right),$$

*then there exists a family $(Y_\alpha)_{\alpha \in A}$ of measurable sets satisfying $Y_\alpha \subseteq X_\alpha$ and $\mu(Y_\alpha) = \kappa(\alpha)$ for every $\alpha \in A$, and $\mu(Y_\alpha \cap Y_{\alpha'}) = 0$ whenever $\alpha \neq \alpha'$.*

Inspired by Theorems 1 and 2, Hilton and Johnson [6] considered, for finite graphs, a common generalization of both these theorems. For an atomless measure space $(X, \mathcal{M}, \mu)$ and for demands $\kappa : V(G) \to \mathbb{R}^+$, an assignment $L(v) \in \mathcal{M}$ satisfy the *generalized Hall condition* if

$$\sum_{v \in V(H)} \kappa(v) \leq \int_X \alpha(x, L, H) \, \mathrm{d}\, \mu \tag{2}$$

holds for every subgraph $H$ of $G$, where $\alpha(x, L, H)$ is the number of vertices $v$ of $H$ such that $x \in L(v)$. Note that the function $\alpha(x, L, H)$ is integrable since $L(v) \in \mathcal{M}$ for every vertex $v$. Hilton and Johnson [6] proved the following.

**Theorem 3 (Measurable CGL Theorem [6]).** *The following statements are equivalent for a finite graph $G$.*

- *For every $\kappa : V(G) \to \mathbb{R}^+$, an atomless measure space $(X, \mathcal{M}, \mu)$ and $L : V(G) \to \mathcal{M}$, the generalized Hall condition* (2) *is sufficient and necessary for the existence of a coloring $c : V(G) \to \mathcal{M}$ such that for every $v \in V(G)$, $c(v)$ is a subset of $L(v)$ of measure $\kappa(v)$, and $c(v) \cap c(v') = \emptyset$ for every pair of adjacent vertices $v$ and $v'$;*
- *$G$ is the line graph of a forest;*
- *every block of $G$ is a clique and each cut-vertex belongs exactly to two blocks;*
- *$G$ does not contain $K_4^-$, $K_{1,3}$ or a cycle of length more than* 3 *as an induced subgraph.*

The proof of the harder implication of Theorem 3 given by Hilton and Johnson [6] is six pages long, and it is quite involved. Our method allows us to present a shorter proof in Section 5.

# 2 Notation

In this section, we introduce notation used throughout the paper. All measure spaces considered in this paper are atomless. Carathéodory established that if $(X, \mathcal{M}, \mu)$ is such a measure space and $A \in \mathcal{M}$, then for every real $\beta \in (0, \mu(A))$ there exists a measurable set $B \subset A$ such that $\mu(B) = \beta$. The reader is refered to the book of Fremlin [5] for more details.

It turns out that in our considerations it is useful to work with demand and list size functions both in the measure and the ordinary setting. Let us first introduce the notation for multicolorings. Let $\kappa : V(G) \to \mathbb{R}^+$ and $\lambda : V(G) \to \mathbb{R}^+$ be two functions from the vertex set of a graph $G$ to non-negative real numbers. A $\lambda$-*list assignment* is a function $L$ that assigns to each vertex $v$ of $G$ a set $L(v)$ of at least $\lambda(v)$ colors. A mapping $c$ that assigns to each vertex a set $c(v) \subseteq L(v)$ is a $\kappa$-*coloring with respect to* $L$ if $|c(v)| \geq \kappa(v)$ for every vertex $v \in V(G)$, and $c(v) \cap c(v') = \emptyset$ for every pair of adjacent vertices $v$ and $v'$. If $G$ admits a $\kappa$-coloring with respect to every $\lambda$-list assignment, we say that $G$ is $(\kappa, \lambda)$-*choosable*.

We slightly abuse the notation and in case that $t$ is a real number, $t\lambda$ is the function assigning to a vertex $v$ the number $t\lambda(v)$, and $t\kappa$ is the function assigning to $v$ the number $t\kappa(v)$. A *$\delta$-approximate $\lambda$-list assignment* is a $(1-\delta)\lambda$-list assignment, and an *$\varepsilon$-close $\kappa$-coloring* is a $(1-\varepsilon)\kappa$-coloring.

The just introduced definitions readily translate to measurable colorings. A *measurable $\lambda$-list assignment* is a mapping $L : V(G) \to \mathcal{M}$ such that $\mu(L(v)) \geq \lambda(v)$. A mapping $c$ that assigns to each vertex a measurable set $c(v) \subseteq L(v)$ is a *measurable $\kappa$-coloring with respect to $L$* if $\mu(c(v)) \geq \kappa(v)$ for every $v \in V(G)$ and $c(v) \cap c(v') = \emptyset$ for every pair of adjacent vertices $v$ and $v'$. The definitions of measurably $(\kappa, \lambda)$-choosable graphs, $\delta$-approximate measurable $\lambda$-list assignment and $\varepsilon$-close measurable $\kappa$-coloring are analogous. We also slightly extend our notation and for $t \in \mathbb{R}^+$, we call $c$ a $t$-coloring if it is a $\kappa$-coloring where $\kappa \equiv t$. Similarly, we define a $t$-list assignment. Observe that using this extended notation, the measurable list chromatic number of $G$ is the smallest $t$ such that $G$ has a 1-coloring for every $t$-list assignment.

# 3 Limit theorem

In this section, we establish a sufficient condition for a finite graph to be measurably choosable. It is the core of our arguments presented in the next sections.

**Theorem 4.** *Let $G$ be a finite graph and let $\kappa$ and $\lambda$ be two mappings from $V(G)$ to $\mathbb{R}^+$. Suppose that for every real $\varepsilon > 0$ there exists a real $\delta > 0$ and a positive integer $t$ such that every $\delta$-approximate $(t'\lambda)$-list assignment admits an $\varepsilon$-close $(t'\kappa)$-coloring for every integer $t' \geq t$. Then, $G$ is measurably $(\kappa, \lambda)$-choosable.*

*Proof.* Fix an atomless measure space $(X, \mathcal{M}, \mu)$, and a measurable $\lambda$-list assignment $L_0 : V(G) \to \mathcal{M}$. Let $\overline{L_0} : 2^{V(G)} \to \mathcal{M}$ be the function that assigns to a subset $U$ of $V(G)$ the set of the elements of $X$ contained in all the sets $L_0(u)$ for $u \in U$, and in none of the sets $L_0(u)$ for $u \notin U$. Formally,

$$\overline{L_0}(U) = \left(\bigcap_{u \in U} L_0(u)\right) \setminus \left(\bigcup_{u \notin U} L_0(u)\right).$$

Finally, let $\overline{\lambda_0} : 2^{V(G)} \to \mathbb{R}^+$ be the function that assigns to a subset $U$ of $V(G)$ the measure of the set $\overline{L_0}(U)$. Note that for every vertex $u$,

$$\sum_{\substack{U \subseteq V(G) \\ u \in U}} \overline{\lambda_0}(U) = \mu(L_0(u)) \geq \lambda(u). \tag{3}$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive reals that converges to 0, e.g., $\varepsilon_i = \frac{1}{i+1}$. Further, let $\delta_i$ be the value of $\delta$ from the statement of the theorem for $\varepsilon = \varepsilon_i$ and $t_i$ the value of $t$. Finally, set $t'_i = \max\{t_i, 1/(\delta_i m_0)\}$ where $m_0$ is the smallest non-zero value of $\overline{\lambda_0}(U)$ for a subset $U$ of $V(G)$. For every positive integer $i$ and every non-empty subset $U$ of $V(G)$, we fix a set $A_{i,U}$ of $\lfloor \overline{\lambda_0}(U)t'_i \rfloor$ elements in such a way that the sets $A_{i,U}$ and $A_{i,U'}$ are disjoint whenever $U \neq U'$. Finally, for a vertex $u \in V(G)$ we set

$$L_i(u) = \bigcup_{\substack{U\subseteq V(G)\\ u\in U}} A_{i,U} \tag{4}$$

Since for every $i$, the sets $A_{i,U}$ are pairwise disjoint, we infer from (3) and (4) that

$$\begin{aligned}|L_i(u)| = \sum_{\substack{U\subseteq V(G)\\ u\in U}} |A_{i,U}| &\geq \sum_{\substack{U\subseteq V(G)\\ u\in U}} \left(\overline{\lambda_0}(U)t'_i - 1\right) \geq \sum_{\substack{U\subseteq V(G)\\ u\in U}} \left(\overline{\lambda_0}(U)t'_i - \delta_i t'_i m_0\right)\\ &\geq \sum_{\substack{U\subseteq V(G)\\ u\in U}} (1-\delta_i)\overline{\lambda_0}(U)t'_i \geq (1-\delta_i)t'_i \sum_{\substack{U\subseteq V(G)\\ u\in U}} \overline{\lambda_0}(U)\\ &= (1-\delta_i)t'_i\lambda(u),\end{aligned}$$

where the sums are taken only over the sets $U$ with $\overline{\lambda_0}(U) > 0$. Hence, the list assignment $L_i$ is a $\delta_i$-approximate $(t'_i\lambda)$-list assignment. By the assumption of the theorem, there exists an $\varepsilon_i$-close $(t'_i\kappa)$-coloring $c_i$ for this assignment, i.e., $c_i(u)$ is a subset of $L_i(u)$ with $|c_i(u)| \geq (1-\varepsilon_i)\kappa(u)t'_i$ for every vertex $u$ of $G$ and $c_i(u)\cap c_i(u') = \emptyset$ for every pair of adjacent vertices $u$ and $u'$.

Let us now define a function $\sigma_i : 2^{V(G)} \times 2^{V(G)} \to \mathbb{N}_0$ so that $\sigma_i(U, W)$ is the number of elements of the set $A_{i,U}$ assigned precisely to the elements of the set $W$, i.e.,

$$\sigma_i(U, W) = \left|\left(A_{i,U} \cap \bigcap_{w\in W} c_i(w)\right) \setminus \bigcup_{w\notin W} c_i(w)\right|.$$

Note that if $\sigma_i(U, W) > 0$, then $W$ is necessarily an independent set of vertices of $G$, which is a subset of $U$. Observe that for every $u \in V(G)$

$$|c_i(u)| = \sum_{\substack{U,W\subseteq V(G)\\ u\in W}} \sigma_i(U, W) \geq (1-\varepsilon_i)\kappa(u)t'_i, \tag{5}$$

and for every non-empty subset $U$ of $V(G)$,

$$\sum_{W \subseteq V(G)} \sigma_i(U, W) \leq |A_{i,U}| \leq \overline{\lambda_0}(U) t'_i. \tag{6}$$

Finally, let $\sigma'_i(U, W) = \sigma_i(U, W)/t'_i$.

Each of the functions $\sigma'_i(U, W)$ can be viewed as a $2^{2|V(G)|}$-dimensional vector of real numbers from the interval $[0, \overline{\lambda_0}(U)] \subseteq [0, \max_{v \in V(G)} \lambda(v)]$. Using this correspondence, the functions $\sigma'_i$ form a sequence of $2^{2|V(G)|}$-dimensional vectors contained in a bounded and closed subspace of $\mathbb{R}^{2|V(G)|}$. Thus, by the theorem of Heine-Borel, there exists a converging subsequence of $(\sigma'_i)_i$. Let $\sigma_0(U, W)$ be the function equal to such a limit vector.

As (5) and (6) hold for every $i$, we obtain for every vertex $u$

$$\sum_{\substack{U, W \subseteq V(G) \\ u \in W}} \sigma_0(U, W) \geq \kappa(u), \tag{7}$$

and for every non-empty subset $U$ of $V(G)$

$$\sum_{W \subseteq V(G)} \sigma_0(U, W) \leq \overline{\lambda_0}(U). \tag{8}$$

By (8) and the fact that the considered measure space is atomless, there exist disjoint subsets $K_{U,W}$ of $\overline{L_0}(U)$ such that $\mu(K_{U,W}) = \sigma_0(U, W)$. Observe that the subsets $K_{U,W}$ are mutually disjoint, as the sets $\overline{L_0}(U)$ are.

We are now ready to define the desired measurable coloring. For every vertex $u$ of $G$, we set

$$c_0(u) = \bigcup_{\substack{U, W \subseteq V(G) \\ u \in W}} K_{U,W}.$$

By (7), we obtain $\mu(c_0(u)) \geq \kappa(u)$ for every vertex $u$, as the sets $K_{U,W}$ are disjoint. It remains to argue that the obtained coloring is proper and colors each vertex with a subset of its list.

Let $u$ and $u'$ be two adjacent vertices. Then, as noted earlier, for all subsets $U$ and $W$ of $V(G)$ such that $\{u, u'\} \subseteq W$, it holds that $\sigma'_i(U, W) = 0$ for every index $i$. Hence, $\sigma_0(U, W) = 0$ and so $K_{U,W} = \emptyset$. Since the sets $K_{U,W}$ are mutually disjoint, we infer that $c(u) \cap c(u') = \emptyset$.

It remains to show that $c_0(u) \subseteq L_0(u)$ for every vertex $u$. Indeed, since $c_0(u)$ is a union of sets $K_{U,W}$ such that $u \in W \subseteq U$ and $K_{U,W} \subseteq \overline{L_0}(U)$, the inclusion $c_0(u) \subseteq L_0(u)$ follows by the definition of $\overline{L_0}(U)$. $\square$

A similar limit statement appears also in the proof of the Measurable CGL Theorem [6], where the problem is solved with the help of the Krein-Milman theorem—by considering extreme points of a certain nonempty compact convex set in $\mathcal{L}^\infty(X)$ which corresponds to measurable $\kappa$-colorings. We also note that our technique cannot be translated to infinite graphs. In particular, it does not seem to yield an elementary proof of Theorem 2.

# 4 Fractional colorings

Theorem 4 can be used to establish a relation between measurable choosability and fractional colorings. In order to apply Theorem 4, we have to show that every graph with fractional chromatic number $\chi_f$ is almost $t$-choosable for every $(\chi_f t)$-list assignment. As mentioned in the Introduction, a statement of this form was proved by Alon et al. [2]. Our arguments essentially follow the lines of their proof but we decided to include a full argument since we need a statement that slightly differs from theirs, and we also want the presented arguments to be complete.

**Theorem 5.** *If a finite graph $G$ has fractional chromatic number $\chi_f$, then $G$ is measurably $(1, \chi_f)$-choosable. In particular, $\chi_{m,\ell}(G) = \chi_f$.*

To prove this theorem, we shall use the Chernoff Bound [7]. The *binomial random variable* $\mathrm{BIN}(n, p)$ is the sum of $n$ independent zero-one random variables, each being 1 with probability $p$. A simple and well-known corollary [1] of the Chernoff Bound yields that for every $\delta \in [0, 1]$

$$\mathbf{Pr}\left(\mathrm{BIN}(n,p) < (1-\delta)np\right) < \exp\left(-\frac{np}{3}\delta^2\right).$$

*Proof of Theorem 5.* By Theorem 4, it is enough to show that for every real $\varepsilon > 0$ there exists a real $\delta > 0$ and a positive integer $t$ such that every $\delta$-approximate $(\chi_f t')$-list assignment admits an $\varepsilon$-close $t'$-coloring for every integer $t' \geq t$.

Since $G$ is finite, there exist two integers $p$ and $q$ such that $\chi_f = p/q$. Thus, there exists a $q$-coloring $c_f : V(G) \to \binom{\{1,\dots,p\}}{q}$. Let $\varepsilon \in (0, 1/3)$. We set $\delta = \varepsilon/2$ and

$$t = \left\lceil \frac{16q\ln(pn)}{\varepsilon^2} \right\rceil \tag{9}$$

where $n$ is the number of vertices of $G$.

Consider now an integer $t' \geq t$ and a $\delta$-approximate $(\chi_f t')$-list assignment $L$. Let $A$ be the union of the lists $L(v)$ taken over all the vertices $v \in V(G)$. We assert that there exists a partition of $A$ into sets $A_1, \ldots, A_p$ such that for every index $i$ and every vertex $v$, $|L(v) \cap A_i| \geq (1-\varepsilon)t'/q$. This assertion is established using a probabilistic argument.

Consider a random partition of $A$ into $p$ parts $A_1, \ldots, A_p$, where each element of $A$ is included to one of the sets $A_1, \ldots, A_p$ uniformly and independently at random. The random variable $|L(v) \cap A_i|$ is the binomial random variable $\mathrm{BIN}\left(|L(v)|, 1/p\right)$. Hence the Chernoff bound implies that

$$\mathbf{Pr}\left(|L(v) \cap A_i| < (1-\delta)\frac{|L(v)|}{p}\right) \leq \exp\left(-\frac{\delta^2 |L(v)|}{3p}\right).$$

Since $|L(v)| \geq (1-\delta)\chi_f t' = (1-\delta)t'p/q$ and $(1-\delta)^2 = (1-\varepsilon) > 2/3$, we infer from (9) that

$$\mathbf{Pr}\left(|L(v) \cap A_i| < (1-\varepsilon)\frac{t'}{q}\right) < \exp\left(-\frac{\varepsilon^2(2-\varepsilon)t'}{24q}\right) < \frac{1}{pn},$$

which proves the assertion, since there are $p$ choices for $i$ and $n$ choices for $v \in V(G)$. Let us fix such a partition $A_1, \ldots, A_p$ for the rest of the proof.

We are now ready to define an $\varepsilon$-close $t'$-coloring as follows. For every vertex $v \in V(G)$, let

$$c(v) = L(v) \cap \bigcup_{i \in c_f(v)} A_i \quad \text{for every vertex } v \in V(G).$$

The inclusion $c(v) \subseteq L(v)$ follows from the definition. Since the subsets $A_i$ are disjoint, $|c(v)| \geq q(1-\varepsilon)t'/q = (1-\varepsilon)t'$. Finally, assume that $v$ and $v'$ are adjacent vertices of $G$. Since the sets $A_i$ form a partition of the set $A$ and $c_f(v) \cap c_f(v') = \emptyset$, we deduce that $\bigcup_{i \in c_f(v)} A_i$ and $\bigcup_{i \in c_f(v')} A_i$ are disjoint. Consequently, $c(v)$ and $c(v')$ are also disjoint. We conclude that $c$ is an $\varepsilon$-close $t'$-coloring of $G$ for the list assignment $L$. $\square$

# 5 Proof of the Measurable CGL Theorem

The technique used in the proof of Theorem 4 allows us to give a shorter proof of Theorem 3.

*Proof of Theorem 3.* If $G$ contains $K_4^-$, $K_{1,3}$ or a cycle of length more than three as an induced subgraph, then a measurable list assignment $L$ and a mapping $\kappa : V(G) \to \mathbb{R}^+$ that satisfy the generalized Hall condition and such that $G$ has no $\kappa$-coloring for $L$ can be constructed (see [6] for details). In the rest, we focus on proving that if $G$ contains no $K_4^-$, no $K_{1,3}$ and no cycle of length more than three as induced subgraphs, then $G$ is measurably $\kappa$-choosable for a list assignment $L$ whenever $\kappa$ and $L$ satisfy the generalized Hall condition.

Let us fix an atomless measure space $(X, \mathcal{M}, \mu)$, a mapping $\kappa : V(G) \to \mathbb{R}^+$ and a list assignment $L : V(G) \to \mathcal{M}$ that satisfy the generalized Hall condition. For every $\varepsilon > 0$, we show that there exists an $\varepsilon$-close $\kappa$-coloring $c : V(G) \to \mathcal{M}$ with $c(v) \subseteq L(v)$. Once this statement is established, we can define $\overline{L}$ and eventually construct $\sigma_0$ as in the proof of Theorem 4 corresponding to a $\kappa$-coloring of $G$. Since all the arguments are analogous to those presented in the proof of Theorem 4, we omit further details on this final step.

It remains to establish the existence of an $\varepsilon$-close $\kappa$-coloring $c_\varepsilon$ for every $\varepsilon > 0$. Our argument involves the function $\overline{L} : 2^{V(G)} \to \mathcal{M}$ as defined in the proof of Theorem 4, i.e., for every $U \subseteq V(G)$, $\overline{L}(U)$ is the set of those points that are contained exactly in $\bigcup\limits_{u \in U} L(u)$. Let

$$x_0 = \frac{\varepsilon}{2} \cdot \min \left\{ \min_{\substack{v \in V(G) \\ \kappa(v) > 0}} \kappa(v), \min_{\substack{U \subseteq V(G) \\ \mu(\overline{L}(U)) > 0}} \mu(\overline{L}(U)) \right\}. \tag{10}$$

The value of $x_0$ represents the measure corresponding to one color in the list assignment defined in the sequel. For $U \subseteq V(G)$, choose a set $A_U$ of $\lfloor \mu(\overline{L}(U))/x_0 \rfloor$ colors so that the sets $A_U$ are pairwise disjoint. By the choice of $x_0$, it holds that

$$|A_U| \geq \frac{\mu(\overline{L}(U))}{x_0} - 1 \geq \left(1 - \frac{\varepsilon}{2}\right) \frac{\mu(\overline{L}(U))}{x_0}. \tag{11}$$

We define the list assignment $L'$ by setting

$$L'(v) = \bigcup_{\substack{U \subseteq V(G) \\ v \in U}} A_U$$

for every $v \in V(G)$. Note that $|L'(v)| \geq (1-\varepsilon/2)\mu(L(v))/x_0$ by (11). Finally, define $\kappa'(v) = \lfloor (1-\varepsilon/2)\kappa(v)/x_0 \rfloor$ and observe that $\kappa'(v) \geq (1-\varepsilon)\kappa(v)$ by the choice of $x_0$.

We verify that the function $\kappa'$ and the list assignment $L'$ satisfy the generalized Hall condition. Consider a subgraph $H$ of $G$. Since $L$ and $\kappa$ satisfy the generalized Hall condition, it holds that

$$\sum_{v \in V(H)} \kappa(v) \leq \int_x \alpha(x, L, H) \,\mathrm{d}\,\mu(x) = \sum_{U \subseteq V(G)} \alpha(H[U \cap V(H)]) \cdot \mu(\overline{L}(U)). \tag{12}$$

We infer from (11) that

$$\sum_{v \in V(H)} \kappa(v) \leq \frac{x_0}{1-\varepsilon/2} \sum_{U \subseteq V(G)} \alpha(H[U \cap V(H)]) \cdot |A_U|, \tag{13}$$

and consequently that

$$\begin{aligned}\sum_{v \in V(H)} \kappa'(v) = \sum_{v \in V(H)} \left\lfloor \frac{1-\varepsilon/2}{x_0} \kappa(v) \right\rfloor \leq \sum_{U \subseteq V(G)} \alpha(H[U \cap V(H)]) \cdot |A_U| \\ = \sum_{x \in \cup_{v \in V(H)} L'(v)} \alpha(x, L', H).\end{aligned}$$

Hence, $\kappa'$ and $L'$ satisfy the generalized Hall condition. By Theorem 1, there exists a $\kappa'$-coloring $c'$ for the list assignment $L'$. Based on $c'$, for every $U, W \subseteq V(H)$, we define $\sigma(U,W)$ to be the number of elements of $A_U$ assigned precisely to the vertices contained in $W$. Moreover, we let $\sigma'(U,W) = \sigma(U,W)x_0$.

The $\varepsilon$-close $\kappa$-coloring $c_\varepsilon$ is defined based on $\sigma'(U,W)$ analogously to the construction presented in the proof of Theorem 4. More precisely, for every subsets $U$ and $W$ of $V(G)$, we let $K_{U,W}$ be a subset of $\overline{L}(U)$ of measure $\sigma'(U,W)$, such that the sets $K_{U,W}$ are pairwise disjoint. Now, for each vertex $v \in V(G)$, we define $c_\varepsilon(v)$ to be the union of the sets $K_{U,W}$ for all subsets $U$ and $W$ of $V(G)$ such that $v \in W$. Note that $c_\varepsilon$ is $\varepsilon$-close since $\kappa'(v)x_0 \geq (1-\varepsilon)\kappa(v)$ for every vertex $v$ of $G$.

The construction of the coloring $c_\varepsilon$ for every $\varepsilon > 0$ allows to finish the proof of the theorem, by defining $\overline{L}$ and $\sigma_0(U,W)$ in an analogous manner as in the proof of Theorem 4. □

# 6 Conclusion

In this section, we briefly discuss a possible extension of our main result (Theorem 5) to infinite graphs. First, we show that there exists a locally finite bipartite graph $G$ that is not measurably 2-choosable.

**Proposition 6.** *There exists a locally finite bipartite graph $G$ with a countable number of vertices that is not measurably 2-choosable.*

*Proof.* Let us define the graph $G$ as follows.

$$V(G) = \bigcup_{i\in\mathbb{N}} \left\{ w_i^j \;:\; j = 1, \ldots, \binom{2^i}{2^{i-1}} \right\} \cup \{a_i \;:\; i \in \mathbb{N}\},$$
$$E(G) = \left\{ w_i^j w_{i'}^{j'} \;:\; |i - i'| = 1 \right\} \cup \{a_i w_i^j\} \cup \{a_i a_{i+1}\}.$$

The graph $G$ is obviously a locally finite bipartite graph with a countable number of vertices. The measure space which we consider is the interval $[0, 3] \subseteq \mathbb{R}$ enhanced with the Lebesgue measure $\lambda$.

Set $L(a_i) = [0, 2]$ for all $i \in \mathbb{N}$. For $i \in \mathbb{N}$, the vertices $\{w_i^j \mid j = 1, \ldots, \binom{2^i}{2^{i-1}}\}$ are assigned all sets of the form $S \cup [2, 3]$ where $S$ is a subset of the interval $[0, 2]$, $\lambda(S) = 1$ and $S$ can be written as a union of a finite number of closed intervals of the type $[x2^{-i+1}, y2^{-i+1}]$ where $x, y \in \mathbb{N}_0$ (there are $\binom{2^i}{2^{i-1}}$ choices for $S$).

Suppose now that there is a proper 1-coloring $c$ of $G$ with $c(v) \subseteq L(v)$ for every vertex $v$ of $G$. By altering the sets $c(a_i)$ by sets of measure zero if needed, we may assume that $c(a_{2i-1}) = c(a_{2j-1})$ and $c(a_{2i}) = c(a_{2j})$ for all positive integers $i$ and $j$. Furthermore, there exist closed sets $S_1$ and $S_2$ that are unions of closed intervals of the type $[x2^{-\ell+1}, y2^{-\ell+1}]$ for some number $\ell \in \mathbb{N}$, and $\lambda(S_1 \cap c(a_1)) \geq 3/4$ and $\lambda(S_2 \cap c(a_2)) \geq 3/4$.

We can assume without loss of generality that $\ell$ is odd and consider vertices $w_\ell^i$ and $w_{\ell+1}^j$ such that $S_1 \subseteq L(w_\ell^i)$ and $S_2 \subseteq L(w_{\ell+1}^j)$. Since $\lambda(c(a_\ell) \cap S_1) \geq 3/4$, it follows that $\lambda(c(w_\ell^i) \cap [2, 3]) \geq 3/4$. Similarly, $\lambda(c(w_{\ell+1}^j) \cap [2, 3]) \geq 3/4$. We conclude that the sets $c(w_\ell^i)$ and $c(w_{\ell+1}^j)$ are not disjoint which contradicts that $c$ is proper. $\square$

We have seen that if $G$ is an infinite graph, then $G$ need not be measurably $\chi_f(G)$-choosable. We conjecture that if $G$ is locally finite, then $G$ is measurably $(\chi_f(G) + \varepsilon)$-choosable for every $\varepsilon > 0$, and thus $\chi_f(G) = \chi_{m,l}(G)$.

**Conjecture 1.** *If $G$ is a locally finite graph with a countable number of vertices, then $\chi_f(G) = \chi_{m,l}(G)$.*

The assumption that $G$ is locally finite is necessary—it is easy to observe that $K_{\omega,\omega}$, the complete bipartite graph with two countable parts, is not measurably $k$-choosable for any $k \geq 2$.

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