A NEW COMBINATORIAL APPROACH TO THE CONSTRAINT SATISFACTION PROBLEM DICHOTOMY CLASSIFICATION

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Abstract. We introduce a new general polynomial-time construction- the *fibre construction*- which reduces any constraint satisfaction problem CSP(H) to the constraint satisfaction problem CSP(P), where P is any subprojective relational structure. As a consequence we get a new proof (not using universal algebra) that CSP(P) is NP-complete for any subprojective (and so for any projective) relational structure. The fibre construction allows us to prove the NP-completeness part of the conjectured Dichotomy Classification of CSPs, previously obtained by algebraic methods. We show that this conjectured Dichotomy Classification is equivalent to the dichotomy of whether or not the template is subprojective. This approach is flexible enough to yield NP-completeness of coloring problems with large girth and bounded degree restrictions thus reducing the Feder-Hell-Huang and Kostočka-Nešetřil-Smolíková problems to the Dichotomy Classification of coloring problems.

1. Introduction

Many combinatorial problems can be expressed as Constraint Satisfaction Problems (CSPs). This concept originated in the context of Artificial Intelligence (see e.g. [23]) and is very active in several areas of Computer Science. CSPs includes standard satisfiability problems and many combinatorial optimization problems, thus are also a very interesting class of problems from the theoretical point of view. The whole area was revitalized by Feder and Vardi [9], who reformulated CSPs as homomorphism problems (or H-coloring problems) for relational structures. Motivated by the results of [29] and [13], they formulated the following.

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Conjecture 1.1. (Dichotomy) *Every Constraint Satisfaction Problem is either P or NP-complete.*

Schaefer [29] established the dichotomy for CSPs with binary domains, and Hell-Nešetřil [13] established the dichotomy for undirected graphs; it follows from [9] that the dichotomy for CSPs can be reduced to the dichotomy problem for H-coloring for oriented graphs. This setting, and related problems, have motivated intensive research in descriptive complexity theory. This is surveyed, for example, in [6], [14] and [11].

Recently the whole area was put into yet another context by Peter Jeavons and his collaborators, in [15] and [4], when they recast the complexity of CSPs as properties of algebras and polymorphisms of relational structures. In particular, they related the complexity of CSPs to a Galois correspondence between polymorphisms and definable relations (obtained by Bodnarčuk et al. [1] and by Gaiger [10]; see [27] and [28]). This greatly simplified elaborate and tedious reductions of particular problems and led to the solution of the dichotomy problem for ternary CSPs [2] and other results which are surveyed, for example, in [4] and [12]. This approach to studying CSPs via certain algebraic objects yields, in particular, that for every *projective* structure H the corresponding CSP(H) is an NP-complete problem [16], [15]. It also led to Conjecture 5.5, from [4], which strengthens Dichotomy Conjecture 1.1 by actually conjecturing what the dichotomy is.

The success of these general algebraic methods gave motivation for some older results to be restated in this new context. For example, [3] treats H-coloring problems for undirected graphs in such a way that the dichotomy between the tractable and NP-complete cases of H-coloring problem agrees with Conjecture 5.5.

In this paper we propose a new approach to the dichotomy problem. We define a general construction- the *fibre construction*- which allows us to prove in a simple way that for every projective structure H, CSP(H) is NP-complete. In fact we define a *subprojective* structure and prove that for every subprojective relational structure H, CSP(H) is NP-complete. We then show (Theorem 2.2) that this dichotomy (to be or not to be a subprojective structure; Conjecture 2.4) coincides with the dichotomy for CSPs that is conjectured in [4], thus reproving in a combinatorial way the main result yielded by algebraic methods. This is stated as Corollaries 2.3 and 5.9.

The fibre construction lends easily to restricted versions of CSPs, so allows us to address open problems from [8] and [17]. In particular, we reduce the Feder-Hell-Huang conjecture that NP-complete CSPs are NP-complete for instances of bounded degree, to the Dichotomy Classification Conjecture 2.4.

Our approach is motivated by the *Sparse Incomparability Lemma* [25] and *Müller's Extension Theorem* [24] (both these results are covered in [14]). Strictly speaking, we do not need these results for the dichotomy results– Theorem 2.2 and Corollary 2.3– but they provided an inspiration for early forms of the fibre construction in [30] and [31] and for the general case presented here. However, we do need a strengthening of these results to address the question of [17]. This is stated in Section 7.

The fibre construction is simple, and is a refinement of gadgets, or indicator constructions [13, 14], via familiar extremal combinatorial results [24, 25, 26]). However, the simplicity becomes obscured by the notation when dealing with general relational structures. Thus we find it useful to outline the fibre construction in Section 3 by presenting a simple case of it, which, nonetheless, contains all the essential ingredients of the general fibre construction.

In Section 2 we introduce all the definitions and state the main results. In Section 4 we prove Theorem 2.2. In Section 5 we relate this to the Dichotomy Classification Conjecture. In Section 6.1 we consider colorings of bounded degree graphs and of graphs with large girth. Section 7 contains some background material which was both motivation and prerequisite to our paper. It also contains an extension of some of these motivativing results.

2. Definitions and statement of results

We work with finite relational structures of a given type (or signature). A *type* is a vector $K = (k_i)_{i \in I}$ of positive integers, called *arities*. A *relational structure* $\mathcal{H}$ of type K, consists of a finite vertex set $V = V(\mathcal{H})$, and a k_i-ary relation $R_i = R_i(\mathcal{H}) \subset V^{k_i}$ on V, for each $i \in I$. An element of R_i is called an k_i-tuple. Thus a (di)graph is just a relational structure of type $K = (2)$. Its edges (arcs) are 2-tuples in the 2-ary relation R.

Throughout the paper, we will use script letters, such as $\mathcal{G}, \mathcal{H}$ and $\mathcal{P}$, to represent relational structures except in the case that we are talking specifically of graphs.

Given two relational structures $\mathcal{G}$ and $\mathcal{H}$ of the same type, an $\mathcal{H}$-*coloring of* $\mathcal{G}$ is a map $\phi : V(\mathcal{G}) \to V(\mathcal{H})$ such that for all $i \in I$ and every k_i-tuple $(v_1, \ldots, v_{k_i}) \in R_i(\mathcal{G})$, $(\phi(v_1), \ldots, \phi(v_{k_i}))$ is in $R_i(\mathcal{H})$. For a fixed relational structure $\mathcal{H}$ (sometimes called *template*), CSP($\mathcal{H}$) is the following problem decision problem:

Problem CSP($\mathcal{H}$)
Instance: A relational structure $\mathcal{G}$;
Question: Does there exists an $\mathcal{H}$-coloring of $\mathcal{G}$?

We write $\mathcal{G} \to \mathcal{H}$ to mean that $\mathcal{G}$ has an $\mathcal{H}$-coloring.

A relational structure $\mathcal{H}$ is a *core* if its only $\mathcal{H}$-colorings are automorphisms. It is well known, (see, for example, [14]) that $\mathcal{G} \to \mathcal{H}$ if and only if $\mathcal{G}' \to \mathcal{H}'$, where $\mathcal{G}'$ and $\mathcal{H}'$ are the cores of $\mathcal{G}$ and $\mathcal{H}$ respectively. Therefore, in the sequel, we only consider relational structures that are cores.

All relational structures of a given type form a category with nice properties. In particular, this category has products and powers which are defined explicitly as follows:

Given a relational structure $\mathcal{H}$, and a positive integer d, the d-ary *power* $\mathcal{H}^d$ of $\mathcal{H}$ is the relational structure of the same type as $\mathcal{H}$, defined as follows.

- $V(\mathcal{H}^d) = \{(v_1, \ldots, v_d) \mid v_1, \ldots, v_d \in V(\mathcal{H})\}$.
- For $i \in I$, $((v_{1,1}, v_{1,2}, \ldots, v_{1,d}), \ldots, (v_{k_i,1}, \ldots, v_{k_i,d}))$ is in $R_i(\mathcal{H}^d)$ if and only if all of $(v_{1,1}, v_{2,1}, \ldots, v_{k_i,1}), \ldots, (v_{1,d}, \ldots, v_{k_i,d})$ are in $R_i(\mathcal{H})$.

An $\mathcal{H}$-coloring of $\mathcal{H}^d$ (i.e. a homomorphism $\mathcal{H}^d \to \mathcal{H}$) is called a *$d$-ary polymorphism* of $\mathcal{H}$. A d-ary polymorphism ϕ is called a *projection* if there exists some $i \in 1, \ldots, d$ such that $\phi((v_1, \ldots, v_d)) = v_i$ for any $v_1, \ldots, v_d \in V(\mathcal{H})$. Let Pol($\mathcal{H}$), Aut($\mathcal{H}$) and Proj($\mathcal{H}$) be the sets of polymorphisms, automorphisms and projections (of all arities) of $\mathcal{H}$. A relational structure $\mathcal{H}$ is *projective* if for every $\phi \in \mathrm{Pol}(\mathcal{H})$, $\phi = \sigma \circ \pi$ for some $\sigma \in \mathrm{Aut}(\mathcal{H})$ and some $\pi \in \mathrm{Proj}(\mathcal{H})$. It is shown in [21], that almost all relational structures are projective.

The following definition of graphs that are, in a sense, locally projective, is our principal definition.

Definition 2.1. A subset S of $V(\mathcal{H})$ is called *projective* if for every $\phi \in \mathrm{Pol}(\mathcal{H})$, ϕ restricts on S^d, where d is the arity of ϕ, to the same function as does $\sigma \circ \pi$ for some $\sigma \in \mathrm{Aut}(\mathcal{H})$ and some $\pi \in \mathrm{Proj}(\mathcal{H})$. S is called *non-trivial* if $|S| > 1$. A relational structure $\mathcal{H}$ is called *subprojective* if it is a core and it contains a non-trivial projective subset.

Note that any subset of a projective set is again projective. A structure is projective if and only if the set of all its vertices is projective. It is easy to find subprojective structures which fail to be projective.

The main tool of the paper is the following general indicator construction which we call the *fibre construction*. This construction extends a construction first used in a Ramsey theory setting in [30], and then proved in [31] in the present form, for $\mathcal{H} = K_3$ and $\mathcal{P}$ being projective. The construction is outlined in Section 3 and is described in full in Section 4.

Theorem 2.2. *Let $\mathcal{H}$ be any relational structure, and let $\mathcal{P}$ be any subprojective relational structure. Then there exists a polynomial time construction, the fibre construction $\mathcal{M}_{\mathcal{H}}^{\mathcal{P}}$, which provides for any instance $\mathcal{G}$ of* CSP($\mathcal{H}$)*, an instance $\mathcal{M}_{\mathcal{H}}^{\mathcal{P}}(\mathcal{G})$ of $\mathcal{P}$ such that*

$$\mathcal{G} \to \mathcal{H} \iff \mathcal{M}_{\mathcal{H}}^{\mathcal{P}}(\mathcal{G}) \to \mathcal{P}.$$

Note that $\mathcal{H}$ and $\mathcal{P}$ need not be of the same type. Since CSP(K_3) is NP-complete, taking $\mathcal{H}$ to be K_3 gives the following.

Corollary 2.3. *For any subprojective relational structure $\mathcal{P}$, the problem* CSP($\mathcal{P}$) *is NP-complete.*

In [15] and [16], algebraic techniques were used to show that CSP($\mathcal{P}$) is NP-complete for any projective relational structure $\mathcal{P}$. In [4] and [5], the techniques of [15] were extended to show that CSP($\mathcal{H}$) is NP-complete for a certain class of relational structures whose description requires some algebraic definitions that we present in Section 5. It is conjectured in [4] (see Conjecture 5.5) that CSP($\mathcal{H}$) is polynomial time solvable for any other relational structure $\mathcal{H}$.

In Section 5 we show that any relational structure $\mathcal{H}$ for which CSP($\mathcal{H}$) is shown to be NP-complete in [4] is subprojective (Proposition 5.4). Thus we give a simple combinatorial proof of the results of [4], as they relate to CSPs. Further, we use a reformulation of Conjecture 5.5 by Larose and Zádori [19] to show that it is equivalent to the following.

Conjecture 2.4. (Dichotomy Classification Conjecture) *The problem* CSP($\mathcal{H}$) *is NP-complete if and only if $\mathcal{H}$ is subprojective.*

The fibre construction also has immediate applications to restricted versions of CSP complexity.

The *degree* of a vertex v in a relational structure $\mathcal{G}$ is the number of tuples it occurs in in $\bigcup R_i$, and the maximum degree, over all vertices in $\mathcal{H}$, is denoted by $\Delta(\mathcal{G})$. $\mathcal{G}$ is called *b-bounded* if $\Delta(\mathcal{G}) \leq b$.

It is conjectured in [8] that for any relational structure $\mathcal{H}$, if CSP($\mathcal{H}$) is NP-complete, then there is some finite b such that CSP($\mathcal{H}$) is NP-complete when restricted to b-bounded instances.

In [31], this was shown to be true in the case of graphs and projective relational structures $\mathcal{H}$. Furthermore, explicit bounds were given on $b(\mathcal{H})$, which is the minimum b such that CSP($\mathcal{H}$) is NP-complete when restriced to b-bounded instances.

In Section 6.1, we observe the following corollary of the proof of Theorem 2.2. The same was shown for projective relational structures in [31].

Corollary 2.5. *For any subprojective relational structure $\mathcal{P}$,*

$$b(\mathcal{P}) < (4 \cdot \Delta(\mathcal{P})^6).$$

This greatly improves the bound on $b(H)$ from [31] in the case of graphs H. More importantly, this shows that if the Dichotomy Classification Conjecture 2.4 (and, equivalently, Conjecture 5.5 of [4]) is true, then so is the conjecture from [8].

Degrees and short cycles are classical restrictions for coloring problems. Recall that the *girth* $g(G)$ of a graph G is the length of the shortest cycle in G. In Section 6.1 we use an extenstion (from Section 7) of the *Sparse Incomparability Lemma* [25] to prove the following about sparse graphs.

Theorem 2.6. *Let H be a subprojective graph, and ℓ a positive integer. Then the problem* CSP(H) *is NP-complete when restricted to graphs with girth $\geq \ell$.*

This solves a problem of [17] where the question of CSPs restricted to instances with large girth was studied. This result can be generalized further to relational structures but we decided to stop here.

3. Outline of the Fibre Construction - Undirected Graph Case

Indicator constructions are various graph constructions that are often used to reduce H-coloring for some structure H to H'-coloring, for some other graph H'. Such constructions were used to great effect in, for example, [13].

One of the difficulties with indicator constructions is that one uses many ad hoc tricks to find a constrution for particular graphs H' or H.

The fibre construction, provided by Theorem 2.2, is an indicator construction that will suffice for all reductions.

We outline the proof Theorem 2.2 by describing the fibre construction for the simple case where $\mathcal{H}$ is C_5 and $\mathcal{P}$ is K_3. Thus we give a polynomial time construction that gives, for any instance G of $\mathrm{CSP}(C_5)$, an instance $M(G)$ of $\mathrm{CSP}(K_3)$ such that

$$G \to C_5 \iff M(G) \to K_3.$$

Ours is not the most elegant known reduction of C_5-coloring to K_3-coloring, but it has one advantage: it can be generalized to cover all $\mathrm{CSP}(\mathcal{H})$.

3.1. **Notation.** Throughout the paper, we will often define indexed sets of vertices such as $W^* = [w_1^*, \dots, w_d^*]$. A copy W^a of the set W^* will mean the indexed set $W^a = [w_1^a, \dots, w_d^a]$. Given two copies W^a and W^b of the same set W^* we say that we *identify W^a and W^b index-wise* to mean we identify the vertices w_i^a and w_i^b for $i = 1, \dots, d$. When we define a function f on W^*, we will assume it to be defined on any copy W^a of W^* by $f(w_\alpha^a) = f(w_\alpha^*)$ for all $\alpha = 1, \dots, d$. We will often refer to a function f on an indexed set W^* as a *pattern* of W^*. In the case that the image of f is contained in the vertex set of some graph H we speak about *H-pattern* of W^*.

3.2. **The Fibre Gadget - a special case.** Our construction consists of two parts. In the first part we build a fibre gadget M which depends only on C_5 and K_3. The important features of this gadget is that it will contain two copies W^a and W^b of an indexed set W^*, and that the following are true for a chosen set $F = \{f_x \mid x \in V(C_5)\}$ of distinct K_3-patterns of W^*.

(i) Any K_3-coloring of M, restricted to W^a, (or to W^b) is a pattern in F.

(ii) For any K_3-coloring ϕ of M, ϕ restricts on W^a to f_x and on W^b to f_y for some edge xy of C_5.

(iii) For any edge xy (or yx) of C_5, there is a K_3-coloring ϕ of M that restricts on W^a to f_x and on W^b to f_y.

We do not construct M here as it is a special case of the fibre gadget provided by Lemma 4.1.

The name 'fibre gadget' comes from the relation of the vertices of W^* to the set of K_3-patterns F. We view $w \in W^*$ as a fibre in $V(K_3)^{|F|}$, whose i^{th} postition corresponds to its image under the i^{th} pattern f_{x_i} in F.

3.3. **The Fibre Construction - a special case.** In the second part of the construction, we take an instance G of $\mathrm{CSP}(C_5)$ and construct $M(G)$, from $|E(G)|$ copies of M, as follows. (See Figure 1.)

(i) For each vertex v of G let W^v be a copy of W^*.

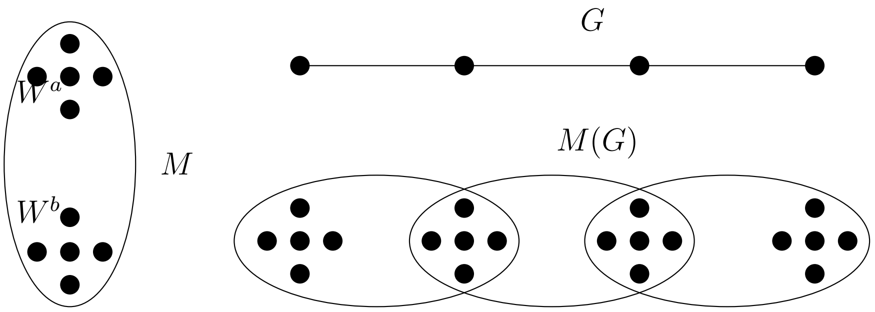

FIGURE 1. Fibre Construction

(ii) For each edge uv of G let M^{uv} be a copy of the fibre gadget M. Index-wise, identify W^u with the copy of W^a in M^{uv} and W^v with the copy of W^b.

Given a K_3-coloring ϕ of $M(G)$, ϕ will restrict on W^v for each vertex v of G to a pattern in F. Thus $\phi' : V(G) \to V(C_5)$ is well defined by $\phi'(v) = x$ where ϕ restricts on W^v to the pattern f_x. Moreover, by property (ii) of the fibre gadget M, ϕ' is a C_5-coloring of G.

On the other hand, given a C_5-coloring ϕ' of G we define a K_3-coloring ϕ of $M(G)$ as follows. For all vertices v of G, let ϕ be $f_{\phi'(v)}$ on the set W^v. For every edge uv of G, the sets W^u and W^v are already colored by ϕ, and we must extend this coloring to M^{uv}. Now ϕ restricts on W^u to $f_{\phi'(u)}$ and on W^v to $f_{\phi'(v)}$, and $\phi'(u)\phi'(v)$ is an edge of C_5, so by property (iii) of M, ϕ can be extended to M^{uv}. Thus ϕ can be extended to a K_3-coloring of $M(G)$.

3.4. **Remark.** This outline gives only the idea of the general proof. There are several obstacles. For example, in the general case of relational structures, we will need a different fibre gadget for each relation. And, of course, our relations need not be symmetric. In the general case, the set F in the Fibre Gadget will be S-patterns, instead of K_3-patterns, and we will generally only be able to define them up to a permutation of S. To make sure that the permutation of S is constant over all copies of M (or at least all copies of M in a component of $M(G)$) we will ensure that for any pattern f_i in F, $\sigma \circ f_i$ is not in F for any non-identity permutation σ of S.

These are just technicalities which can be handled with care.

4. Proof of Theorem 2.2

We follow the same strategy outlined in Section 3. In particular, we use patterns as introduced in 3.1.

4.1. **The Fibre Gadget.**

The following lemma generalises the fibre gadget that was introduced in 3.2.

Lemma 4.1. *Let $\mathcal{H}$ be any relational structure, and $\mathcal{P}$ be any subprojective relational structure. Let S be a non-trivial projective subset of $\mathcal{P}$. Then there exists an indexed set W^* of independent vertices, and a set $F = \{f_x \mid x \in V(\mathcal{H})\}$ of S-patterns of W^* such that for every relation R of $\mathcal{H}$ there is an instance $\mathcal{M}_R$ of $\mathcal{P}$ satisfying the following conditions, where d is the arity of R.*

- (i) *$V(\mathcal{M}_R)$ contains copies $W^1, \ldots, W^d$ of W^*.*
- (ii) *For every f_x in F, $\sigma \circ f_x$ is not in F for any non-identity $\sigma \in \mathrm{Aut}(\mathcal{P})$.*
- (iii) *For any $\mathcal{P}$-coloring ϕ of $\mathcal{M}_R$, there is some $\sigma \in \mathrm{Aut}(\mathcal{P})$ and some $(x_1, \ldots, x_d) \in R$, such that for $i = 1, \ldots, d$, ϕ restricts to $\sigma \circ f_{x_i}$ on W^i.*
- (iv) *For any d-tuple $(x_1, \ldots, x_d) \in R$ there is a $\mathcal{P}$-coloring ϕ of $\mathcal{M}_R$ that restricts on W^i to f_{x_i} for $i = 1, \ldots, d$.*

The proof depends on the following simple lemma which is motivated by a result of Müller, which will be stated as Theorem 7.1.

Lemma 4.2. *Let $\mathcal{P}$ be a subprojective relational structure, and S be a non-trivial projective subset of $\mathcal{P}$. Let W be an indexed set, and let $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ be a set of S patterns of W. Then there exists a relational structure $\mathcal{M}$, with $W \subset V(\mathcal{M})$, such that the set of $\mathcal{P}$-colorings of $\mathcal{M}$, when restricted to W, is exactly*

$$\{\alpha \circ \gamma \mid \alpha \in \mathrm{Aut}(\mathcal{P}), \gamma \in \Gamma\}.$$

Moreover, assume that the set Γ satisfies the following condition (*).

> *For any pair $w \neq w' \in W$, there exists some $\gamma \in \Gamma$ for which $\gamma(w) \neq \gamma(w')$.*

Then we can put $\mathcal{M} \cong \mathcal{P}^d$.

Proof of Lemma 4.2. First assume that Γ satisfies (*). Put $\mathcal{M} = \mathcal{P}^d$ and for each $w \in W$, identify w with the vertex $(\gamma_1(w), \ldots, \gamma_d(w))$ of $\mathcal{M}$. By condition (*), these are distinct elements of $V(\mathcal{M})$.

Since S is a projective subset of $\mathcal{P}$, the only $\mathcal{P}$-colorings of $\mathcal{M} = \mathcal{P}^d$ restrict on S^d, which contains W, to $\alpha \circ \pi$ where α is an automorphism of $\mathcal{P}$ and π is a projection. But the projections restrict on W to exactly the maps of Γ, so the lemma follows.

Now consider the case that Γ does not satisfy (*). Let W' be the maximum subset of W for which (*) holds on the restriction of Γ to W'. Apply this Lemma to W' and to the restrictions of patterns in Γ to W'. We get the structure $\mathcal{M}'$, $\mathcal{M}' \cong \mathcal{P}^d$, containing the vertices W' with all above properties. For every vertex $w \in W \setminus W'$ that has the same images under all $\gamma \in \Gamma$ as some w' in W', add a new vertex w to $\mathcal{M}'$ with all the same neighbours as w'. Call this new relational structure $\mathcal{M}$.

To complete the proof of the lemma, we show:

Claim 4.3. *$\phi(w) = \phi(w')$ for every $w \in W \setminus W'$, and every $\mathcal{P}$-coloring ϕ of $\mathcal{M}$. Thus ϕ restricts on W to some $\alpha \circ \gamma$.*

This claim is obvious for $d = 1$ as in this case it suffices to use the fact that $\mathcal{P}$ is a core. So assume $d > 1$. The function $\psi : V(\mathcal{M}) \to V(\mathcal{M})$ that exchanges w and w', and fixes all other vertices, is clearly an automorphism. If $\phi(w) \neq \phi(w')$, then $\phi \circ \psi$ is $\mathcal{P}$-coloring of $\mathcal{M}$ that when restricted to S^d differs from ϕ only on w'. We show that this is impossible by showing that for a $\mathcal{P}$ coloring ϕ of $\mathcal{M}$, $\phi(w')$ is uniquely determined by the definition of ϕ on the rest of S^d.

Let $w' = (s_1, \dots, s_d)$, where $s_1, \dots, s_d \in S$, and assume that not all s_i are the same. Let T be the subset of S^d consisting of w' and $(s_1, \dots, s_1)$ and all d-tuples that can be created from these two d-tuples by permuting the set $\{s_1, \dots, s_d\}$. Let $d' = |\{s_1, \dots, s_d\}|$. Since $\phi = \alpha \circ \pi$, ϕ takes $(d'-1)!$ elements of T to each of d' images. Thus the image of w' is determined by counting the images of the other elements of T.

If $w' = (s, \dots, s)$ for some $s \in S$, then the same argument works using any other non-constant d-tuple that has s in some slot.

This finishes the proof of the claim and of Lemma 4.2.

□

Proof of Lemma 4.1. Let $\mathcal{H}$ be a relational structure, and $\mathcal{P}$ be a subprojective relational structure with the non-trivial projective subset S. As S is non-trivial, it contains a two element subset $S' = \{0, 1\}$. Let $W^* = [w^*_x \mid x \in V(\mathcal{H})]$ be an indexed set of vertices. Let $F = \{f_x \mid x \in V(\mathcal{H})\}$ where

f_x is the $\{0,1\}$-pattern of W^* defined by

$$f_x(w_y^*) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Now let R be any d-ary relation of $\mathcal{H}$. Let $W = \bigcup_{i=1}^d W^i$, where W^i is a copy of W^*, and let $\Gamma = \{\gamma_r \mid r = (x_1, \ldots, x_d) \in R\}$ where γ_r is the $\{0,1\}$-pattern of W defined by

For $i = 1, \ldots, d$, γ_r restricted to W^i is equal to f_{x_i}.

Apply Lemma 4.2 to W and Γ and let $\mathcal{M}_R$ be the instance of $\mathcal{P}$ that it returns.

We must show that $\mathcal{M}_R$ satisfies the conditions (i - iv) of the lemma. Since the graph $\mathcal{M}_R$ provided by Lemma 4.2 contains $W = \bigcup_{i=1}^d W^i$, condition (i) is satisfied. Any $f_x \in F$ maps one vertex of W^* 1 and the rest to 0. This property is not preserved under any non-identity permutation of $V(\mathcal{P})$ so condition (ii) is met. Conditions (iii) and (iv) come directly from our definition of Γ and the conclusion of Lemma 4.2. □

4.2. **The Fibre Construction.** The fibre gadgets are put together with the following construction, which formalises the process outlined in 3.3.

Construction 4.4. *Let $\mathcal{H}$ be a relational structure, and $\mathcal{P}$ be a subprojective relational structure. Let W^*, F, and $\mathcal{M}_R$ for every relation R of $\mathcal{H}$, be as in Lemma 4.1.*

Let $\mathcal{G}$ be an instance of CSP($\mathcal{H}$)*, and construct $M_{\mathcal{H}}^{\mathcal{P}}(\mathcal{G})$ as follows.*

(i) *For each vertex v of $\mathcal{G}$ let W^v be a copy of W^*.*

(ii) *For each tuple $r = (v_1, \ldots, v_d)$ in each relation R of $\mathcal{G}$ (where the relation R has arity d) let $\mathcal{M}_R^r$ be a copy of $\mathcal{M}_R$. For $i = 1, \ldots d$ identify W^{v_i} with W^i in $\mathcal{M}_R^r$ index-wise.*

So $\mathcal{M}_{\mathcal{H}}^{\mathcal{P}}(G)$ consists of $|V(\mathcal{G})|$ copies of W^* and $|R|$ copies of M_R for each relation R of $\mathcal{G}$. All vertices are distinct unless identified above.

4.3. **Proof of Theorem 2.2.**

Proof. Let $\mathcal{H}$ be a relational structure and $\mathcal{P}$ be a subprojective relational structure with nontrivial projective subset S. For any instance $\mathcal{G}$ of CSP($\mathcal{H}$) let $\mathcal{M} = \mathcal{M}_{\mathcal{H}}^{\mathcal{P}}(\mathcal{G})$ be the structure defined by the Fibre Construction (4.4).

We show that

$$\mathcal{G} \to \mathcal{H} \iff \mathcal{M} \to \mathcal{P}.$$

Since Construction 4.4 is polynomial in $|V(\mathcal{G})|$ this will prove the theorem. Disconnected components of $\mathcal{G}$ correspond to disconnected components of $\mathcal{M}$, it is thus enough to prove the theorem for connected structures $\mathcal{G}$.

Let ϕ be a $\mathcal{P}$-coloring of $\mathcal{M}$. By property (iii) of Lemma 4.1 there is some $\sigma_r \in \mathrm{Aut}(\mathcal{P})$ for each tuple r in each relation R, such that for each copy of W^* in $\mathcal{M}_R^r$, ϕ restricts on W^* to $\sigma_r \circ f_i$ for some f_i in F. By the connectedness of $\mathcal{G}$, and property (ii) of Lemma 4.1, all of these σ_r are the same permutation. We assume, wlog, that they are all the identity. Thus ϕ restricts on the copy W^v of W^* in $\mathcal{M}$ for each $v \in V(\mathcal{G})$, to some pattern in F.

Define $\phi' : V(\mathcal{G}) \to V(\mathcal{H})$ by $\phi'(v) = x$ where ϕ restricts on W^v to f_x. For any tuple $r = (v_1, \ldots, v_d)$ in any relation R of $\mathcal{G}$, this implies that $(\phi'(v_1), \ldots, \phi'(v_d)) = (x_1, \ldots, x_d)$ where ϕ restricts on W^{v_i} to x_i. But for $i = 1, \ldots, d$, W^{v_i} is identified with the copy of W^i in $\mathcal{M}_R^r$, and so by property (iii) of Lemma 4.1, $(x_1, \ldots, x_d)$ is in the relation R of $\mathcal{H}$. Thus ϕ' is an $\mathcal{H}$-coloring of $\mathcal{G}$.

On the other hand, let ϕ' be an $\mathcal{H}$-coloring of $\mathcal{G}$, and define $\phi : V(\mathcal{M}) \to V(\mathcal{P})$ as follows. For all $v \in V(\mathcal{G})$, set ϕ equal to $f_{\phi'(v)}$ on W^v. Now we must show that for each tuple $r = (v_1, \ldots, v_d)$ or each relation R of $\mathcal{G}$, ϕ can be extended to a $\mathcal{P}$-coloring of $\mathcal{M}_R^r$.

The copies $W^1, \ldots, W^d$ of W^* in $\mathcal{M}_R^r$ are identified with the copies $W^{v_1}, \ldots, W^{v_d}$ respectively. Thus ϕ restricts on them to the patterns $f_{\phi'(v_1)}$, $\ldots, f_{\phi'(v_d)}$ respectively.

□

5. CSP Dichotomy Conjecture - algebraic approach

In [4], the universal algebra approach of [15] is extended to show that CSP($\mathcal{H}$) is NP-complete for a large class of CSPs. A conjecture is made that CSP($\mathcal{H}$) is polynomial time solvable for all other CSPs $\mathcal{H}$. In [19], this conjecture is then transported to the language of posets. In this section, we recall the necessary definitions of [4] and [19] to show that all $\mathcal{H}$ for which they show CSP($\mathcal{H}$) to be NP-complete, are subprojective.

We present only those definitions that we must, so this is by no means a good introduction to the powerful techniques used in [4] and [19].

5.1. **Universal Algebra.** An algebra $\mathcal{A} = (A, F)$ consists of a non-empty set A, and a set F of finitary operations on A. It is *finite* if A is finite. Given a relational structure $\mathcal{H}$, recall that Pol($\mathcal{H}$) is the set of polymorphisms of

$\mathcal{H}$. This defines an algebra $\mathcal{A}_{\mathcal{H}} = (V(\mathcal{H}), \mathrm{Pol}(\mathcal{H}))$. We say that $\mathcal{A}_{\mathcal{H}}$ is NP-complete if $\mathrm{CSP}(\mathcal{H})$ is.

The following two definitions are directly from [4].

Definition 5.1. Let $\mathcal{A} = (A, F)$ be an algebra and B a subset of A such that, for any $f \in F$ and for any $b_1, \ldots, b_d \in B$, where d is the arity of f, we have $f(b_1, \ldots, b_d) \in B$. Then the algbera $\mathcal{B} = (B, F|_B)$ is called a *subalgebra of* $\mathcal{A}$, where $F|_B$ consists of the restrictions of all operations in F to B.

Definition 5.2. Let $\mathcal{B} = (B, F_1)$ and $\mathcal{C} = (C, F_2)$ be such that $F_1 = \{f_i^1 \mid i \in I\}$ and $F_2 = \{f_i^2 \mid i \in I\}$, where both f_i^1 and f_i^2 are d_i-ary, for all $i \in I$. Then $\mathcal{C}$ is a *homomorphic image of* $\mathcal{B}$ if there exists a surjection $\psi : B \to C$ such that the following holds for all $i \in I$, and all $b_1, \ldots, b_{d_i} \in B$.

$$\psi \circ f_i^1(b_1, \ldots, b_{d_i}) = f_i^2(\psi(b_1), \ldots, \psi(b_{d_i})).$$

Given an algebra $\mathcal{C} = (C, F)$, the *term operators* of $\mathcal{C}$ refer to the set of finitary operators of C that preserve the same relations on C as F does. Thus all operators in F are term operators. A d-ary operator f of F is *essentially unary* if $f = f' \circ \pi$ for some projection $\pi : C^d \to C$ and some non-constant function $f' : C \to C$. Because this f' is non-constant, if F has any essentially unary operators, then $|C| \geq 2$.

The following is Corollary 7.3 in [4].

Theorem 5.3. *A finite algebra $\mathcal{A}$ is NP-complete if it has a subalgebra $\mathcal{B}$ with a homomorphic image $\mathcal{C}$, all of whose term operators are essentially unary.*

Using our main theorem, the following reproves Theorem 5.3 in the case that $\mathcal{A} = \mathcal{A}_{\mathcal{H}}$ for some core relational structure $\mathcal{H}$.

Proposition 5.4. *Let $\mathcal{H}$ be a relational structure such that $\mathcal{A}_{\mathcal{H}}$ has a subalgebra $\mathcal{B} = \mathcal{A}(B, F_1 = \mathrm{Pol}(\mathcal{H})|_B)$ with a homomorphic image $\mathcal{C} = \mathcal{A}(C, F_2)$, all of whose term operators are esseníally unary. Then $\mathcal{H}$ is subprojective.*

Proof. Let ψ be the surjective homomorphism of $\mathcal{B}$ to $\mathcal{C}$. Since all the elements of $F_{\mathcal{C}}$ are essentially unary, $|C| \geq 2$, so there exist b and b' in B such that $\psi(b) \neq \psi(b')$. We show that $\{b, b'\}$ is a projective subset of $\mathcal{H}$.

Let ϕ be a d-ary polymorphism in $\mathrm{Pol}(\mathcal{H})$. Then because $\mathcal{B}$ is a subalgebra of $\mathcal{A}_{\mathcal{H}}$, ϕ restricts on B to some member f^1 of F_1. We now consider f^1 further restricted to $\{b, b'\}$, we must show that it restricts to the same function as $\sigma \circ \pi$ for some $\sigma \in \mathrm{Aut}(\mathcal{H})$ and $\pi \in \mathrm{Proj}(H)$.

Let $b_1, \ldots, b_d \in \{b, b'\}$. Then because $\mathcal{C}$ is the homomorphic image of $\mathcal{B}$ via the homomorphism ψ, we have that

$$\psi \circ f^1(b_1, \ldots, b_d) = f^2(\psi(b_1), \ldots, \psi(b_d))$$

for some $f^2 \in F_2$. By assumption, f^2 is essentially unary, so $f^2 = g \circ \pi$ for some projection π and some non-constant function g. Thus $\psi \circ f^1 = g \circ \pi \circ (\psi \times \cdots \times \psi) = g \circ \psi \circ \pi$. Since $\psi(b) \neq \psi(b')$, ψ is invertible when restricted to $\{b, b'\}$, thus $f^1 = \psi^{-1} \circ g \circ \psi \circ \pi$. Letting $\sigma = \psi^{-1} \circ g \circ \psi$ we have shown that f^1, so ϕ restricts on $\{b, b'\}^d$ to the same function as $\sigma \circ \pi$. This was for any $\phi \in \mathrm{Pol}(\mathcal{H})$, and so $\{b, b'\}$ is a projective subset of $\mathcal{H}$. □

Conjecture 5.5. ([4]) *For a relational structure $\mathcal{H}$,* CSP($\mathcal{H}$) *is NP-complete if and only if $\mathcal{A}_{\mathcal{H}}$ has a subalgebra $\mathcal{B} = \mathcal{A}(B, F_1 = \mathrm{Pol}(\mathcal{H})|_B)$ with a homomorphic image $\mathcal{C} = \mathcal{A}(C, F_2)$, all of whose term operators are essenially unary.*

5.2. **Taylor Operations.** An idempotent d-ary operation ϕ on an algebra $\mathcal{A} = (A, F)$ is called a Taylor operation if for every $a \neq b \in A$ and every $i \in 1, \ldots, d$, there is some way to fill in the slots in the following equation, with some choice of a and b, such that it is true. (Here, the shown a and b are both in the i^{th} slot of ϕ.)

$$\phi(-, -, \ldots, -, a, -, \ldots, -, -) = \phi(-, -, \ldots, -, b, -, \ldots, -, -)$$

The following result immediately follows from a result of Taylor [32] that characterises all algebras without any Taylor operations (see Theorem 4 and Proposition 5 of [19]).

Theorem 5.6 ([19])**.** *For a relational structure $\mathcal{H}$, $\mathcal{A}_{\mathcal{H}}$ has a subalgebra with a homomorphic image all of whose term operators are essentially unary, if and only if there are no Taylor operations among the term operators of $\mathcal{A}_{\mathcal{H}}$.*

With this theorem, Larose and Zádori get that the following statement is equivalent to Conjecture 5.5.

Conjecture 5.7. *The problem* CSP($\mathcal{H}$) *is NP-complete if and only if there are no Taylor operations among the term operators of $\mathcal{A}_{\mathcal{H}}$.*

We can now prove the following.

Proposition 5.8. *Let $\mathcal{H}$ be a subprojective relational structure. Then there are no Taylor operations among the term operations of the algebra $\mathcal{A}_{\mathcal{H}}$.*

Proof. Since $\mathcal{H}$ is subprojective, it has some 2 element projective subset $\{a, b\}$. Let ϕ be any d-ary term operation, i.e., polymorphism, of $\mathcal{H}$. Then ϕ restriced to $\{a, b\}^d$ is equal to $\alpha \circ \pi$ for some projection π and some automorphism α. Assume wlog that $\pi = \pi_1$. Then

$$\phi(a, -, -, \dots, -) = \alpha(a) \neq \alpha(b) = \phi(b, -, -, \dots, -)$$

for any choice of a and b in the empty slots. Thus ϕ is not a Taylor operation. □

By Propositions 5.4 and 5.8, and Theorem 5.6 we get the following.

Corollary 5.9. *The Conjectures 2.4, 5.5, and 5.7 are equivalent.*

6. Restricted CSPs

6.1. **Degree Bounded CSPs.** We mentioned in the introduction, that because $\mathrm{CSP}(K_3)$ is NP-complete, taking $\mathcal{H} = K_3$, Corollary 2.3 follows from Theorem 2.2. In fact, $\mathrm{CSP}(K_3)$ is NP-complete for 4-bounded instances G. Using this we now prove Corollary 2.5.

Proof. When $\mathcal{H} = K_3$, the set Γ in the proof of Lemma 4.1 contains six pattens, and satisfies condition (*) of Lemma 4.2. Thus the one fibre gadget $\mathcal{M}_R$ used in the construction $G \to \mathcal{M}^{\mathcal{P}}_{\mathcal{H}}(G)$ is $\mathcal{P}^6$, and so has maximum degree $\Delta(\mathcal{P})^6$. Since any 4-bounded instance G yields a $(4 \cdot \Delta(\mathcal{P})^6)$-bounded structure $\mathcal{M}^{\mathcal{P}}_{\mathcal{H}}(G)$, $\mathrm{CSP}(\mathcal{P})$ is NP-complete for $(4 \cdot \Delta(\mathcal{P})^6)$-bounded instances. □

With some work we could reduce the 4 in the above proof to a 3 and could reduce the exponent 6 to the minimum number of tuples in all relations of a relational structure $\mathcal{H}$ for which $\mathrm{CSP}(\mathcal{H})$ is NP-complete. We will not include this though, as even it is probably not tight. In fact, we conjecture that $b(\mathcal{H}) \leq \Delta(\mathcal{H}) + o(\Delta(\mathcal{H}))$. The highest known value of $b(\mathcal{H})$, in terms of $\Delta(\mathcal{H})$, is for the complete graphs. It follows from [7] and [22] that $b(K_k)$ is about $k + \sqrt{k}$ for large enough k.

6.2. **Girth.** The results in this subsection are for graphs. The following lemma is proved in [24] in the case that P is a complete graph, and is proved in [26] without item (iii) in the case that P is projective. In both of these cases, $S = V(P)$.

Lemma 6.1. *Let P be a subprojective graph with projective subset S, and let $\ell \geq 3$ be an integer. Let W be an indexed set, and let $\Gamma = \{\gamma_1, \dots, \gamma_d\}$ be a set of S patterns of W. Then there exists a relational structure M with $W \subset V(M)$, such that the following are true:*

(i) *The set of P-colorings of M, when restricted to W, is exactly*

$$\{\alpha \circ \gamma \mid \alpha \in \mathrm{Aut}(P), \gamma \in \Gamma\}.$$

(ii) *M has girth at least ℓ.*
(iii) *The distance, in M, between any two vertices of W is at least ℓ.*

Proof. This lemma follows from Lemma 4.2, and Theorem 7.5 which is a local form of the main result of [26]. The result will be stated in Section 7. □

Using this in place of Lemma 4.2 in our construction, we now prove Theorem 2.6.

Proof. Let P be a graph with projective subset S, and let G be an instance of CSP(P). Let $\ell \geq 3$ be an integer. Assume that Lemma 6.1 is used in place of 4.2 in the proof of Lemma 4.1, and then use this version of Lemma 4.1 in Construction 4.4. The effect of this is that our one fibre gadget $\mathcal{M}_R$ has the added property that it has girth ℓ, and distance ℓ between vertices of W. We verify that the graph $\mathcal{M}^P_{K_3}(G)$ provided by Construction 4.4 has girth ℓ. This will imply that CSP(P) is NP-complete for graphs of girth ℓ.

Assume that $\mathcal{M}^P_{K_3}(G)$ contains a cycle C of length less than ℓ. Since the fibre gadget $\mathcal{M}_R$ has girth ℓ, C must contain edges of more than one copy of $\mathcal{M}_R$. Thus C has to contain at least two vertices of W in some copy of $\mathcal{M}_R$. However, these vertices are distance at least d apart, so this is a contradiction, and $\mathcal{M}^P_{K_3}(G)$ contains no cycles of length less than ℓ. □

7. Coloring Theoreoms - Combinatorial Background

There are two results which underlie our construction, these results go back to [24] and [25]. In a response to a problem of Erdős, Müller proved [24] the following.

Theorem 7.1. *Let A be a set, k, l positive integers, $k \geq 3$, and let $\pi_1, \ldots, \pi_t$ be distinct partitions of A each into at most k nonempty classes. Then there exists a graph M' with the following properties.*

(i) *$g(M') > \ell$;*
(ii) *$\chi(M') = k$;*
(iii) *up to a permutation of colors, every coloring of M' by k colors is the unique extension of one of the partitions π_i for an $1 \leq i \leq t$.*

This theorem has been generalized in [26] as follows. We need a definition: A graph H is said to be *pointed for* graph M (or *M-pointed*) if the following

$$\begin{tikzcd} M \arrow[r, "f"] & H \\ M' \arrow[u, "c"] \arrow[ur, "g"'] & \end{tikzcd}$$

FIGURE 2.

is true. For any two homomorphisms $g, g' : M \to H$, if $g(x) = g'(x)$ for all $x \neq x_0$ (for some fixed vertex $x_0 \in V(M)$), then $g(x_0) = g'(x_0)$.

Theorem 7.2. *For every graph M and every choice of positive integers k and l there exists a graph M' together with a surjective homomorphism $c : M' \to M$ with the following properties:*

(i) $g(M') > \ell$*;*
(ii) *For every graph H with at most k vertices and there exists a homomorphism $g : M' \to H$ if and only if there exists a homomorphism $f : M \to H$.*
(iii) *For every M-pointed graph H with at most k vertices and for every homomorphism $g : M' \to H$ there exists a unique homomorphism $f : M \to H$ such that $g = f \circ c$.*

The conditions (ii) and (iii) may be expressed saying that for any choice of $g : M' \to H$ or $f : M \to H$, there is a unique way to complete the commuting diagram in Figure 2.

This result generalizes the following result known as the Sparse Incomparability Lemma, proved in [25].

Theorem 7.3. *For every pair M, H of graphs such that H is M-colorable and M fails to be H-colorable there exists a graph M' with the following properties:*

(i) $g(M') > \ell$
(ii) *M' is M-colorable and M' fails to be H-colorable.*

For the proof of Theorem 2.6, we needed a generalization of Theorem 7.2. We now provide this, as Theorem 7.5, via the following localization of the notion of M-pointed.

Definition 7.4. Let M, H be graphs. Subsets S_M of $V(M)$ and S_H of $V(H)$ are said to be (M,H)-*pointed subsets* if the following is true. For any two homomorphisms $g, g' : M \to H$, if $g(x) = g'(x) \in S_H$ whenever $x \neq x_0$ and $x \in S_M$ (for some fixed vertex $x_0 \in S_M$), then $g(x_0) = g'(x_0) \in S_H$.

Theorem 7.5. *For every graph M and every choice of positive integers k and ℓ there exists a graph M' with the following properties.*

(i) $g(M') > \ell$;
(ii) *For every graph H with at most k vertices and there exists a homomorphism $g : M' \to H$ if and only if there exists a homomorphism $f : M \to H$.*

Furthermore, there exists a surjective homomorphism $c : M' \to M$ such that for every (M,H)-pointed subsets $S_M \subset V(M), S_H \subset V(H)$, where H has at most k vertices, and for every homomorphism $g : M' \to M$, the following properties also hold.

(iii) *There exists an H-coloring f of M such that g and $f \circ c$ restricts to the same function on $c^{-1}(S_M)$. Moreover, if there are two such H-colorings f_1 and f_2, then $f_1(x) = f_2(x)$ for every $x \in S_M$.*
(iv) *There exists a set $\{s' \in c^{-1}(s) \mid s \in S_M\}$ of represetatives of the sets $c^{-1}(s)$ that are pairwise distance at least ℓ apart.*

Proof. We follow the proof of Theorem 7.2 from [26] very closely, and we refer to this paper for many of the details.

Where M has a vertices $\{1, \dots, a\}$ and q edges, let $V_1, \dots, V_a$ be disjoint sets of n vertices each. Let M_0 be the graph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_a$, and edge set

$$\{xy \mid x \in V_i, y \in V_j, ij \in E(M)\}.$$

Thus M_0, which is often referred to as the n-blowup of M, has qn^2 edges. Let $\mathcal{M}$ be the set of all subgraphs of M_0 with $m = \lfloor qn^{1+\varepsilon} \rfloor$ edges, where $0 < \varepsilon < 1/\ell$. Let $\delta = \min\{\varepsilon\ell, 1/k\}$.

Asymptotically, almost all graphs G of $\mathcal{M}$ satisty the following properties.

(a) G has at most n^δ cycles of length $\leq \ell$, moreover, these cycles are vertex disjoint.
(b) For any two non-empty sets $A \subset V_i$ and $B \subset V_j$ of $V(G)$ (with ij in M) such that $|A| + |B| \geq \delta n$, the subgraph of G induced by

$A \cup B$ is not a matching (set of mutually disjoint edges,) with fewer than n^δ edges.

(c) There is a choice of vertices $\{v_1, \ldots, v_a\}$, such that $v_i \in V_i$, and for any $1 \leq i \neq j \leq a$, the distance in G between v_i and v_j is at least ℓ.

It was shown in [26], using standard calculations, that asymptotically, almost all graphs G of $\mathcal{M}$ satisfy properties (a) and (b), thus we prove that almost all graphs of $\mathcal{M}$ satisfy properties (a) - (c), by proving the following claim.

Claim 7.6. *Almost all graphs G of $\mathcal{M}$ satisfy property (c) above.*

Proof. For a graph G chosen uniformly at random from $\mathcal{M}$, the probability that a given vertex u is distance ℓ or less from a vertex v is less than $n^{\ell\varepsilon-1}$. Thus the probability that a given set of a vertices $\{v_1, \ldots, v_a\}$, with $v_i \in V_i$ for all i, fail to satisfy property (c) is less than $a^2 n^{\ell\varepsilon-1}$. As $\varepsilon < 1/\ell$, this goes to zero as n goes to infinity, so not only do almost all graphs G of $\mathcal{M}$ satisfy property (c), almost all choices of the set $\{v_1, \ldots, v_a\}$ in almost all G satisfy (c). □

We now continue with the proof of Theorem 7.5. Let G be any graph of $\mathcal{M}$ that satisfies the properties (a), (b) and (c). It is clear that we can remove a matching of size at most n^δ from G and end up with a graph M' having the following corresponding properties.

(a') $g(M') > \ell$

(b') For any two non-empty sets $A \subset V_i$ and $B \subset V_j$ of $V(M')$ (with ij in M) such that $|A| + |B| \geq \delta n$, there is at least one edge of M' with both endpoints in $A \cup B$.

(c') Same as (c).

We now verify that M' satisfies properties (i) - (iv) of the theorem. Property (i) is given by property (a').

Letting $c : M' \to M$ be the M-coloring defined by

$$c(v) = i \text{ where } v \in V_i,$$

it is clear that for every graph H, and every H-coloring f of M, $g = f \circ c$ is an H-coloring of M'. To finish the proof that M' satisfies property (ii), it suffices to show that for any graph H with at most k vertices, and any H-coloring g of M' there is an H-coloring f of M.

Let such an H-coloring g of M' be given, and define $f : M \to H$ as follows. For each vertex i of M, there exists, by the pigeonhole principle, a vertex h of H such that $|V_i \cap g^{-1}(h)| \geq n/k > \delta n$. Let $f(i) = h$ for any such h. We now show that f is an H-coloring of M. Let ij be an edge of M. There is an edge of M' whose endpoints map to $f(i)$ and $f(j)$ (under g), and so f maps ij to an edge of H. Indeed, the sets $A = V_i \cap g^{-1}(f(i))$ and $B = V_j \cap g^{-1}(f(j))$ both have size at least n^δ and so by property (b'), there is an edge of M' with one endpoint in A and one in B. This edge clearly maps to $f(i)f(j)$, and so property (ii) is proved.

To show property (iii) of the theorem, assume that $S_M \subset V(M)$ and $S_H \subset V(H)$ are (M, H)-pointed subsets, where H has at most k vertices, and assume that g is an H-coloring of M'.

The main point is that for any vertex s in S_M, g is constant on the set V_s. Indeed let v be any vertex of V_s and define $f_v : V(M) \to V(H)$ by letting $f_v(s) = v$, and otherwise letting f_v be defined as f in the proof of property (ii). That is, for $i \neq s$, set $f_v(i) = h$ for some vertex h of H such that $|V_i \cap g^{-1}(h)| \geq n/k > \delta n$. By almost the same argument as before, we get that f_v is an H-coloring of M. Thus if g is not constant on V_j we get different H-colorings of M that differ only on $j \in S_M$. This contradicts the fact that S_M and S_H are (M, H)-pointed.

The statement that g and $f \circ c$ restrict on S_M to the same function, uniquely determines the function f on S_M, so we just have to show that there exists such an f. Because g is constant on V_s for all $s \in S_M$, the function f defined as in the proof of property (ii) is such that g and $f \circ c$ restrict on $c^{-1}(S_M)$ to the same function. Thus M' has property (iii).

Property (iv) of the theorem follows directly from property (c'), so the theorem is proved.

□

8. Summary and Concluding Remarks

We introduced a new Dichotomy Classification Conjecture and showed that this combinatorial conjecture is equivalent to other conjectured dichotomies derived by algebraic methods. It turns out that this approach to the complexity of CSPs is flexible enough to yield a solution to problems about CSPs restricted to structures with bounded degrees or with large girth. As our reduction (yes, one reduction is enough) is explicit (and easy) we expect that our results will have other applications and will inspire a new approach to the Dichotomy Conjecture itself. In particular, as we want to demonstrate in a forthcomming paper this also yields a new approach to the (undirected) graph H-coloring problem.

References

[1] V.G. Bodnarčuk, L.A. Kaluzhnin, V.N. Kotov, B.A. Romov, *Galois theory for Post algebras I-II* (in Russian), *Kibernetika*, **3** (1969), 1-10 and **5** (1969), 1-9. English version: *Cybernetics*, (1969), 243-252 and 531-539.

[2] A. Bulatov: A dichotomy theorem for constraints on a three element set. In: FOCS'02, 2002, 649–658.

[3] A. Bulatov: *H-Coloring Dichotomy Revisited*, Theoret. Comp. Sci. 349,1 (2005), 31-39.

[4] A. Bulatov, P. Jeavons, A. Krokhin, *Classifying the complexity of constraints using finite algebras.* SIAM J. Comput. 34 (2005), no. 3, 720–742.

[5] A. Bulatov, P. Jeavons, A. Krokhin, *Constraint satisfaction problems and finite algebras*, In: Proceedings of the 27th International Coll. on Automata, Languages and Programming- ICALP'00, LNCS 1853, Springer, 2000, pp. 272–282.

[6] N. Creignou, S. Khanna, M. Sudan, **Complexity Classifications of Boolean Constraint Satisfaction Problems**, SIAM Monographs on Discrete Mathematics and Applications, SIAM, 2001.

[7] T. Emden-Weinert, S. Hougardy, B. Kreuter, *Uniquely colourable graphs and the hardness of colouring graphs of large girth.* Combin. Probab. Comput. 7 (1998), no. 4, 375–386.

[8] T. Feder, P. Hell, J. Huang, *List Homomorphisms of Graphs with Bounded Degree.* Submitted.

[9] T. Feder, M. Vardi, *The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory.* SIAM J. Comput. 28 (1999), no. 1, 57–104.

[10] D. Geiger, *Closed systems of functions and predicates*, Pacific. Journal of Math. 27 (1968), 95–100.

[11] P. Hell, *Algorithmic aspects of graph homomorphisms*, In: Survey in Combinatorics 2003, Cambridge University Press 2003, pp. 239–276.

[12] P. Hell: *From Graph Colouring to Constraint Satisfaction: There and Back Again*, in M. Klazar, J. Kratochvil, M. Loebl, J. Matousek, R. Thomas, P. Valtr (eds.) Topics in Discrete Mathematics. Dedicated to Jarik Nesetril on the Occasion of his 60th Birthday, Springer Verlag, 2006, pp. 407-432.

[13] P. Hell, J. Nešetřil, *On the complexity of H-colouring* J. Combin. Theory B 48 (1990) 92-100.

[14] P. Hell, J. Nešetřil, **Graphs and Homomorphisms**, Oxford University Press, 2004

[15] P. G. Jeavons, *On the algebraic structure of combinatorial problems.* Theoret. Comput. Sci. 200 (1998), no. 1-2, 185–204.

[16] P. G. Jeavons, D. A. Cohen, M. Gyssens, *Closure properties of Constraints*, Journal of the ACM, 44 (1997), 527–548.

[17] A. Kostochka, J. Nešetřil, P. Smolíková, *Colorings and homomorphisms of degenerate and bounded degree graphs.* Graph theory (Prague, 1998). Discrete Math. 233 (2001), no. 1-3, 257–276.

[18] B. Larose, L. Zádori, *Finite posets and topological spaces in locally finite varieties.* Algebra Universalis 52 (2004), no. 2-3, 119–136.

[19] B. Larose, L. Zádori, *The Complexity of the Extendibility Problem for Finite Posets* SIAM J. Discrete Math. 17 (2003) no. 1, 114-121.

[20] B. Larose, C. Tardif: *Strongly rigid graphs and projectivity*, Multiple-Valued Logic **7** (2001), 339–361.
[21] T.Łuczak, J. Nešetřil, *A probabilistic approach to the dichotomy problem.* SIAM J. Comput. 36 (2006), no. 3, 835–843
[22] M. Molloy, B. Reed, *Colouring graphs when the number of colours is nearly the maximum degree.* Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing, 462–470 (electronic), ACM, New York, 2001.
[23] U. Montanari, *Networks of constraints: Fundamental properties and applications to picture processing*, Information Sciences 7(1974), 95–132.
[24] V. Müller *On colorings of graphs without short cycles*, Discrete Math. 26 (1979),165–176.
[25] J. Nešetřil, V. Rödl, *Chromatically optimal rigid graphs*, J. Comb. Th. B,46 (1989), 122–141.
[26] J. Nešetřil, X. Zhu, *On sparse graphs with given colorings and homomorphisms.* J. Combin. Theory Ser. B 90 (2004), no. 1, 161–172.
[27] N. Pippenger, **Theories of Computability**, Cambridge University Press, 1997.
[28] R. Pöschel, L. A. Kalužnin, **Funktionen- und Relatrionenalgebren**, DVW, Berlin, 1979.
[29] T. J. Schaefer, *The complexity of satisfiability problems.* In: Proceedings of the 10th ACM Symposium on Theory of Computing (STOC78), 1978, 216–226.
[30] M. Siggers, *On Highly Ramsey Infinte Graphs.* Submitted, (2006).
[31] M. Siggers, *Dichotomy for Bounded Degree H-colouring.* Submitted, (2007)
[32] W. Taylor, *Varieties obeying homotopy laws,* Canad. J. Math., 29 (1977), pp. 498-527.

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