On Finite Maximal Antichains in the Homomorphism Order

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1 Introduction

The relation of existence of a homomorphism on the class of all relational structures of a fixed type is reflexive and transitive; it is a quasiorder. There are standard ways to transform a quasiorder into a partial order – by identifying equivalent objects, or by choosing a particular representative for each equivalence class. The resulting partial order is identical in both cases.

Properties of this partial order (the *homomorphism order*) have been intensively studied in algebraic, category theory, random and combinatorial context, see [5]. Particular interest has been paid to density and universality. Here, we are interested in the characterisation of all *finite maximal antichains* in the homomorphism order.

We show that for structures with at most two relations all finite maximal antichains correspond to what is known as finite homomorphism dualities (see [4, 8]). In addition, we examine the *splitting property* of finite maximal antichains in the homomorphism order (see [2]). We derive a structural condition which implies that most finite maximal antichains split. This was previously known for digraphs [3] and structures with at most one relation [4].

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2 Definitions

2.1 Relational structures

A *type* Δ is a sequence $(\delta_i : i \in I)$ of positive integers; I is a finite set of indices. A (finite) *relational structure* A of type Δ is a pair $\big(X, (R_i : i \in I)\big)$, where X is a finite nonempty set and $R_i \subseteq X^{\delta_i}$; that is, R_i is a δ_i-ary relation on X.

In this abstract, we are interested only in relational structures with no unary relations, i.e. $\delta_i \geq 2$ for all $i \in I$.

If $A = \big(X, (R_i : i \in I)\big)$, the *base set* X is denoted by $\underline{A}$ and the relation R_i by $R_i(A)$. We often refer to a relational structure of type Δ as Δ-structure. The elements of the base set are called *vertices* and the elements of the R_i's are called *edges*.

The *shadow* of a Δ-structure A is the undirected multigraph $\mathrm{Sh}(A)$ whose vertices are the elements of $\underline{A}$ and there is one edge from a to b for each edge in some $R_i(A)$ of arity $\delta_i \geq 2$ such that $(a_1, \ldots, a_{\delta_i}) \in R_i(A)$ with $a_j = a$, $a_{j+1} = b$ for some $1 \leq j < \delta_i$.

A Δ-structure A is called *connected* if its shadow $\mathrm{Sh}(A)$ is connected.

A Δ-structure A is called a Δ-*tree* or simply a *tree* if $\mathrm{Sh}(A)$ is a tree; it is called a Δ-*forest* or just a *forest* if $\mathrm{Sh}(A)$ is a forest. A Δ-tree P is called a Δ-*path* if every edge of P intersects at most two other edges.

2.2 Homomorphisms

Let A and A' be two relational structures of the same type Δ. A mapping $f : \underline{A} \to \underline{A'}$ is a *homomorphism* from A to A' if for every $i \in I$ and for every $m_1, m_2, \ldots, m_{\delta_i} \in \underline{A}$ the following implication holds:

$$(m_1, m_2, \ldots, m_{\delta_i}) \in R_i(A) \quad \Rightarrow \quad \big(f(m_1), f(m_2), \ldots, f(m_{\delta_i})\big) \in R_i(A').$$

The fact that f is a homomorphism from A to A' is denoted by $f : A \to A'$. If there exists a homomorphism from A to A', we say that A is *homomorphic* to A' and write $A \to A'$; otherwise we write $A \nrightarrow A'$. If A is homomorphic to A' and at the same time A' is homomorphic to A, we say that A and A' are *homomorphically equivalent* and write $A \sim A'$. If on the other hand there exists no homomorphism from A to A' and no homomorphism from A' to A, we say that A and A' are *incomparable*. Note that the composition of homomorphisms is a homomorphism as well and that homomorphic equivalence is indeed an equivalence relation on the class of all Δ-structures.

A finite Δ-structure C is called a *core* if it is not homomorphic to any proper substructure of C. A substructure C of A is called the *core of* A if it is a core and A and C are homomorphically equivalent.

For a fixed type $\Delta = (\delta_i : i \in I)$, all Δ-structures (objects) and their homomorphisms (morphisms) form a category. Finite products and finite sums exist in this category; sums are disjoint unions of structures. (See [5] for a general introduction to relational structures and their homomorphisms.)

2.3 Height labelling and balanced structures

We say that a Δ-structure A is *balanced* if A is homomorphic to a Δ-forest.

Let A be a Δ-structure and let $\daleth$ be a labelling of its vertices with $(\sum_{i\in I} \delta_i - |I|)$-tuples of integers, indexed by $(i,1), (i,2), \ldots, (i, \delta_i - 1)$, $i \in I$.

We say that $\daleth$ is a *height labelling* of A if whenever $(x_1, x_2, \ldots, x_{\delta_i}) \in R_i(A)$ and $1 \le j < \delta_i$, then

$$\begin{aligned} \big(\daleth(x_{j+1})\big)_{(i,j)} &= \big(\daleth(x_j)\big)_{(i,j)} + 1, \quad \text{and} \\ \big(\daleth(x_{j+1})\big)_{(i',j')} &= \big(\daleth(x_j)\big)_{(i',j')} \quad \text{for } (i',j') \ne (i,j). \end{aligned}$$

Proposition 2.1. *If A is a balanced Δ-structure, then A has a height labelling. If a height labelling of a connected structure exists, then it is unique up to an additive constant vector.*

2.4 Homomorphism duality

Let $\mathcal{F}$ and $\mathcal{D}$ be two finite sets of Δ-structures such that no homomorphisms exist among the structures in $\mathcal{F}$ and among the structures in $\mathcal{D}$. We say that $(\mathcal{F}, \mathcal{D})$ is a *finite homomorphism duality* (often just a *finite duality*) if for every Δ-structure A we have

$$\exists F \in \mathcal{F} : F \to A \Leftrightarrow \forall D \in \mathcal{D} : A \nrightarrow D.$$

Theorem 2.2 ([8]). *If $\big(\{F\}, \{D\}\big)$ is a finite homomorphism duality, then F is homomorphically equivalent to a Δ-tree. Conversely, if F is a Δ-tree with more than one vertex, then there exists a unique (up to homomorphic equivalence) structure D such that $\big(\{F\}, \{D\}\big)$ is a finite homomorphism duality.*

Theorem 2.3 ([4]). *If $(\mathcal{F}, \mathcal{D})$ is a finite homomorphism duality, then all elements of $\mathcal{F}$ are homomorphically equivalent to Δ-forests and $\mathcal{D}$ is determined by $\mathcal{F}$ uniquely up to homomorphic equivalence. Conversely, for any* *finite collection $\mathcal{F}$ of Δ-forests there exists $\mathcal{D}$ such that $(\mathcal{F}, \mathcal{D})$ is a finite homomorphism duality.*

3 Finite maximal antichains

The set of all (non-isomorphic) cores with the relation $\to$ is a partially ordered set, denoted by $\mathcal{C}(\Delta)$; we speak of the *homomorphism order* of relational structures.

We use the slightly unusual notation $A \to B$ instead of the more common $A \le B$ for the homomorphism partial order. Where convenient, however, we use $A < B$ to denote that $A \to B$ and at the same time $B \nrightarrow A$.

A set $\mathcal{Q}$ of Δ-structures is an *antichain* if any two distinct elements of $\mathcal{Q}$ are incomparable; it is a *maximal antichain* if moreover for any Δ-structure A there exists $Q \in \mathcal{Q}$ such that $A \to Q$ or $Q \to A$. A finite maximal antichain $\mathcal{Q}$ *splits* if there are disjoint sets $\mathcal{F}$ and $\mathcal{D}$ such that $\mathcal{F} \cup \mathcal{D} = \mathcal{Q}$ and for any Δ-structure A there exists $F \in \mathcal{F}$ such that $F \to A$ or there exists $D \in \mathcal{D}$ with $A \to D$.

We are going to investigate which finite maximal antichains in the homomorphism order split.

Definition 3.1. Let $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_n\}$ be a finite maximal antichain in $\mathcal{C}(\Delta)$. Recursively, define the sets $\mathcal{F}_0$, $\mathcal{F}_1$, …, $\mathcal{F}_n$ in this way:

1. Let $\mathcal{F}_0 = \emptyset$.
2. For $i = 1, 2, \dots, n$: check whether there exists a Δ-structure X satisfying

   (i) $Q_i < X$,

   (ii) $F \nrightarrow X$ for any $F \in \mathcal{F}_{i-1}$, and

   (iii) $Q_j \nrightarrow X$ for any $j > i$.

   If such a structure X exists, let $\mathcal{F}_i = \mathcal{F}_{i-1} \cup \{Q_i\}$, otherwise let $\mathcal{F}_i = \mathcal{F}_{i-1}$.
3. Finally, let $\mathcal{F} = \mathcal{F}_n$ and $\mathcal{D} = \mathcal{Q} \setminus \mathcal{F}$.

The definition directly implies the following two properties of the partition of $\mathcal{Q}$ into $\mathcal{F}$ and $\mathcal{D}$:

Lemma 3.2. *Let $\mathcal{Q}$ be a finite maximal antichain and $\mathcal{F}$, $\mathcal{D}$ be defined in 3.1. If $Q \in \mathcal{Q}$, X is a Δ-structure, and $Q < X$, then there exists $F \in \mathcal{F}$ such that $F < X$.*

Lemma 3.3. *Let $\mathcal{Q}$ be a finite maximal antichain and $\mathcal{F}$, $\mathcal{D}$ be defined in 3.1. For every F in $\mathcal{F}$ there exists a Δ-structure $\check{F}$ such that $F < \check{F}$ and moreover F is the only element of $\mathcal{F}$ that is homomorphic to $\check{F}$.*

Next, we examine the properties of the structures in $\mathcal{F}$.

Lemma 3.4. *Let $\mathcal{Q}$ be a finite maximal antichain and $\mathcal{F}$, $\mathcal{D}$ be defined in 3.1. If $F \in \mathcal{F}$, then F is balanced.*

Sketch of proof. Let $F \in \mathcal{F}$ be arbitrary. We use a tool called the "sparse incomparability lemma" (the concept is based on [1, 7]). In particular, there exists a Δ-structure H that is locally a tree (any substructure with at most $|\underline{F}|$ vertices is a forest) and with prescribed existence of homomorphisms to a finite number of structures. Specifically, we want that $H \to \check{F}$ and $H \nrightarrow Q$ for any $Q \in \mathcal{Q}$. Then $F \to H$ because the antichain is maximal and because of Lemmas 3.2 and 3.3, and the image of F by this homomorphism is a forest. $\square$

The following lemma describes "obstacles to splitting".

Lemma 3.5. *Let $\mathcal{Q}$ be a finite maximal antichain and $\mathcal{F}$, $\mathcal{D}$ be defined in 3.1. Then exactly one of the following conditions holds:*

(1) The pair $(\mathcal{F}, \mathcal{D})$ is a finite duality and $\mathcal{Q}$ splits.

(2) There exists a structure Y such that $Q \nrightarrow Y$ for any $Q \in \mathcal{Q}$ and $Y \nrightarrow D$ for any $D \in \mathcal{D}$.

We will now investigate those maximal antichains that satisfy (2). The structure Y has to be comparable with some element of the maximal antichain $\mathcal{Q}$, and because of the condition (2) there exists $F \in \mathcal{F}$ such that $Y < F$.

Every Δ-path has a height labelling; we say that a core Δ-path P is a *forbidden path* if it has two edges of the same kind whose vertices are not labelled the same. (This property does not depend on what height labelling we choose, see Proposition 2.1.)

Lemma 3.6. *Let $\mathcal{Q}$ be a finite maximal antichain in $\mathcal{C}(\Delta)$ and let $\mathcal{F}$, $\mathcal{D}$ be defined in 3.1. If Y is a Δ-structure such that $Y \nrightarrow D$ for any $D \in \mathcal{D}$ and $Y < F$ for some $F \in \mathcal{F}$, and P is a forbidden path, then $P \nrightarrow Y$.*

Sketch of proof. Construct an unbalanced structure W with the property that if any sufficiently small (in terms of the number of vertices) structure maps to W then it maps to P. This can be done by taking a long cycle-like structure that contains P.

Then consider the sum $W+Y$. It is comparable with some some element of the maximal antichain $\mathcal{Q}$. However, $W+Y \nrightarrow D$ for any $D \in \mathcal{D}$ because of Y, and $W+Y \nrightarrow F$ for any $F \in \mathcal{F}$ because W is not balanced. Therefore $F \to W+Y$ for some $F \in \mathcal{F}$, and the construction of W is such that $F \to P+Y$. As $F \nrightarrow Y$ (by the definition of Y), necessarily $P \nrightarrow Y$. $\square$

Lemma 3.7. *Let C be a connected Δ-structure. If no forbidden path is homomorphic to C, then C is homomorphic to a tree with at most one edge of each kind.*

Sketch of proof. First we observe that if no forbidden path is homomorphic to C, then C has a height labelling. In this labelling, any two edges of the same kind get identical labels. We construct a tree that contains vertices with exactly the same labels as are those used for vertices of C; the height labelling of C is then a homomorphism to this tree. $\square$

Let D^* be the sum of all Δ-trees with at most one edge of each kind.

As a direct consequence of the two preceding lemmas we get:

Proposition 3.8. *If Y satisfies the condition (2) of 3.5, then $Y \to D^*$.*

This shows that the cases when the antichain does not split are very specific (and one would like to say they are rather rare):

Theorem 3.9. *Let $\mathcal{Q}$ be a finite maximal antichain in $\mathcal{C}(\Delta)$. Suppose that every element $Q \in \mathcal{Q}$ has the property that whenever $Y < Q$ and $Y \to D^*$ then there exists a Δ-structure X such that $Y < X < Q$ and $X \nrightarrow D^*$. Then the antichain $\mathcal{Q}$ splits.*

Further examination reveals that in the case of structures with at most two relations there are no infinite increasing chains below D^*. From that we can conclude that all elements of $\mathcal{F}$ are Δ-forests and thus we get the following theorem.

Theorem 3.10. *Let $\Delta = (\delta_i : i \in I)$ be a type such that $|I| \leq 2$. Then all finite maximal antichains in the homomorphism order $\mathcal{C}(\Delta)$ are exactly the sets*

$$\mathcal{Q} = \mathcal{F} \cup \{D \in \mathcal{D} : D \nrightarrow F \textit{ for any } F \in \mathcal{F}\}$$

where $(\mathcal{F}, \mathcal{D})$ is a finite homomorphism duality.

The case of three or more relations ($|I| \geq 3$) is presently open. There may be a "quantum leap" here as indicated by the following result, which can be deduced from [6].

Proposition 3.11. *Let $\Delta = (2, 2, 2)$. Then the suborder of $\mathcal{C}(\Delta)$ induced by all structures homomorphic to D^* is a universal countable partial order; that is, any countable partial order is an induced suborder of this order.*

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