Removing degeneracy may require unbounded dimension increase

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Abstract

Many geometric algorithms are formulated for input objects in general position; sometimes this is for convenience and simplicity, and sometimes it is essential for the algorithm to work at all. For arbitrary inputs this requires removing degeneracies, which has usually been solved by relatively complicated and computationally demanding perturbation methods.

The result of this paper can be regarded as an indication that the problem of removing degeneracies has no simple "abstract" solution. We consider *LP-type problems*, a successful axiomatic framework for optimization problems capturing, e.g., linear programming and the smallest enclosing ball of a point set. We prove that in order to remove degeneracies of an LP-type problem, we sometimes have to increase its combinatorial dimension by an arbitrarily large amount.

The proof consists of showing that certain posets cannot be covered by pairwise disjoint copies of Boolean algebras under some restrictions on their placement. To this end, we prove that certain systems of linear inequalities are unsolvable, which seems to require surprisingly precise calculations.

1 Introduction

Geometric computation and degeneracy. Many descriptions of algorithms in computational geometry or in geometric optimization, as well as numerous proofs in discrete geometry, start with a sentence similar to "Let us assume that the given points are in general position." General position may mean that no three among the points are collinear, or we may also require than no four are cocircular, etc., depending on the considered problem. Violations of general positions, such as three points on a line, are referred to as *degeneracies*.

Assuming the input to be nondegenerate (i.e., in general position) usually simplifies the description, analysis, and implementation of a geometric algorithm significantly. For many algorithms, this assumption can be avoided with some extra work and careful attention to detail (a case study, arguing in favor of expending such extra work, is Burnikel et al. [BMS94]). However, for some algorithms, the nondegeneracy assumption is not only a convenient simplification, but rather an essential condition for correctness and/or for running time analysis, which seems difficult to circumvent—we will mention an example below.

General methods have been developed for removing degeneracies in geometric algorithms, based on *infinitesimal perturbations* of the input (Edelsbrunner and Mücke [EM90], Yap [Yap90], Emiris and Canny [EC92]). Roughly speaking, the coordinates of each input object are changed by a suitable function of a real parameter $\varepsilon > 0$, and the considered algorithm is executed with these new input objects, treating ε as a formal quantity, smaller than any concrete nonzero real number occurring in the algorithm. These approaches can actually be implemented, but they have several drawbacks: They slow down the computations significantly (typically by a large constant factor, but sometimes even much more), they increase space requirements, and sometimes it may be difficult or impossible to reconstruct the correct result for the original input from the result for the perturbed input—see [BMS94] for a discussion.

Removing degeneracies means "breaking ties" in some sense. Of course, the ties cannot be broken arbitrarily, since geometric algorithms almost always depend on some kind of global consistency of the input. Still, one might hope for some simpler, perhaps combinatorial, way of removing degeneracies. The present work was motivated by this question, and it can be regarded as an indication that a simple, general, and efficient combinatorial method is unlikely to exist.

LP-type problems. We are going to investigate the problem of removing degeneracies in a class of optimization problems known as LP-type problems (or "generalized linear programming problems"). This axiomatic framework, invented by Sharir and Welzl in 1992 [SW92], has become a well-established tool in the field of geometric optimization; see [MSW96, Ame94, Ame96, BSV03, GW01, Cha04, Hal04, GMRŠ06] for more applications and results on LP-type problems, as well as e.g. [SW00, GS06, Fis05, Sch04] for the investigation of other, related frameworks.

Once it is shown that a particular optimization problem is an LP-type problem and certain algorithmic primitives are implemented for it, several efficient algorithms are immediately at disposal: the Sharir–Welzl algorithm, two other randomized optimization algorithms due to Clarkson [Cla95] (see [GW96, CM96] for a discussion of how it fits the LP-type framework), a deterministic version of it [CM96], and an algorithm for computing the minimum solution that violates at most k of the given n constraints [Mat95] (this is the promised example of an algorithm where nondegeneracy appears crucial).

An LP-type problem is given by a finite set H of *constraints* and a *value* $w(G) \in \mathbf{R}$ for every subset $G \subseteq H$. Intuitively, $w(G)$ is the minimum value of a solution that satisfies all constraints in G. As our running example, we will use the problem of computing the smallest disk containing a given planar point set. Here H is a finite point set in $\mathbf{R}^2$ and $w(G)$ is the radius of the smallest circular disk that encloses all points of G. The general definition is as follows:

Definition 1.1 *An* LP-type problem *is a pair (H, w), where H is a finite set and $w\colon 2^H \to \mathbf{R}$ is a mapping satisfying the following two conditions:*[1]

Monotonicity:	*For all $F \subseteq G \subseteq H$ we have $w(F) \leq w(G)$.*
Locality:	*For all $F \subseteq G \subseteq H$ and all $h \in H$* *with $w(F) = w(G) = w(F \cup \{h\})$* *we have $w(G \cup \{h\}) = w(G)$.*

For the smallest enclosing disk problem, monotonicity is obvious, while

[1] Actually, the usual definition of an LP-type problem is more general: the mapping w can also attain a special value $-\infty$, which is considered smaller than all real numbers, and for which the locality axiom is not required. Moreover, instead of $\mathbf{R}$, one can use an arbitrary linearly ordered set, but this brings nothing new, just sometimes a more convenient notation. We will stick to the definition above since it is simpler, and it will be easy to check that the more general definition doesn't change anything in our result.

verifying locality requires the nontrivial but well known geometric result that the smallest enclosing disk is unique for every set.

The most important parameter of an LP-type problem, essentially controlling the behavior of algorithms dealing with the given problem, is the combinatorial dimension.

Definition 1.2 *Let (H, w) be an LP-type problem and let $G \subseteq H$. A* basis *of G is any inclusion-minimal subset $B \subseteq G$ with $w(B) = w(G)$. A set $B \subseteq H$ is called a* basis in (H, w) *if it is a basis of some $G \subseteq H$. The* combinatorial dimension *of (H, w) is the maximum cardinality of a basis.*

If (H, w) is a smallest enclosing disk problem, then the combinatorial dimension is at most 3 (since for every point set G in the plane there is a subset B of at most 3 points of G such that G and B have the same smallest enclosing disk). Similarly, a higher-dimensional version, the smallest enclosing ball problem of a point set in $\mathbf{R}^d$, has combinatorial dimension at most $d + 1$.

Degeneracy in LP-type problems. What should be considered a degeneracy in the smallest enclosing disk problem? A reasonable answer is a subproblem with an "overdetermined" solution, which means a set G whose minimum enclosing disk is determined by two distinct inclusion-minimal subsets $B_1, B_2 \subseteq G$. For example, B_1 and B_2 can be two different diametrical pairs determining the same disk. Nondegeneracy for an arbitrary LP-type problem can be defined in a similar way [Mat95].

Definition 1.3 *We call an LP-type problem (H, w)* nondegenerate *if $w(B_1) \neq w(B_2)$ for any two distinct bases B_1, B_2.*

Consequently, in a nondegenerate LP-type problem, every $G \subseteq H$ has exactly one basis.

For removing degeneracies, we want to break the ties $w(B_1) = w(B_2)$ by slightly modifying the values of w, while retaining all strict inequalities among the original values:

Definition 1.4 *An LP-type problem (H, w') is a* refinement *of an LP-type problem (H, w) on the same set of constraints if for all $F, G \subseteq H$ with $w(F) < w(G)$ we have $w'(F) < w'(G)$.*

We thus formalize "removing degeneracies" of an LP-type problem (H, w) as the question of finding a nondegenerate refinement of (H, w).

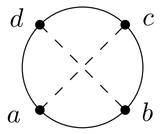

Figure 1: A degenerate LP-type problem where removing degeneracy increases dimension.

At first sight it might seem that in order to produce a nondegenerate refinement, it should suffice to impose some suitable linear order on every group of bases sharing the same value of w—perhaps one could even take an arbitrary ordering.

However, some thought reveals that things are not that simple. As was observed in [Mat95], sometimes we also have to create *new* bases, and even larger ones than those present in (H, w). Namely, consider the smallest enclosing disk problem with $H = \{a, b, c, d\}$ forming the vertices of a square (Fig. 1). The set H has two bases $B_1 = \{a, c\}$ and $B_2 = \{b, d\}$, and the combinatorial dimension of the problem is 2. We will refer to this particular 2-dimensional LP-type problem as the *square example* and denote it by (H_0, w_0). It is easily checked (we will do so in Section 2) that any nondegenerate refinement has dimension at least 3.

Here we exhibit LP-type problems where removing degeneracies requires arbitrarily large increase of the dimension.

Theorem 1.5 *There exists a positive constant $\varepsilon > 0$ such that for infinitely many values of D, there is an LP-type problem of combinatorial dimension D, for which every nondegenerate refinement has combinatorial dimension at least $(1 + \varepsilon)D$.*

The example of an LP-type problem as in the theorem is obtained by an "iterated join" of the square example. We also show that an essentially equivalent example can be represented as a linear program in the usual sense (a highly degenerate linear program).

The result can also be understood as telling us that for degenerate LP-type problems, the combinatorial dimension doesn't convey a full "dimensionality" information about the problem. An alternative dimension parameter might be the smallest possible dimension of a nondegenerate refinement; however, this appears quite hard to determine. In actual geometric problems, we can resort to the dimension of the ambient space, which doesn't grow when degeneracies are removed by perturbation methods. But this brings us back to our initial point—geometric perturbations are not easy to deal with.

The main open question is, can the smallest possible dimension of a nondegenerate refinement be bounded by some function of the dimension of the original degenerate LP-type problem? In particular, does every 2-dimensional LP-type problem have a nondegenerate refinement of dimension bounded by a universal constant? We suspect that it might not be the case, but it seems that the methods of the present paper are not sufficient to yield such a result.

2 Structure of nondegenerate LP-type problems

Let (H, w) be an LP-type problem. We consider the partially ordered set (poset) $(2^H, \subseteq)$, a Boolean algebra. For every $x \in \mathbf{R}$, the system $\mathcal{P}_x = \{G \subseteq H : w(G) = x\}$ is a subposet of $(2^H, \subseteq)$, and the $\mathcal{P}_x$ for all $x \in \mathbf{R}$ form a disjoint cover of 2^H. Monotonicity implies that a $\mathcal{P}_x$ has no "holes": If $F \subset M \subset G$ and $x = w(F) = w(G)$, then $w(M) = x$ as well. The following lemma shows that for *nondegenerate* LP-type problems, each $\mathcal{P}_x$ is actually a copy of a Boolean algebra.

Lemma 2.1 (Cube lemma) *Let (H, w) be a nondegenerate LP-type problem. For every $x \in \mathbf{R}$ with $\mathcal{P}_x \neq \emptyset$ there exist two (uniquely determined) sets $B, C \subseteq H$ such that $\mathcal{P}_x = \{F \subseteq H : B \subseteq F \subseteq C\}$. The set B is the basis of all $F \in \mathcal{P}_x$.*

We call the set $\{F \subseteq H : B \subseteq F \subseteq C\}$ a *cube*, we use the notation $[B, C]$ for it, we call B the *bottom vertex* and C the *top vertex* of the cube $[B, C]$, and $|C \setminus B|$ is the *dimension* of the cube.

Proof. We choose $G \in \mathcal{P}_x$ arbitrarily, we let B be the basis of G, and we set

$$C = \Big\{h \in H : w(B) = w(B \cup \{h\})\Big\}.$$

We claim that this choice of B and C satisfies the desired conditions. First we check that $w(B) = w(C)$; this is a simple consequence of locality. Letting $C \setminus B = \{c_1, \ldots, c_m\}$, we check by induction that $w(B) = w(B \cup \{c_1, \ldots, c_i\})$,

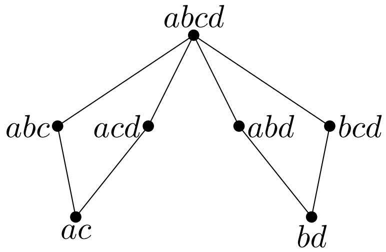

Figure 2: The poset $\mathcal{P}_{w_0(H_0)}$ for the square example.

$i = 0, 1, \ldots, m$. Indeed, the induction step from i to $i+1$ follows immediately from the locality axiom with $F = B$, $G = B \cup \{c_1, \ldots, c_i\}$, and $h = c_{i+1}$.

Now we know that $[B, C] \subseteq \mathcal{P}_x$, so let us assume $w(F) = w(B)$ for some $F \subseteq H$. Let B' be a basis of F; we have $w(B') = w(F) = w(B)$, and thus $B = B'$ by nondegeneracy. In particular, $B \subseteq F$. For every $f \in F$ we have $w(B) \leq w(B \cup \{f\}) \leq w(F) = w(B)$, so $w(B) = w(B \cup \{f\})$, and hence $f \in C$. Thus $\mathcal{P}_x \subseteq [B, C]$. The lemma is proved. $\square$

To see how this lemma can be used, let us check the claim made in the introduction: every nondegenerate refinement of the square example (H_0, w_0) has dimension at least 3. The poset $\mathcal{P}_{w_0(H_0)}$ of all subsets of H_0 with the same smallest enclosing circle as that of H_0 consists of all subsets of $\{a, b, c, d\}$ containing $\{a, c\}$ or $\{b, d\}$; see Fig. 2.

In any nondegenerate refinement, $\mathcal{P}_{w_0(H_0)}$ has to be expressed as a disjoint union of cubes, and if the dimension of the refinement were 2, all of these cubes would have to have a 2-element set as the bottom vertex. Such a covering is obviously impossible, since in order to cover $\{a, b, c, d\}$, we have to use a 2-dimensional cube, say $[\{a, c\}, \{a, b, c, d\}]$, and any covering of the remaining sets $\{b, d\}$, $\{a, b, d\}$, and $\{b, c, d\}$ by disjoint cubes must use at least one of the 0-dimensional (single-vertex) cubes $[\{a, b, d\}, \{a, b, d\}]$ and $[\{b, c, d\}, \{b, c, d\}]$.

3 The construction

We begin by defining a binary operation on LP-type problems.

Definition 3.1 *Let (H_1, w_1) and (H_2, w_2) be LP-type problems, and assume $H_1 \cap H_2 = \emptyset$. We define a new LP-type problem, denoted by $(H, w) = (H_1, w_1) * (H_2, w_2)$ and called the* join[2] *of (H_1, w_1) and (H_2, w_2): $H := H_1 \cup H_2$ and $w(G) := w_1(G \cap H_1) + w_2(G \cap H_2)$ for all $G \subseteq H$.*

Lemma 3.2 *The join $(H, w) = (H_1, w_1) * (H_2, w_2)$ is indeed an LP-type problem, and $\dim(H, w) = \dim(H_1, w_1) + \dim(H_2, w_2)$.*

Proof. First we observe that if $F \subseteq G$ and $w(F) = w(G)$, then $w_i(F \cap H_i) = w_i(G \cap H_i)$, $i = 1, 2$. Indeed, since $F \cap H_i \subseteq G \cap H_i$, we have $w_i(F \cap H_i) \leq w_i(G \cap H_i)$, and to get equality of the sum, equality must hold in both components.

Now we verify the axioms for (H, w). Monotonicity is obvious, and for locality, let $F \subseteq G \subseteq H$ and $h \in H$ satisfy $w(F) = w(G) = w(F \cup \{h\})$. Supposing $h \in H_1$, we have $w_1(F \cap H_1) = w_1(G \cap H_1) = w_1((F \cap H_1) \cup \{h\})$ by the observation above, and locality in (H_1, w_1) yields $w_1((G \cap H_1) \cup \{h\}) = w_1(G)$. Then $w(G \cup \{h\}) = w_1((G \cap H_1) \cup \{h\}) + w_2(G \cap H_2) = w_1(G \cap H_1) + w_2(G \cap H_2) = w(G)$, so (H, w) is indeed an LP-type problem.

Now we check $\dim(H, w) \geq \dim(H_1, w_1) + \dim(H_2, w_2)$. Let B_i be a basis in (H_i, w_i) witnessing $\dim(H_i, w_i)$. It suffices to check that $B = B_1 \cup B_2$ is a basis in (H, w); that is, $w(A) < w(B)$ for every proper subset of B. Letting $A_i = A \cap H_i$, we have $A_i \subseteq B_i$ with at least one of the inclusions proper, say $A_1 \subset B_1$. Since B_1 is a basis, we have $w_1(A_1) < w_1(B_1)$ and $w(A) < w(B)$ follows.

For the opposite inequality $\dim(H, w) \leq \dim(H_1, w_1) + \dim(H_2, w_2)$, we choose a basis B in (H, w) with $|B| = \dim(H, w)$ and set $B_i = B \cap H_i$. It suffices to check that B_i is a basis in (H_i, w_i). Let us consider a proper subset $A_1 \subset B_1$; then $w_1(B_1) + w_2(B_2) = w(B_1 \cup B_2) > w(A_1 \cup B_2) = w_1(A_1) + w_2(B_2)$, and we get $w_1(A_1) < w_1(B_1)$ as needed. The lemma is proved. $\square$

The example. For the proof of Theorem 1.5 we define, for a natural number m, an LP-type problem $\mathcal{L}_m$ as the m-fold join of the square ex-

[2]The definition is, in some sense, "non-canonical", since by changing w_1 and w_2 in a way that is immaterial for the combinatorial structure of (H_1, w_1) and (H_2, w_2) we might change the combinatorial structure of the join. This is not important for our purposes. However, for a cleaner definition we could set $w(G) := \varphi(w_1(G \cap H_1), w_2(G \cap H_2))$, where $\varphi\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a suitable function that is strictly increasing in both arguments and "generic" in the sense that for all $G_1, G_1' \subseteq H_1$ and $G_2, G_2' \subseteq H_2$ we have $\varphi(w_1(G_1), w_2(G_2)) = \varphi(w_1(G_1'), w_2(G_2'))$ only if $w_1(G_1) = w_1(G_1')$ and $w_2(G_2) = w_2(G_2')$. This would lead to the "most nondegenerate" join.

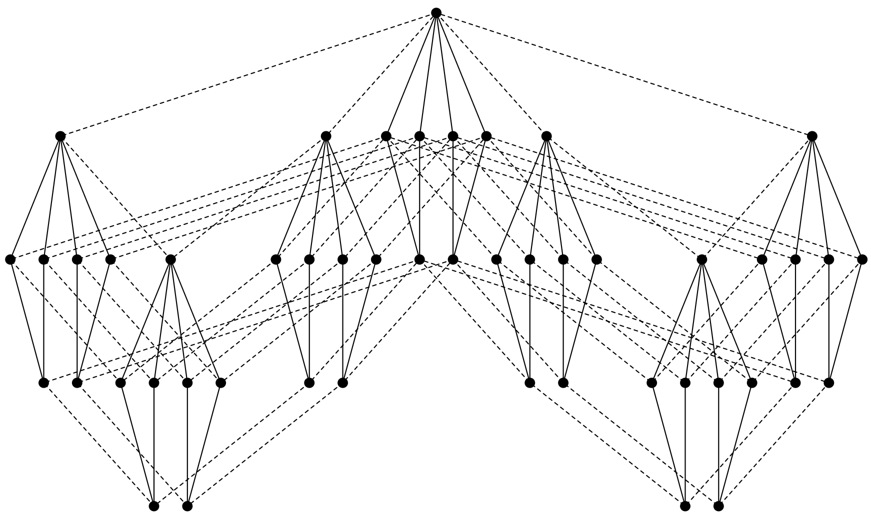

Figure 3: The poset $\mathcal{P}_{w(H)}$ for $m = 2$.

ample (H_0, w_0). More formally, we choose distinct elements $a_1, a_2, \ldots, a_m$, $b_1, \ldots, b_m$, $c_1, \ldots, c_m$, $d_1, \ldots, d_m$, we let $H_i = \{a_i, b_i, c_i, d_i\}$, and we let $w_i\colon H_i \to \mathbf{R}$ be a "copy" of the value function w_0 from the square example, defined on H_i. We let $\mathcal{L}_m = (H, w) = (H_1, w_1) * (H_2, w_2) * \cdots * (H_m, w_m)$. We have $|H| = 4m$ and by the above lemma, $\mathcal{L}_m$ is an LP-type problem of combinatorial dimension $D = 2m$.

We want to bound from below the dimension of any nondegenerate refinement of $\mathcal{L}_m$. Similar to the warm-up argument for (H_0, w_0), any nondegenerate refinement $\mathcal{L}' = (H, w')$ of $\mathcal{L}_m$ of dimension D' yields a covering of the poset $\mathcal{P}_{w(H)} = \{G \subseteq H : w(G) = w(H)\}$ by disjoint cubes $[B_j, C_j]$, where each bottom vertex B_j satisfies $|B_j| \leq D'$. We will deal with this combinatorial problem in the next two sections.

The case $m = 2$. The 4-dimensional LP-type problem $\mathcal{L}_2$ is analyzed in [Ško06], and it is shown that every nondegenerate refinement has dimension at least 6. The corresponding poset $\mathcal{P}_{w(H)}$ is illustrated in Fig. 3. Interestingly, this $\mathcal{P}_{w(H)}$ does admit a cover by disjoint cubes with bottom vertices of cardinality at most 5; see Fig. 4. However, the covers corresponding to a nondegenerate refinement have to satisfy additional condition,

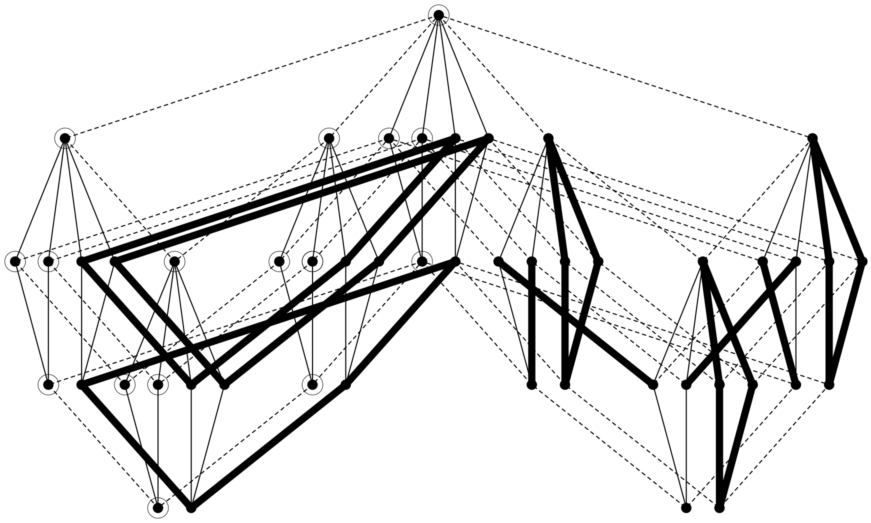

Figure 4: A covering of $\mathcal{P}_{w(H)}$ by disjoint cubes; a 4-dimensional cube is marked by circles around its vertices.

called *acyclicity*, and a case analysis in [Ško06] verifies that every acyclic cover must have a bottom vertex of cardinality 6 or larger. Here we won't define acyclicity; we just remark that arbitrary covers by disjoint cubes correspond to nondegenerate *violator spaces*, which is a generalization of LP-type problems investigated in [GMRŠ06]. One can thus say that $\mathcal{L}_2$ has a 5-dimensional nondegenerate refinement in the realm of violator spaces, but not in the realm of LP-type problems. On the other hand, the subsequent proof of Theorem 1.5 doesn't use acyclicity in any way and thus it applies equally well to violator spaces.

4 Setting up a linear system

The basic strategy for the proof of Theorem 1.5 is simple. Let $\mathcal{L}_m = (H, w)$ be the example constructed above and let us suppose that the poset $\mathcal{P} := \mathcal{P}_{w(H)} \subset 2^H$ can be covered by disjoint cubes $[B_j, C_j]$ with $|B_j| \leq D'$. Since $\dim(H, w) = D = 2m$, we have $2m \leq |B_j| \leq |C_j| \leq |H| = 4m$ for all j. Let

$x_{d,k}$ denote the number of cubes with $|B_j| = 2m + d$ and $|C_j| = 2m + k$, $d \le \Delta := D' - 2m$, $d \le k \le 2m$. A cube $[B_j, C_j]$ with $|B_j| = 2m + d$ and $|C_j| = 2m + k$ contains sets of cardinality $2m + \ell$, $d \le \ell \le k$, and the number of sets of this cardinality in $[B_j, C_j]$ equals $\binom{k-d}{\ell-d}$ (this formula is actually valid for all ℓ if we adopt the convention that $\binom{a}{b} = 0$ for $b < 0$ or $b > a$). If we let

$$F(m, \ell) = |\{G \in \mathcal{P} : |G| = 2m + \ell\}|,$$

we get that the $x_{k,d}$ have to satisfy the following system of linear equations:

$$\sum_{d=0}^{\Delta} \sum_{k=\max(d,\ell)}^{2m} \binom{k-d}{\ell-d} x_{d,k} = F(m, \ell), \qquad \ell = 0, 1, \ldots, 2m. \tag{1}$$

We are going to prove that with $\Delta = \lceil \varepsilon D \rceil$, where ε is a sufficiently small positive constant, this system of equations for variables $x_{k,d}$ has no *nonnegative real* solution, provided that m is sufficiently large. First we evaluate $F(m, \ell)$.

Lemma 4.1 *We have*

$$F(m, \ell) = \sum_{s} \binom{m}{s, \ell - 2s, m - \ell + s} 2^{m+\ell-3s},$$

with the sum being over all s with $0 \le 2s \le \ell$ and $s \ge \ell - m$ (here $\binom{n}{k_1,k_2,k_3} = \frac{n!}{k_1!k_2!k_3!}$ is the multinomial coefficient, $k_1 + k_2 + k_3 = n$).

Proof. First we observe, reasoning as in the proof of Lemma 3.2, that a set $B \subseteq H$ is a basis of H in $\mathcal{L}_m$ if and only if each $B_i = B \cap H_i$ is a basis of H_i in (H_i, w_i). Hence the bases of H are the sets B with $B \cap H_i = \{a_i, c_i\}$ or $B \cap H_i = \{b_i, d_i\}$ for all $i = 1, 2, \ldots, m$. A set $G \subseteq H$ is in $\mathcal{P}$ iff it contains at least one of these bases; i.e., if it contains at least one of the pairs $\{a_i, c_i\}, \{b_i, d_i\}$ for all i.

For $G \in \mathcal{P}$ of cardinality $2m + \ell$ let $s_r = |\{i \in \{1, 2, \ldots, m\} : |G \cap H_i| = r\}|$, $r = 2, 3, 4$. We have $s_2 + s_3 + s_4 = m$ and $2s_2 + 3s_3 + 4s_4 = |G| = 2m + \ell$. Calculation shows that $s_2 = m - \ell + s_4$ and $s_3 = \ell - 2s_4$.

For counting the number of possible ways of choosing G, we first fix $s = s_4$. Then s_2 and s_3 are fixed as well, and there are $\binom{m}{s_2,s_3,s_4}$ ways to choose the indices i contributing to each s_r (in other words, to choose which are the H_i where G takes 2, 3, or 4 elements, respectively). Knowing that $|G \cap H_i| = 2$, there are two possibilities for $G \cap H_i$, for $|G \cap H_i| = 3$ we have 4 possibilities, and for $|G \cap H_i|$ there is just one possibility. Therefore, once $|G \cap H_i|$ has been fixed for all i, there are $2^{s_2} \cdot 4^{s_3} = 2^{m+\ell-3s_4}$ possibilities for G. Summation over $s = s_4$ yields the statement of the lemma (the conditions on the range of s in the summation correspond to the obvious restrictions $s_2, s_3, s_4 \geq 0$) $\square$

5 Unsolvability of the linear system

We recall that for finishing the proof of Theorem 1.5, it suffices to show that for $\Delta := \lceil 2\varepsilon m \rceil$ and m sufficiently large, the linear system (1) has no nonnegative solution $x = (x_{d,k})_{d=0}^{\Delta}{}_{k=d}^{2m}$.

Before starting with the formal proof, which is a sequence of somewhat frightening calculations, we say a few words about how it was found. We started by testing the solvability for concrete values of parameters via linear programming. We used the function `LinearProgramming` in Mathematica, which uses arbitrary precision arithmetic and computes the solution exactly; this allowed us to deal with m up to about 1000 (other LP solvers we tried failed for large instances because of insufficient accuracy). By the Farkas lemma, the unsolvability is always witnessed by a linear combination of the equations that has nonnegative coefficients on the left-hand side and negative right-hand side. By minimizing the sum of absolute values of (suitably normalized) coefficients providing such a linear combination, we found that the unsolvability was witnessed, in all examples we tried, by a linear combination of only 3 of the equations. For simplifying the analytic approach, we then tried 3 *consecutive* equations, and found that such combinations work as well, provided that the index of the middle equation is chosen in a suitable range. These numerical results encouraged us to try finer and finer estimates, until we finally reached the following proof.

Proof of the unsolvability of (1). We set, somewhat arbitrarily, $t = \frac{1}{2}m$, assuming m even (we suspect that $t = \tau m$ for any fixed $\tau \in (0, 1)$ would work, but we haven't checked). We will show that for sufficiently large m already the system of the three consecutive equations with $\ell = t-1$, t, and $t+1$ has no nonnegative solution. To this end, we find a linear combinations of these three equations, with suitable coefficients α, β, γ, such that the resulting equation has all coefficients on the left-hand side nonnegative, while the right-hand side is strictly negative. We normalize the coefficients so that $\beta = -1$. Then, explicitly, we need the following:

$$\alpha\binom{k-d}{t-d-1} - \binom{k-d}{t-d} + \gamma\binom{k-d}{t-d+1} \geq 0, \quad 0 \leq d \leq \Delta \tag{2}$$

and

$$\alpha F(m,t-1) - F(m,t) + \gamma F(m,t+1) < 0. \tag{3}$$

In (2), a binomial coefficient $\binom{a}{b}$ is to be interpreted as 0 for $b > a$.

The choice of suitable α and γ turns out to be surprisingly subtle. Namely, we need to choose $\alpha = \alpha_0 + \alpha_1/t$ and $\gamma = \gamma_0 + \gamma_1/t$, where

$$\alpha_0 = \frac{\sqrt{10}+2}{4} \approx 1.29057, \quad \gamma_0 = \frac{\sqrt{10}-2}{6} \approx 0.193713$$

are uniquely determined real constants and α_1, γ_1 are constants in certain ranges. For concreteness we set $\alpha_1 = 1$ and $\gamma_1 = \frac{1}{8}$.

We get (2) from the following lemma:

Lemma 5.1 *There is a positive constant $\varepsilon > 0$ such that, with this choice of t, α, and γ, Equation (2) holds for all $d \leq \Delta = \lceil 2\varepsilon m \rceil$ and for all $k \geq d$, provided that m, and hence t, are sufficiently large.*

Proof. We use the substitution $y = k - d$ and $x = t - d$. We thus want to show $\alpha\binom{y}{x-1} - \binom{y}{x} + \gamma\binom{y}{x+1} \geq 0$.

For $y < x - 1$ all three terms are 0, and so we may assume $y \geq x - 1$. We rewrite the left-hand side to

$$\frac{y!}{(x+1)!(y-x+1)!}\Big(\alpha x(x+1) - (x+1)(y-x+1) + \gamma(y-x+1)(y-x)\Big).$$

Let us denote by $f(\alpha, \gamma, y, x)$ the expression in parentheses; we want to show that it is nonnegative.

Let us choose constants $\alpha_1' < \alpha_1$ and $\gamma_1' < \gamma_1$. Assuming ε in the lemma sufficiently small, we have d sufficiently small compared to x, and hence $\alpha = \alpha_0 + \alpha_1/(x+d) \geq \alpha_0 + \alpha_1'/x$ and $\gamma = \gamma_0 + \gamma_1/(x+d) \geq \gamma_0 + \gamma_1'/x$.

Since f is increasing in α and in γ (for the relevant y and x), it suffices to check that

$$f(\alpha_0 + \frac{\alpha_1'}{x}, \gamma_0 + \frac{\gamma_1'}{x}, y, x) \geq 0,$$

and we will verify this for all sufficiently large *real* x and all real y. One of the properties of α_0 and γ_0 needed here is $\alpha_0\gamma_0 = \frac{1}{4}$. Things can be simplified a little by the substitution $y = x(z+1)$. Then $f(\alpha_0 + \frac{\alpha_1'}{x}, \gamma_0 + \frac{\gamma_1'}{x}, x(z+1), x)$ is a polynomial in x and z. For x fixed it is a quadratic polynomial in z, and the coefficient of z^2 is $x^2/4\alpha_0 + \gamma_1' x > 0$ (this calculation and the subsequent ones were done using Mathematica). Therefore, it has a unique minimum, which can be found by setting the first derivative (according to z) to 0. This minimum occurs at

$$z_0 = z_0(x) = \frac{x^2 + (1-\gamma_0)x - \gamma_1'}{2x(\gamma_0 x + \gamma_1')}$$

(the expression was simplified using $\alpha_0\gamma_0 = \frac{1}{4}$). Substituting this into $f(\alpha_0 + \frac{\alpha_1'}{x}, \gamma_0 + \frac{\gamma_1'}{x}, z(x+1), x)$ yields a function of x of the form

$$\frac{-\gamma_0 - 2\gamma_0^2 + 4\alpha_1'\gamma_0^2 + \gamma_1'}{4\gamma_0^2} x + O(1),$$

the $O(.)$ notation referring to $x \to \infty$. Calculation shows that the coefficient of x is a positive real number (for α_1' and γ_1' sufficiently close to α_1 and γ_1, respectively). Hence f is indeed positive for the considered values of the variables. $\square$

Remark. It is easy to check that if α, γ are positive *constants*, then the inequality $f(\alpha, \gamma, x, y) \geq 0$ holds for all y and all sufficiently large x if and only if $\alpha\gamma > \frac{1}{4}$. However, for such α and γ (3) fails. We are thus forced to choose α and γ depending on x such that $\alpha\gamma \to \frac{1}{4}$ as $x \to \infty$.

We now proceed to establish (3). We set

$$Q(m,t,s) = \binom{m}{s, t-2s, m-t+s} 2^{m+t-3s},$$

so that $F(m,t) = \sum_s Q(m,t,s)$. First we look for the s maximizing $Q(m,t,s)$. Let

$$r(m,t,s) = \frac{Q(m,t,s)}{Q(m,t,s-1)} = \frac{(t-2s+1)(t-2s+2)}{8s(m-t+s)}$$

be the ratio of two consecutive terms. As a function of s it is decreasing, and so $Q(m,t,s)$ is maximum for the largest s with $r(m,t,s) \geq 1$.

We stick to our choice $t = \frac{1}{2}m$. It is more convenient to use t as a parameter; let us write $\tilde{r}(t,s) = r(2t,t,s)$ and $\tilde{Q}(t,s) = Q(2t,t,s)$, and let us note that $m - t = t$. If we let $\sigma = (\sqrt{10}-3)/2 \approx 0.0811388$ be the positive root of the equation $(1-2\sigma)^2 = 8\sigma(1+\sigma)$ and $s_0 = \lfloor \sigma t \rfloor$, then $\tilde{r}(t, s_0) = 1 + O(t^{-1})$. Next, we need an estimate on the rate of decrease of $\tilde{Q}(t, s_0 + a)$ as $|a|$ increases.

Lemma 5.2 *Let $c_0 = \frac{4}{1-2\sigma} + \frac{1}{\sigma} + \frac{1}{1+\sigma} \approx 18.0244$. Suppose that $a = o(t^{2/3})$. Then*

$$\frac{\tilde{Q}(t, s_0 + a)}{\tilde{Q}(t, s_0)} = (1 + o(1))e^{-c_0 a^2/2t},$$

where $o(.)$ refers to $t \to \infty$ and the convergence is uniform in a.

Proof. We will be summing over $j = 1, 2, \ldots, a$ in the proof. Let us write $\xi = j/t$; thus $\xi = o(1)$. We have

$$\begin{aligned}
\tilde{r}(t, s_0 + j) &= (1 + O(t^{-1}))\frac{\tilde{r}(t, s_0 + j)}{\tilde{r}(t, s_0)} \\
&= (1 + O(t^{-1}))\frac{\left(1 - \frac{2j}{t-2s_0+1}\right)\left(1 - \frac{2j}{t-2s_0+2}\right)}{\left(1 + \frac{j}{s_0}\right)\left(1 + \frac{j}{t+s_0}\right)} \\
&= (1 + O(t^{-1}))\frac{\left(1 - \frac{2}{1-2\sigma}\xi\right)^2}{\left(1 + \frac{1}{\sigma}\xi\right)\left(1 + \frac{1}{1+\sigma}\xi\right)} \\
&= (1 + O(t^{-1}) + O(\xi^2))e^{-c_0\xi}.
\end{aligned}$$

Then

$$\begin{aligned}
\ln\frac{\tilde{Q}(t, s_0 + a)}{\tilde{Q}(t, s_0)} &= \sum_{j=1}^{a} \ln \tilde{r}(t, s_0 + j) \\
&= \left(\sum_{j=1}^{a} -\frac{c_0 j}{t}\right) + O\left(\frac{a}{t}\right) + O\left(\frac{a^3}{t^2}\right) \\
&= -\frac{c_0 a^2}{2t} + O\left(\frac{a}{t} + \frac{a^3}{t^2}\right).
\end{aligned}$$

The lemma follows. □

Next, we consider the expression $\tilde{D}(t, s) = \alpha Q(m, t-1, s) - Q(m, t, s) + \gamma Q(m, t+1, s)$ with $m = 2t$, $\alpha = \alpha_0 + \alpha_1/t$, and $\gamma = \gamma_0 + \gamma_1/t$ as above.

The idea is to show that for s close to s_0 we have $\tilde{D}(t,s)$ negative, while for s further from s_0 it can be positive but it is sufficiently small compared to $-\tilde{D}(t,s_0)$. Again, the calculation has to be done rather precisely in order to work.

Lemma 5.3 *Let us suppose that $a = o(t)$, and let $s_0 = \lfloor \sigma t \rfloor$ be as above. Then*

$$\tilde{D}(t, s_0 + a) = \tilde{Q}(t, s_0 + a) \cdot (1 + o(1)) \frac{C}{t} \left(\frac{c_1}{t} a^2 - 1 + o(1) \right),$$

where C is a certain positive constant whose value will not be important, $c_1 = (14584\sqrt{10} + 46192)/5877 \approx 15.70710522$, the $o(.)$ notation refers to $t \to \infty$, and the convergence is uniform in a.

Proof. Similar to the proof of Lemma 5.1 we rewrite

$$\tilde{D}(t,s) = \tilde{Q}(t,s) \cdot \frac{1}{2(t+1-2s)(t+s+1)} g(\alpha, \gamma, t, s),$$

with $g(\alpha,\gamma,t,s) = \alpha(t-2s+1)(t-2s) - 2(t-2s+1)(t+s+1) + 4\gamma(t+s)(t+s+1)$. With the constant σ as above, $g(\alpha_0 + \alpha_1/t, \gamma_0 + \gamma_1/t, t, \sigma t)$ becomes a polynomial in t, which is a priori quadratic, but the constants α_0 and γ_0 are chosen so that the coefficient at t^2, which equals $14 - 5\sqrt{10} + (26 - 8\sqrt{10})\alpha_0 + (11 - 2\sqrt{10})\gamma_0$, vanishes. (This, together with $\alpha_0\gamma_0 = \frac{1}{4}$, are the two conditions that uniquely determine α_0 and γ_0.) The coefficient of the linear term equals $-c_2 = (191 - 62\sqrt{10})/8 \approx -0.632652$ (and thus $g(\alpha,\gamma,t,s)$ is indeed negative and of order t for s sufficiently near to σt).

More quantitatively, expanding and simplifying gives

$$g(\alpha_0 + \alpha_1/t, \gamma_0 + \gamma_1/t, t, \sigma t + b) = -c_2 t + c_3 b^2 + O(b^2/t + b + 1)$$

with $c_3 = (14 + 5\sqrt{10})/3$. For $a = b + \sigma t - s_0 = b + \sigma t - \lfloor \sigma t \rfloor \le b + 1$ we then obtain

$$g(\alpha_0 + \alpha_1/t, \gamma_0 + \gamma_1/t, t, s_0 + a) = -c_2 t + c_3 a^2 + O(a^2/t + a + 1).$$

Therefore, using $a = o(t)$, we arrive at

$$\begin{aligned} \tilde{D}(t, s_0 + a) &= \tilde{Q}(t, s_0 + a) \cdot \frac{-c_2 t + c_3 a^2 + O(a^2/t + a + 1)}{2(t+1-2s)(t+s+1)} \\ &= \tilde{Q}(t, s_0 + a) \cdot \frac{C}{t} \left(\frac{c_3 a^2}{c_2 t} - 1 + o(1) \right) \end{aligned}$$

as required. $\square$

We are ready to prove (3). For our choice of α, γ, and t we have

$$\alpha F(m,t-1) - F(m,t) + \gamma F(m,t+1) = \sum_a \tilde{D}(t, s_0 + a).$$

For concreteness let us set $a_0 = t^{3/5}$. We will show that

$$\sum_{|a|\le a_0} \tilde{D}(t, s_0+a) \le -\frac{\delta}{\sqrt{t}}\,\tilde{Q}(t, s_0)$$

for a constant $\delta > 0$. Now for $a > a_0$ we have $|\tilde{D}(t, s_0+a)| \le 3\tilde{Q}(t, s_0+a) \le 3\tilde{Q}(t, s_0+a_0)$, and the last expression is smaller that $\tilde{Q}(t, s_0)$ by a factor exponential in t. A similar argument applies for $a < -a_0$ and thus the sum over $|a| > a_0$ is negligible.

Combining Lemmas 5.2 and 5.3, we get that for $|a| \le a_0$ we have

$$\tilde{D}(t, s_0+a) = \tilde{Q}(t, s_0)\frac{C}{t}(1+\phi_t(a))e^{-c_0a^2/2t}\left(\frac{c_1a^2}{t} - 1 + \psi_t(a)\right),$$

where $\phi_t(a)$ and $\psi_t(a)$ are some functions converging to 0 as $t \to \infty$, uniformly in a.

We will show that

$$\sum_{|a|\le a_0} (1+\phi_t(a))e^{-c_0a^2/2t}\left(1 - \frac{c_1a^2}{t} - \psi_t(a)\right) = \Omega(\sqrt{t}\,). \tag{4}$$

Let us fix an arbitrarily small $\nu > 0$ and let us assume that t has been chosen so large that $|\phi_t(a)| \le \nu$, $|\psi_t(a)| \le \nu$ for all a. Then the left-hand side of (4) is bounded from below by

$$\sum_{|a|\le a_0} e^{-c_0a^2/2t}(1 - c_1a^2/t) -$$

$$\sum_{|a|\le a_0} (1+|\phi_t(a)|)e^{-c_0a^2/2t}|\psi_t(a)| - \sum_{|a|\le a_0} |\phi_t(a)|e^{-c_0a^2/2t}$$

$$\ge \sum_{|a|\le a_0} e^{-c_0a^2/2t}\left(1 - \frac{c_1a^2}{t}\right) - 3\nu \sum_{|a|\le a_0} e^{-c_0a^2/2t}.$$

By basic properties of Riemann integration and uniform continuity arguments is routine to check that both of these sums converge to the corresponding integrals as $t \to \infty$. So it suffices to bound from below

$$(1-3\nu)\int_{-a_0}^{a_0} e^{-c_0 a^2/2t}\,\mathrm{d}a - \frac{c_1}{t}\int_{-a_0}^{a_0} a^2 e^{-c_0 a^2/2t}\,\mathrm{d}a.$$

Since $a_0^2/t = t^{1/5} \to \infty$ as $t \to \infty$ and the integrands decrease exponentially in a^2/t, we make only a negligible error by taking both integrals from $-\infty$ to ∞. We have

$$(1-3\nu)\int_{-\infty}^{\infty} e^{-c_0 a^2/2t}\,\mathrm{d}a = (1-3\nu)\sqrt{2\pi t/c_0} \approx 0.590419\sqrt{t}$$

while

$$\frac{c_1}{t}\int_{-\infty}^{\infty} a^2 e^{-c_0 a^2/2t}\,\mathrm{d}a = c_1\sqrt{2\pi}c_0^{-3/2}\sqrt{t} \approx 0.514513\sqrt{t}.$$

This finally proves (3).

6 A geometric representation by a linear program

It turns out that an LP-type problem $\hat{\mathcal{L}}_m = (H, \hat{w})$, which is similar to $\mathcal{L}_m$ and which can also be used as an example establishing Theorem 1.5, can be represented as a linear program. To see that our proof of Theorem 1.5 works for $\hat{\mathcal{L}}_m$ as well, it will be enough to verify that its poset $\mathcal{P}_{\hat{w}(H)}$ of maximum-weight sets is isomorphic to $\mathcal{P}_{w(H)}$ of $\mathcal{L}_m$, and this will follow from the discussion below.

We begin by setting up the following linear program with variables x, y, z ($\varepsilon > 0$ is a very small positive real number):

$$\begin{array}{llcl}
\multicolumn{4}{l}{\text{minimize } z + \varepsilon y + \varepsilon^2 x \text{ subject to}} \\
a: & x + 4y - 2z & \leq & 1 \\
b: & 3x + 8y + 2z & \leq & 5 \\
c: & 3x - 8y + 2z & \leq & -3 \\
d: & -x - 4y - 2z & \leq & -3 \\
 & x, y, z & \geq & 0.
\end{array}$$

The corresponding LP-type problem $(H_0, \hat{w}_0)$ has the set $H_0 = \{a, b, c, d\}$ of four constraints corresponding to the four inequalities of the linear program.

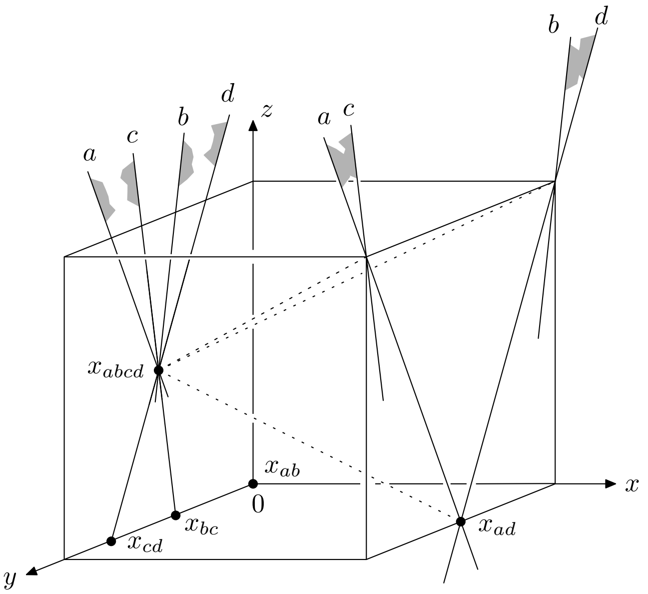

Figure 5: A linear program in $\mathbf{R}^3$ essentially representing the square example.

The value $\hat{w}_0(G)$ of any subset $G \subseteq H_0$ is the minimum of the linear program where the constraints of $H_0 \setminus G$ have been deleted (we stress that the implicit nonnegativity constraints $x, y, z \geq 0$ are always present, even for $G = \emptyset$). In this way, $\hat{w}_0(G)$ is well defined for every G.

The linear program is illustrated in Fig. 5. For better visualization, the picture shows the unit cube $[0,1]^3$, and intersections of the bounding planes of the constraints with the facets $x = 0$ and $x = 1$ of the cube. The minimum of the linear programs containing both the constraints a and c or both the constraints b and d is attained at the point $x_{abcd} = (0, \frac{1}{2}, \frac{1}{2})$; thus, $\hat{w}_0(H_0) = \frac{1}{2}$. It can be checked that for every subset G of constraints containing neither $\{a, c\}$ nor $\{b, d\}$, the minimum is attained at a point with $z = 0$, and thus with $\hat{w}_0 < \frac{1}{2}$ (the picture shows the minima for all G of cardinality 2). Thus $\hat{\mathcal{L}}$ is a 2-dimensional LP-type problem with the poset $\mathcal{P}_{\hat{w}_0(H_0)}$ isomorphic to $\mathcal{P}_{w_0(H_0)}$ for the square example.

Next, we observe that if (H, w) is an LP-type problem corresponding to a linear program with variables $x_1, \ldots, x_n$ and with objective $\min \sum c_i x_i$, and (H', w') is an LP-type problem corresponding to a linear program with variables $x'_1, \ldots, x'_n$ and with objective $\min \sum c'_i x'_i$, then the join $(H, w) * (H', w')$ corresponds to the linear program obtained by putting the constraints of both linear programs together and with objective $\min(\sum c_i x_i + \sum c'_i x'_i)$. Indeed, it suffices to check that the value function in $(H, w) * (H', w')$ coincides with the value function obtained from the combined linear program, and this is immediate. In particular, the m-fold join $\hat{\mathcal{L}}_m$ of m disjoint copies of $(H_0, \hat{w}_0)$ corresponds to the following linear program in $3m$ variables:

$$
\begin{array}{l}
\text{minimize } \sum_{i=1}^{m}(z_i + \varepsilon y_i + \varepsilon^2 x_i) \text{ subject to} \\
\left.\begin{array}{rcl}
x_i + 4y_i - 2z_i & \leq & 1 \\
3x_i + 8y_i + 2z_i & \leq & 5 \\
3x_i - 8y_i + 2z_i & \leq & -3 \\
-x_i - 4y_i - 2z_i & \leq & -3 \\
x_i, y_i, z_i & \geq & 0
\end{array}\right\} i = 1, 2, \ldots, m.
\end{array}
$$

We could have presented the example for Theorem 1.5 in this form, but we find the abstract construction of join more transparent.

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