Coloring Eulerian triangulations of the Klein bottle

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Abstract

We show that an Eulerian triangulation of the Klein bottle has chromatic number equal to six if and only if it contains a complete graph of order six, and it is 5-colorable, otherwise. As a consequence of our proof, we derive that every Eulerian triangulation of the Klein bottle with face-width at least four is 5-colorable.

1 Introduction

A graph is said to be *Eulerian* if all of its vertices have even degree. In this paper we study colorings of *Eulerian triangulations*. These are triangulations of closed surfaces whose graph is Eulerian.

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The celebrated Four Color Theorem [2, 25] asserts that every planar graph, in particular, every triangulation of the plane, is 4-colorable. However, only a handful of planar triangulations are 3-colorable, and it is relatively easy to see that a planar triangulation is 3-colorable if and only if it is Eulerian [26].

Eulerian triangulations are interesting because of their tight connection to 3-colorability of planar graphs. ¿From the algorithmic point of view, it is trivial to decide whether a planar graph is 4-colorable, and an algorithm based on the proof of the Four Color Theorem [24] finds a 4-coloring of a given planar graph in quadratic time. On the other hand, it is NP-complete to decide whether a planar graph is 3-colorable [12]. Knowing this, it is surprising that a planar graph is 3-colorable if and only if it is a subgraph of an Eulerian triangulation. This fact has appeared in [16] and [18]. Other conditions for 3-colorability of plane graphs can be found e.g. in [7].

In this paper we study Eulerian triangulations of more general surfaces, in particular those of the Klein bottle. We refer the reader for a detailed introduction into graph embeddings on surfaces to a recent monograph [21] and we provide here only the basic results that we need in our further exposition.

Going off the plane, the situation concerning chromatic properties of Eulerian triangulations changes drastically. For example, the complete graph K_7 of order seven forms an Eulerian triangulation of the torus – and yet, it needs seven colors to be colored. Even more is true. It can be shown that every graph is a subgraph of some Eulerian triangulation, so the study of chromatic properties of Eulerian triangulations of arbitrary surfaces is as hard as the same problem for arbitrary graphs. However, there are two directions, where one can make some breakthrough. One is to study locally planar graphs, those with large edge-width (see [21] for more details), and the other possibility is to restrict to a fixed surface.

Eulerian triangulations of the projective plane are still well understood. A set C of vertices of a triangulation T is called a *color class* if every face of T has precisely one vertex in C. If C is a color class in the Eulerian triangulation T, then $T - U$ is an *even-faced map*, i.e. an embedded graph whose faces all have even size. Fisk [11] proved that any Eulerian triangulation of the projective plane contains a color class. Mohar [19] extended this result by proving:

Theorem 1 (Mohar [19]). *Let T be an Eulerian triangulation of the projective plane, and let C be a color class in T. Then the chromatic number*

$\chi(T)$ of T is equal to 3 if and only if $T - C$ is bipartite. If $T - C$ is non-bipartite and contains a subgraph which is a quadrangulation of the projective plane, then $\chi(T) = 5$; otherwise $\chi(T) = 4$. Moreover, given T, one can find a color class in T and determine $\chi(T)$ in polynomial time.

The proof of Theorem 1 is based on a discovery of Youngs [31] that quadrangulations of the projective plane are either bipartite or 4-chromatic, but their chromatic number is never equal to 3. It also uses an extension by Gimbel and Thomassen [13] who proved that a graph of girth at least 4 embedded in the projective plane is 3-colorable if and only if it does not contain a non-bipartite quadrangulation of the projective plane.

As a corollary of Theorem 1 we have:

Theorem 2. *Every Eulerian triangulation of the projective plane is 5-colorable.*

This result can be proved directly by first observing that every projective planar graph is 6-colorable. Next, Dirac's theorem [1, 8] asserts that the only obstacle for 5-colorability is the presence of a complete graph of order six: the chromatic number of a projective planar graph is six if and only if it contains a complete graph of order six as a subgraph. Finally, K_6 cannot be a subgraph of an Eulerian triangulation of the projective plane (this follows from Proposition 4).

There are some extensions of these results to more general surfaces. The strongest results in this area have been obtained by DeVos et al. [9]. Other results stem from the corresponding results about even-faced maps. Note that Eulerian triangulations can be obtained from even-faced maps by placing a single vertex into each face of it and joining it to all the other vertices incident with that face.

An important parameter that quantitatively measures local planarity of an embedded graph G is the length of a shortest non-contractible cycle in G, which is called the *edge-width* of G. As an example, let us mention the following result of Hutchinson [14] (cf. also [9] for a strengthening of this result). For every genus g, there exists an integer $r(g)$ such that every graph embedded on an orientable surface of genus g with all faces even and with edge-width at least $r(g)$ is 3-colorable. The claim trivially holds in the plane, since such a graph must be bipartite. For the torus, the original bound of 25 from [14] on $r(1)$ has been improved to 9 by Archdeacon et al. [3]. A recent improvement [17] of $r(1)$ to 6 is best possible since the Cayley graph $C(Z_{13}; 1, 5)$ has chromatic number four and can be embedded on the torus with edge-width five [3]. In fact, this Cayley graph is the only obstacle for a triangle-free even-faced map on the torus to be 3-colorable. The same phenomena appear for other classes of graphs: every Eulerian triangulation of an orientable surface that has sufficiently large edge-width is 4-colorable [14] and every graph embedded on a fixed surface (orientable or non-orientable) with sufficiently large edge-width is 5-colorable [29].

Note that the assumption in the results of Hutchinson [14] that a surface is orientable is essential since every non-orientable surface admits quadrangulations of arbitrarily large edge-width, whose chromatic number is four [3, 9, 20]. However, such non-3-colorable quadrangulations can be characterized [20].

Similarly, Eulerian triangulations on orientable surfaces having sufficiently large edge-width are 4-colorable as proved by Hutchinson, Richter, and Seymour [15]. On the other hand, every non-orientable surface has non-4-colorable Eulerian triangulations of arbitrarily large edge-width [3, 9], and they can also be characterized [22].

As colorings of Eulerian triangulations of the projective plane are well-understood (cf. Theorem 1), we focus on Eulerian triangulations of the Klein bottle, the next simplest non-orientable surface. Our goal is to characterize non-5-colorable Eulerian triangulations of the Klein bottle. In particular, we prove that an Eulerian triangulation of the Klein bottle is 5-colorable if and only if it does not contain the complete graph of order six as a subgraph. Let us mention at this point that graphs considered in this paper can have parallel edges as long as they do not form bigons. As a corollary we prove that every Eulerian triangulation of the Klein bottle with edge-width at least four is 5-colorable. It seems to be more difficult to establish analogous results for 4-colorable Eulerian triangulations of the Klein bottle since there are 5-chromatic Eulerian triangulations with arbitrarily large edge-width; in particular, the list of "forbidden" subgraphs in this case cannot be finite.

Our main result is the following:

Theorem 3. *An Eulerian triangulation G of the Klein bottle is* 5*-colorable if it does not contain a complete graph of order six. If G contains K_6, then it can be obtained from one of the maps T_A, T_B, T_C, T_D and T_E, which are depicted in Figures 1, 11, 14 and 16, by adding vertices and edges and its chromatic number is equal to* 6.

Theorem 3 is a combination of Theorems 28 and 31, which are proved in Sections 8 and 10, respectively.

2 Outline of the proof

Before we start proving our main result, we would like to explain the major steps of the proof. First, we realize that it is enough to prove our results for *proper* triangulations as defined in Section 3. In Section 3, we also define a special type of contraction that can be applied to vertices of degree four. This operation has the property that if it is applied to an Eulerian triangulation, the resulting triangulation is also Eulerian and if the resulting triangulation is 5-colorable, so is the original one. This leads us to the need to analyze Eulerian triangulations without contractible vertices.

Before we proceed with further analysis, we realize that unless the Eulerian triangulation of the Klein bottle contains a vertex of degree two, is 6-regular or of a very special type (all these cases are easy to handle), it contains a vertex of degree four adjacent to a vertex of degree six (Lemma 16). In Section 5, we then analyze Eulerian triangulation with a non-contractible vertex of degree four adjacent to a vertex of degree six and identify one particular configuration that requires a finer analysis—we call such a configuration the *resistant configuration* and the vertex of degree four involved in it a *resistant* vertex. Eulerian triangulations with the resistant configuration are then analyzed in Section 6. All these results are combined in Section 7 where we prove that every minimal (under our operation of the contraction) non-5-colorable Eulerian triangulation of the Klein bottle is isomorphic to one of the five triangulations that are denoted by T_A, T_B, T_C, T_D and T_E and are introduced later.

We observe that each of the five minimal triangulations contains the complete graph K_6 of order six as a subgraph. Moreover, there are only four non-isomorphic embeddings of K_6 in such triangulations. We call these embeddings *bad*. In Section 8, we show that they are the only embeddings of K_6 that can be contained in an Eulerian triangulation of the Klein bottle. In Section 9, we invert the operation of contraction and introduce *expansions* of vertices. We show that if an expansion of a vertex results in a triangulation with no complete graph of order six (and the original graph contained the complete graph of order six as a subgraph), then the obtained graph is 5-colorable. Hence, every Eulerian triangulation of the Klein bottle that is not 5-colorable must contain one of the four bad embeddings of K_6. We formally combine the just explained results in Section 10 in which we state and prove our main result (Theorem 31).

3 Preliminaries

In this section, we introduce notation used throughout this paper. Graphs which we consider have no loops but they can contain multiple edges. If G is a graph and W a subset of its vertices, then $G[W]$ is the subgraph of G induced by W. A *k-vertex* is a vertex of degree k. Similarly, a *k-cycle* is a cycle of length k. If G is embedded on a surface and 2-connected, then a *k-face* is a face bounded by a k-cycle.

Most of our proofs deal with graphs on surfaces. We refer to the monograph [21] for definitions related to graphs embedded on surfaces not mentioned here. If G is such a graph and $C = v_1 \ldots v_\ell$ is a non-contractible ℓ-cycle in G, we can cut the surface along C. This decreases the genus of the surface. In the obtained graph G', the cycle C is either replaced by two copies C' and C'' of the original cycle C or by a 2ℓ-cycle C'. In the former case, the vertices of C' will always be denoted by $v'_1 \ldots v'_\ell$ and the vertices of C'' by $v''_1 \ldots v''_\ell$. In the latter case, the vertices of C' are denoted by $v'_1 \ldots v'_\ell v''_1 \ldots v''_\ell$. The reader is referred for an example to Figure 5 where the Klein bottle was cut along a one-sided cycle vwu_0. Note that the star in the figure denotes a cross-cap. Let us now introduce some additional notation used in our figures. The regions of a graph that are faces are always filled with the white color and those that can contain additional vertices and edges of G with the gray color.

A *triangulation* of a surface is an embedding of a loopless graph such that each face is a 3-face and an *Eulerian triangulation* of the surface is a triangulation in which all vertices have even degrees. A triangulation is *proper* if it has no contractible cycles of length two, every contractible 3-cycle is facial, and every 4-cycle bounds a region that contains at most one vertex of G.

Besides Eulerian triangulations, we are also interested in near-triangulations and Eulerian near-triangulations. An embedding of a graph G in a surface is a *near-triangulation* if all its faces are 3-faces with a possible exception of a single *distinguished* face. An *Eulerian near-triangulation* is a near-triangulation such that all the vertices not incident with the distinguished face have even degrees.

An embedding of a graph G is a *defective Eulerian triangulation* if it is a triangulation and all the vertices of G have even degrees except for a single pair of adjacent vertices. Similarly, a *2-defective Eulerian triangulation* is a triangulation with all the vertices of even degrees except for either two pairs of adjacent vertices or two non-adjacent vertices at distance two.

Proposition 4. *There is no defective Eulerian triangulation of the plane.*

Proof. Suppose that G is a defective Eulerian triangulation of the plane and let v and w be two adjacent vertices of odd degree. Remove the edge vw from G and consider the dual graph G^* of G. Since the degrees of all the vertices of $G \setminus vw$ are even, G^* is bipartite. On the other hand, all its vertices have degree three except for a single vertex of degree four (which corresponds to the face from which the edge vw was removed). Clearly, such a graph cannot exist. $\square$

On the other hand, defective Eulerian triangulations of the projective plane exist. In the next lemma, we show that defective and 2-defective Eulerian triangulations of the projective plane are 5-colorable.

Lemma 5. *Every defective or* 2*-defective Eulerian triangulation of the projective plane is* 5*-colorable.*

Proof. Consider a defective or a 2-defective triangulation G of the projective plane that is not 5-colorable. By Dirac's Theorem [1, 8], G contains a complete graph of order six as a subgraph. Let H be such a subgraph of G. Note that the complete graph of order six has a unique embedding in the projective plane and this embedding is a triangulation.

If G is a defective triangulation, color the edge joining the vertices of odd degree red. If G is 2-defective, color the two edges joining vertices of odd degree red or color a two-edge path between two such vertices red. Note that G contains at most two red edges and a vertex of G has odd degree if and only if it is incident with exactly one red edge.

We now observe that it can be assumed without loss of generality that G has no separating contractible 3-cycle with all inner vertices of even degree. Assume that T is a separating 3-cycle of G and let G_T be the subgraph of G formed by the vertices and edges lying inside the closed disc bounded by T. By Proposition 4, the degrees of all the vertices of G_T are even. Hence, removing the interior of T yields a defective or a 2-defective triangulation G' of the projective plane. Clearly, G is 5-colorable if and only if G' is.

Since each facial cycle of H bounds a 2-cell, each face of H either bounds a face in G or contains a vertex of odd degree (and thus a red edge) in its interior. On the other hand, every vertex v of H is incident either with a red edge (that can be an edge of G contained in H) or with a face containing a red edge in its interior. Otherwise, v would have odd degree and would be incident with no red edges. Since H contains six vertices and G contains at most two red edges, there are exactly two non-empty faces of H each containing a single red edge. Moreover, these faces of H are vertex-disjoint. However, there are no two disjoint facial triangles in the unique embedding of K_6 in the projective plane. This contradiction completes the proof. □

Next, we establish a lemma on the existence of special 5-colorings in plane graphs. This lemma straightforwardly follows from the Four Color Theorem, but we decided to provide a proof independent of it.

Lemma 6. *Let G be a plane graph. For every two non-adjacent vertices w_1 and w_2 of G, G has a 5-coloring that assigns the vertices w_1 and w_2 the same color.*

Proof. Consider a counterexample G of the smallest size. Clearly, such a graph G is connected and simple. By Euler's formula, G has at least three vertices of degree at most five. In particular, G has a vertex w of degree at most five that is neither w_1 and w_2. By the choice of G, $G - w$ has a 5-coloring that assigns the vertices w_1 and w_2 the same color. If the neighbors of w in G are colored with at most four distinct colors, the coloring can be extended to G which is impossible. Hence, the degree of w is five and its five neighbors $v_1, \dots, v_5$ are colored with mutually distinct colors. By symmetry, we can assume that v_5 is colored with the color of w_1 and w_2. Let G_{ij} be the subgraph of G induced by the vertices colored with the color of v_i or v_j, $i, j \in \{1, \dots, 4\}$. Since G is plane, the vertices v_1 and v_3 are not contained in the same component of G_{13} or the vertices v_2 and v_4 are not contained in the same component of G_{24}. By symmetry, we can assume that the former is the case, and switch the two colors used in the component of G_{13} that contains v_3. In this way, we obtain a 5-coloring of G that assigns the vertices w_1 and w_2 the same color and such that the neighbors of w are colored with at most four distinct colors. Such a coloring can be extended to w. We conclude that G is not a counterexample to the statement of the lemma and thus there is no counterexample at all. □

We next focus our attention to Eulerian near-triangulations and show that if the size of the distinguished face is small and G is embedded in the plane, then the parities of degrees of the vertices incident with the distinguished face are very restricted.

Proposition 7. *If G is an Eulerian near-triangulation of the plane with the distinguished face bounded by a two-cycle v_1v_2, then the degrees of both v_1 and v_2 are odd.*

Proof. By the hand-shaking lemma, the parities of the degrees of v_1 and v_2 are the same. If they are both even, we obtain by removing one of the two parallel edges v_1v_2 a defective plane Eulerian triangulation which is impossible by Proposition 4. □

Proposition 8. *If G is an Eulerian near-triangulation of the plane with the distinguished face bounded by a three-cycle $v_1v_2v_3$, then the degrees of v_1, v_2 and v_3 are even.*

Proof. By the hand-shaking lemma, either the degrees of all the vertices v_1, v_2 and v_3 are all even or exactly two of them have odd degree. In the latter case, G is a defective Eulerian triangulation of the plane which is impossible by Proposition 4. □

Arguments similar to those used in the proofs of Propositions 7 and 8 yield the proofs of the next two propositions. We have decided not to include their (short) proofs.

Proposition 9. *If G is an Eulerian near-triangulation of the plane with the distinguished face bounded by a four-cycle $v_1v_2v_3v_4$, then one of the following holds:*

- *the degrees of all the vertices v_1, v_2, v_3 and v_4 are odd,*
- *the degrees of v_1 and v_3 are even, and those of v_2 and v_4 are odd, or*
- *the degrees of v_1 and v_3 are odd, and those of v_2 and v_4 are even.*

Proposition 10. *If G is an Eulerian near-triangulation of the plane with the distinguished face bounded by a 5-cycle $v_1v_2v_3v_4v_5$, then there exists an index i such that the degrees of the vertices v_i and v_{i+1} are odd (the indices are modulo five) and the degrees of the vertices v_j, $j \neq i, i+1$, are even.*

Since we are interested in colorings of Eulerian triangulations, we will also need some tools that allow us to extend precolorings of some of the vertices of a graph to the entire graph.

Proposition 11. *Let G be a plane Eulerian near-triangulation with the distinguished face bounded by a k-cycle $v_1 \ldots v_k$, $k \leq 5$, and let $W = \{v_1, \ldots, v_k\}$. Every precoloring c of the vertices of W that is a proper coloring of $G[W]$ can be extended to a proper 5-coloring of G.*

Proof. If $k \leq 3$, then all the vertices $v_1 \dots v_k$ have mutually distinct colors. Clearly, G is 5-colorable and the colors of such a 5-coloring of G can be permuted to match the colors of $v_1 \dots v_k$. Suppose now that $k = 4$. By Proposition 9, G can be completed to an Eulerian triangulation either by adding an edge or a vertex of degree four. Hence, G is 3-colorable.

Fix a 3-coloring of G. If the vertices $v_1, \dots, v_4$ should be colored with four mutually distinct colors, recolor v_3 and v_4 with the fourth and the fifth color. If two of the vertices, say v_1 and v_3 should have the same color, recolor v_1 and v_3 with the fourth color and v_4 with the fifth color. If, in addition, the colors of v_2 and v_4 should be the same, also recolor v_2 with the fifth color. In this way we obtain a coloring that matches the precoloring of $v_1, \dots, v_4$ (up to a permutation of the colors).

It remains to consider the case $k = 5$. By Proposition 10, we can assume that the degrees of v_3 and v_4 are odd and the degrees of the remaining vertices are even. Consider the plane graph G' obtained by adding edges v_1v_3 and v_1v_4. G' is an Eulerian triangulation of the plane and it is thus 3-colorable. Since all the vertices $v_2, \dots, v_5$ must receive colors different from the color of v_1, the colors of v_2 and v_4 and the colors of v_3 and v_5 are the same. We now recolor some of the vertices $v_1, \dots, v_5$ to match the colors in the precoloring.

Let A_i, $i = 1, \dots, 5$, be the set of the vertices $v_1, \dots, v_5$ precolored with the i-th color. Suppose first that the vertices $v_1, \dots, v_5$ are precolored with three distinct colors. By symmetry, we can assume that $|A_1| = 1$ and $|A_2| = |A_3| = 2$. We proceed as follows: the vertex of A_1 keeps its color and the vertices of A_2 and A_3 are recolored with the fourth and the fifth color, respectively.

If the vertices $v_1, \dots, v_5$ are precolored with four distinct colors, we may assume that $|A_1| = |A_2| = |A_3| = 1$, $|A_4| = 2$ and the vertices contained in A_1 and A_2 are adjacent. In this case, the vertices of $A_1 \cup A_2$ keep their colors, the vertex of A_3 is recolored with the fourth color and the vertices of A_4 with the fifth color.

Finally, if the vertices $v_1, \dots, v_5$ are precolored with five distinct colors, the vertices $v_1, \dots, v_3$ keep their colors and the vertices v_4 and v_5 are recolored with the fourth and the fifth colors. □

Similarly, some special precolorings of plane Eulerian near-triangulations with the distinguished face of length six or seven can also be extended as we show in the next three propositions. Note that the vertex v_6 is *not* assumed to be precolored in the next proposition and its color is determined by the extended coloring.

Proposition 12. *Let G be a plane Eulerian near-triangulation with the distinguished face bounded by a 6-cycle $v_1 \dots v_6$ and let $W = \{v_1, \dots, v_5\}$. Every precoloring c of the vertices of W with five colors that is a proper coloring of $G[W]$ such that $c(v_1) = c(v_4)$ can be extended to a 5-coloring of the entire graph G.*

Proof. Let G' be an isomorphic copy of G, $v'_1 \dots v'_6$ the vertices bounding the distinguished face of G', and H a plane graph obtained from G and G' by identifying the vertices v_i and v'_i for $i = 1, \dots, 6$. Clearly, H is an Eulerian triangulation of the plane and thus it is 3-colorable. Consider the 3-coloring of G that is the restriction of a 3-coloring of H. Recolor the vertices v_1 and v_4 with an unused (fourth) color and the vertex v_5 with another unused (fifth) color. If $c(v_i) = c(v_5)$ for $i \in \{2, 3\}$, also recolor such a vertex v_i with the fifth color. Note that the colors $c(v_2)$ and $c(v_3)$ are distinct from $c(v_1) = c(v_4)$. The coloring of $v_1, \dots, v_5$ now matches the precoloring of $G[W]$ (up to a permutation of the colors). □

Proposition 13. *Let G be a plane Eulerian near-triangulation with the distinguished face bounded by a 6-cycle $v_1 \dots v_6$ and let $W = \{v_1, \dots, v_5\}$. Every precoloring c of the vertices of W with five colors that is a proper coloring of $G[W]$ such that $c(v_1) = c(v_3)$ can be extended to a 5-coloring of the entire graph G.*

Proof. Let G' and H be the graphs as in the proof of Proposition 12 and consider the 3-coloring of G obtained by restricting a 3-coloring of H. Recolor the vertices v_1 and v_3 with an unused (fourth) color and the vertex v_2 with another unused (fifth) color. If $c(v_i) = c(v_1)$ for $i \in \{4, 5\}$, also recolor such a vertex v_i with the fourth color, or if $c(v_i) = c(v_2)$, recolor v_i with the fifth color. The coloring of $v_1, \dots, v_5$ now matches the precoloring. □

Proposition 14. *Let G be a plane Eulerian near-triangulation with the distinguished face bounded by a 7-cycle $v_1 \dots v_7$ and let $W = \{v_1, \dots, v_6\}$. Every precoloring c of the vertices of W with five colors that is a proper coloring of $G[W]$ such that $c(v_1) = c(v_5)$ and $c(v_2) = c(v_6)$ can be extended to a 5-coloring of the entire graph G.*

Proof. As in the previous two proofs, we consider the 3-coloring of G obtained by restricting a 3-coloring of two copies of G pasted along the boundary of the distinguished face. Recolor now the vertices v_1 and v_5 with an unused (fourth) color and the vertices v_2 and v_6 with another unused (fifth) color. If $c(v_3) = c(v_1)$, also recolor v_3 with the fourth color, and if $c(v_4) = c(v_2)$, recolor v_4 with the fifth color. The coloring of the vertices $v_1, \ldots, v_6$ now matches the precoloring of $G[W]$. □

We now introduce a special type of contractions which will be used later in the paper. If G is a triangulation with a vertex v of degree four that is adjacent to vertices v_1, v_2, v_3 and v_4 (in this cyclic order around v), a triangulation obtained by a *contraction of* v_1vv_3 is the triangulation $G.v_1vv_3$ constructed from G in the following way: the edges vv_1 and vv_3 are contracted to a new vertex w, the edges v_2v_1, v_2v and v_2v_3 are identified to a single edge v_2w and the edges v_4v_1, v_4v and v_4v_3 to a single edge v_4w. If $v_1 \neq v_3$ and the vertices v_1 and v_3 are not adjacent, $G.v_1vv_3$ is again a triangulation. Moreover, if the original triangulation is Eulerian, then the obtained triangulation is also Eulerian: the degrees of the vertices v_2 and v_4 decrease by two and the degree of the vertex w is equal to the sum of the degrees of the vertices v_1 and v_3 decreased by four. Similarly, if G is a defective Eulerian triangulation, then $G.v_1vv_3$ is also a defective Eulerian triangulation, and if G is 2-defective, then $G.v_1vv_3$ is either Eulerian, defective or 2-defective Eulerian triangulation.

A vertex v of degree four such that $G.v_1vv_3$ or $G.v_2vv_4$ is a triangulation is said to be *contractible*; otherwise, v is called *non-contractible*. Note that if v is a non-contractible vertex, then $v_1 = v_3$ or the vertices v_1 and v_3 are adjacent, and similarly, $v_2 = v_4$ or the vertices v_2 and v_4 are adjacent. In the case of the Klein bottle, we call a non-contractible vertex v *resistant* if the degrees of v_1, v_2, v_3 and v_4 are at least six, the cycle vv_1v_3 is a two-sided non-separating 3-cycle and vv_2v_4 is a one-sided non-separating 3-cycle.

At the end of this section, let us recall Theorem 3.1 from [6]. The theorem is stated in [6] for simple plane graphs G but the proof readily translates to graphs with multiple edges as long as there are no 2-faces.

Theorem 15 ([6]). *Let G be a plane near-triangulation with the distinguished face bounded by a k-cycle. If all the vertices that are not incident with the distinguished face have degree at least six, then G contains at most $k^2/12 + k/2 + 1$ vertices.*

In particular, if $k = 6$, then G contains at most one vertex that is not incident with the distinguished face and if $k = 8$, then G contains at most two such vertices.

4 Degree structure of Eulerian triangulations of the Klein bottle

In this section, we first describe Eulerian triangulations of the Klein bottle with minimum degree at least four that do not contain a 4-vertex adjacent to a 6-vertex. Such triangulations are either 6-regular or of a very special type:

Lemma 16. *Let G be an Eulerian triangulation of the Klein bottle with minimum degree at least four. Then, either G is 6-regular, or G contains a vertex of degree four adjacent to a vertex of degree at most six, or G contains only vertices of degree four and eight, no two vertices of degree four are adjacent and each vertex of degree eight is adjacent to four vertices of degree four.*

Proof. Assume that neither of the first two cases apply. Let Q be the set of vertices of degree four of G and O the set of vertices of degree at least eight and set $q = |Q|$ and $o = |O|$. Note that all neighbors of each vertex of Q are in the set O by our assumption. Color now the edges incident with the vertices of degree four by blue. Next, the edges that are contained in a 3-face incident with a vertex of degree four and that are not blue are colored red. Let $b = 4q$ be the number of blue edges and r the number of red edges.

Each 3-face incident with a vertex of degree contains one red and two blue edges. Since each vertex of degree four is contained in four faces, there are b 3-faces incident with vertices of degree four. On the other hand, the number of such 3-faces is at most $2r$ (each red edge is contained in at most such faces). We conclude $b \leq 2r$. By Euler's formula, the average degree of G is six. Therefore, $o \leq q$ and the equality holds if only if all the vertices in O have degree exactly eight.

The number of incidences of the vertices in O with red edges and blue edges is $b + 2r \geq 2b = 8q$. Hence, the average degree of G is at least

$$\frac{4q + 8q + 6s}{o + q + s} \geq \frac{12q + 6s}{2q + s} = 6 \qquad (1)$$

where s is the number of vertices of degree six of G. Since the average degree of G is six, all the inequalities $b + 2r \geq 2b$ and (1) are equalities. It follows that $o = q$ and $b + 2r = 2b = 8q$. Therefore, all the vertices of O have degree exactly eight, all the edges incident with vertices of O are red or blue and $b = 2r$. Since two consecutive edges incident with a vertex of

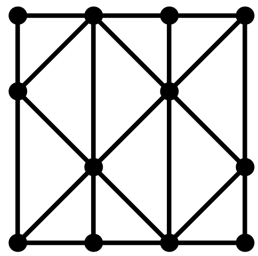


Figure 1: The Eulerian triangulation T_A of the Klein bottle.

O cannot be blue, each vertex of O is incident with four blue and four red edges. In particular, it is not adjacent to a vertex of degree six. Since G is connected, we conclude that G contains no vertices of degree six. The statement of the lemma follows. □

At the end of this section, we briefly deal with one of the cases described in Lemma 16. The structure of 6-regular triangulations of the Klein bottle is well-understood [23, 28] and in particular, the chromatic number of every 6-regular triangulation of the Klein bottle has been determined by Sasanuma [27]. The only 6-chromatic 6-regular triangulation of the Klein bottle is the one obtained from the complete graph of order six by adding a perfect matching (it is depicted in Figure 1). We henceforth refer to this triangulation as the triangulation T_A.

Lemma 17. *A 6-regular triangulation G of the Klein bottle is 5-colorable unless G is the triangulation T_A depicted in Figure 1.*

5 Eulerian triangulations with non-contractible 4-vertices

In this section, we analyze Eulerian triangulations of the Klein bottle that contain a 4-vertex that is neither contractible nor resistant. Those with resistant vertices are analyzed in the next section. We start with the simplest case that a 4-vertex is non-contractible since it is contained in a short separating cycle.

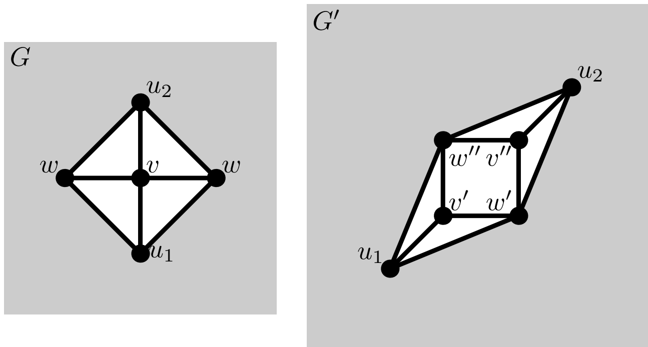


Figure 2: The projective plane graph obtained by cutting along a one-sided 2-cycle vw in the proof of Lemma 19.

Lemma 18. *If G is an Eulerian triangulation of the Klein bottle that contains a surface-separating 2-cycle or 3-cycle, then G is 5-colorable.*

Proof. Cut the surface along the non-contractible 2- or 3-cycle. In case that the length of the cycle is two, remove one of the two parallel edges bounding a new 2-face. In this way, we obtain two Eulerian triangulations of the projective plane or two defective Eulerian triangulations of the projective plane which are 5-colorable by Theorem 2 or by Lemma 5. Their colorings can be easily combined to obtain a coloring of G with five colors. □

We now deal with Eulerian triangulations with a 4-vertex contained in a cycle of length two.

Lemma 19. *If G is a proper Eulerian triangulation of the Klein bottle that contains a surface-non-separating 2-cycle vw such that the degree of v is four, then G is 5-colorable.*

Proof. Let u_1 and u_2 be the two neighbors of v different from w. The 2-cycle vw forms either a one-sided or a double-sided non-separating curve in the Klein bottle. We cut the surface along it.

If the 2-cycle is one-sided, we obtain the projective plane graph G' depicted in Figure 2. Remove the vertices v' and v'' and the edges $w'u_1$ and $w'u_2$ and identify the vertices w' and w''. In this way, we obtain an Eulerian triangulation of the projective plane which is 5-colorable by Theorem 2.

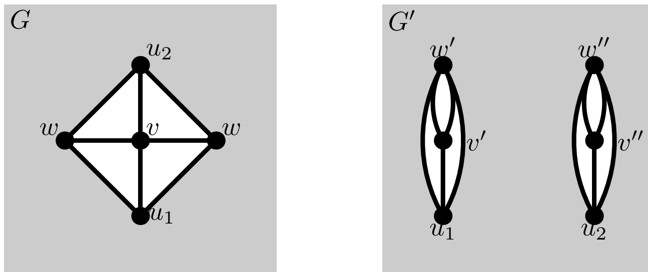


Figure 3: The plane graph obtained by cutting along a two-sided 2-cycle vw in the proof of Lemma 19.

Assigning the same colors to the vertices of $G - v$ yields a proper 5-coloring of G that can be extended to the vertex v (since its degree is four).

If the 2-cycle is double-sided, we obtain the plane graph G' as depicted in Figure 3. By Lemma 6, $G' - \{v', v''\}$ has a 5-coloring that assigns the vertices w' and w'' the same color. Assign now the vertices of G the colors of their counterparts in G' and extend the coloring to v. In this way, we obtain a 5-coloring of G. □

Next, we analyze the case that one neighbor of a non-contractible 4-vertex is also a 4-vertex.

Lemma 20. *Let G be a proper Eulerian triangulation of the Klein bottle with no contractible vertices. If G contains two adjacent vertices of degree four, then G is 5-colorable.*

Proof. By Lemma 18, we may assume that G does not contain separating 2- or 3-cycles. Let v and w be two adjacent vertices of degree four. If v or w is contained in a non-contractible 2-cycle, then G is 5-colorable by Lemma 19. Otherwise, each of v and w is contained in two non-contractible 3-cycles. Let u_0, u_1 and u_2 be their neighbors as depicted in Figure 4. Note that the homotopies of the cycles vu_1u_2 and wu_1u_2 are the same.

If both the cycles vu_1u_2 and vwu_0 are one-sided, then their homotopies are the same. Cutting the surface along the cycle vwu_0 yields a projective plane graph G'. Since the cycles vu_1u_2 and vwu_0 are homotopic, one of the areas bounded by $u_0'u_1u_2$ and $u_0''u_1u_2$ is a 2-cell while the other one contains a cross-cap (hence, we have obtained the left or the middle configuration

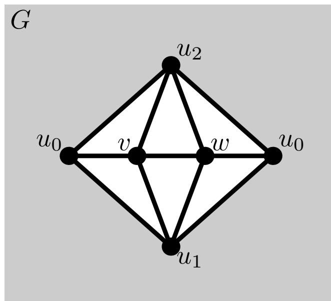


Figure 4: The notation used in the proof of Lemma 20.

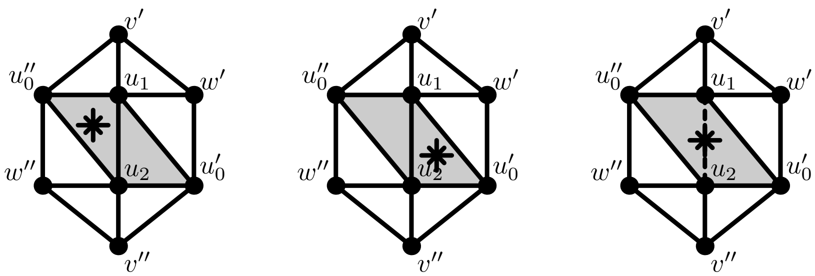


Figure 5: The projective plane graph G' obtained by cutting along the one-sided cycle vwu_0 in the proof of Lemma 20. The left or the middle graph is obtained if the cycle vu_1u_2 is one-sided, and the right one if the cycle is two-sided.

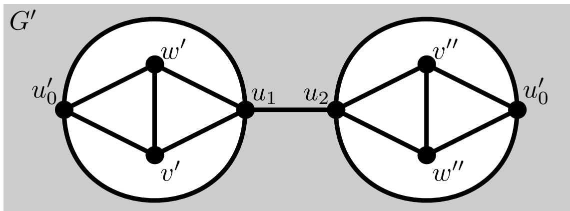


Figure 6: The plane graph G' obtained by cutting along the two-sided cycle vwu_0 in the proof of Lemma 20.

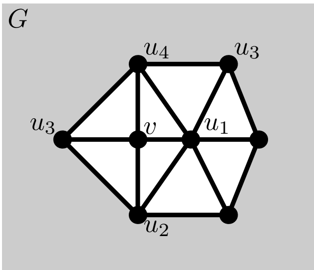
depicted in Figure 5). Color the vertices u_0', u_1 and u_2 with three mutually different colors and color the vertex u_0'' with the same color as u_0'. The coloring can be extended to both the graphs contained in the areas bounded by $u_0'u_1u_2$ and $u_0''u_1u_2$: these graphs are either Eulerian triangulations or defective Eulerian triangulations of the plane or the projective plane and thus they are 5-colorable by Five Color Theorem, Theorem 2 or Lemma 5. The constructed 5-coloring of G' readily yields a 5-coloring of G.

If the cycles vu_1u_2 and wu_1u_2 are double-sided and the cycle vwu_0 is one-sided, we also cut the surface along the cycle vwu_0. This yields the projective plane graph G' depicted in the right part in Figure 5. Remove the vertices v', v'', w' and w'' and the edges $u_0'u_1$ and $u_0'u_2$ and identify the vertices u_0' and u_0'' to a vertex u_0'''. The obtained projective plane triangulation G'' contains two vertices of odd degree: u_1 and u_2. Note that u_1 and u_2 are adjacent and thus G'' is a defective Eulerian triangulation which is 5-colorable by Lemma 5. The 5-coloring of G'' can be extended to the vertices v and w to a 5-coloring of G.

It remains to consider the case that the cycles vu_1u_2 and wu_1u_2 are one-sided and the cycle vwu_0 is double-sided. Let G' be the plane graph obtained by cutting along the cycle vwu_0 (see Figure 6). By Lemma 6, G' has a 5-coloring that assigns the vertices u_0' and u_0'' the same color. This 5-coloring can be easily extended to v and w yielding a 5-coloring of G. □

It remains to analyze one more case when a 4-vertex is neither contractible nor resistant. We do so in the next lemma.

Lemma 21. *Let G be a proper Eulerian triangulation of the Klein bottle with no contractible vertices. If G contains a vertex v of degree four that*



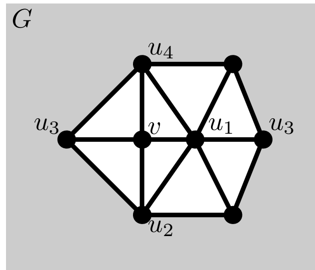


Figure 7: The two possible configurations contained in the graph G in the proof of Lemma 21.

is not resistant and that is adjacent to a vertex of degree six, then G is 5-colorable.

Proof. By Lemmas 18 and 19, we can assume that v is not contained in a 2-cycle. In particular, the vertices u_1, u_2, u_3 and u_4 are mutually distinct. By symmetry, we assume that u_1 is the neighbor of v of degree six. By the assumption of the lemma, both the cycles vu_1u_3 and vu_2u_4 are one-sided. In particular, one of the neighbors of u_1 is the vertex u_3, and G contains one of the two configurations depicted in Figure 7. Hence, cutting along the cycle vu_1u_3 yields one of the four projective plane graphs depicted in Figure 8. Note that one of the two shaded areas in the figure is a 2-cell and one contains a cross-cap since the homotopies of the cycles vu_1u_3 and vu_2u_4 are the same.

We now construct a 5-coloring of G for each of the four cases depicted in Figure 8. In the first case, the projective plane graph bounded by $u_2u_3'u_4$ is either an Eulerian triangulation or a defective Eulerian triangulation. Hence, it is 5-colorable by Theorem 2 or Lemma 5. Next, color the vertex u_3'' by the same color as u_3' and the vertices u_1' and u_1'' with the same color different from the colors of u_2, u_3' and u_4, and extend the coloring to the graph inside the 4-cycle $u_1''u_2u_4u_3''$ by Proposition 11. In this way, we eventually obtain a 5-coloring of G by assigning a color to the vertex v.

In the second case, let G' be the projective plane graph bounded by the cycle $u_2w_1w_2u_3''u_4$. If the vertices w_1 and u_3'' are not adjacent, remove the edge w_2u_3'' and identify the vertices w_1 and u_3''. In this way, we obtain an Eulerian triangulation or a defective Eulerian triangulation of the projective

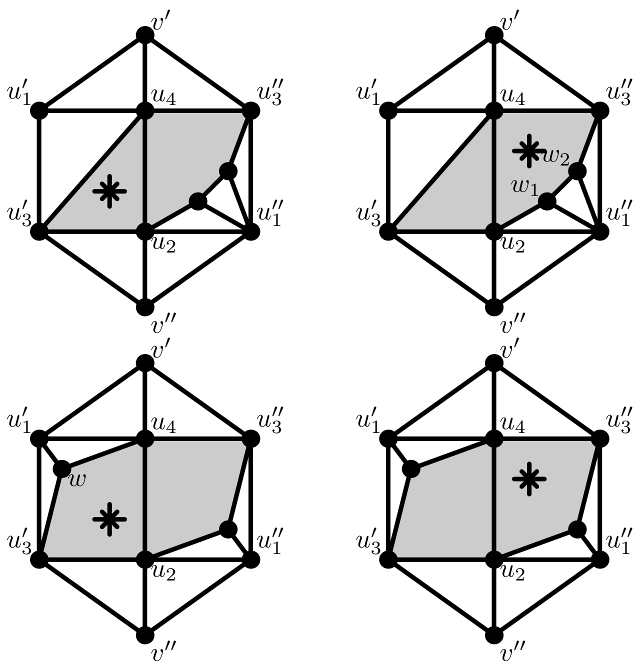


Figure 8: The four projective plane graphs G' that can be obtained by cutting along the cycle vu_1u_3 in the proof of Lemma 21.

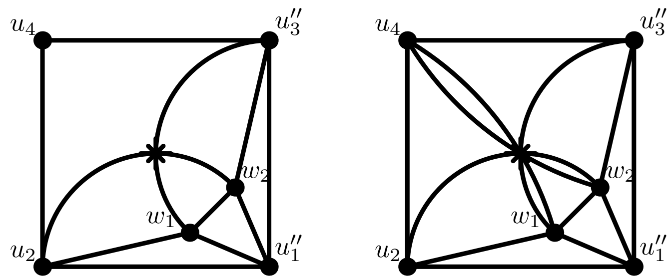


Figure 9: The configurations that appear in G' in the proof of Lemma 21 in the analysis of the second configuration depicted in Figure 9.

plane which is 5-colorable by Theorem 2 or Lemma 5. Next, color the vertices u_1' and u_1'' with a color different from the colors of u_2, $u_3' = u_3''$ and u_4 and extend the obtained coloring to v. In this way, we obtain a 5-coloring of G. We proceed analogously if the vertices w_2 and u_2 are not adjacent.

Assume now that both the vertices w_1 and u_3'' and the vertices w_2 and u_2 are adjacent (see Figure 9). If the vertices u_4 and w_1 are not adjacent, precolor them with the same color and color the vertices u_2, w_2 and u_3'' with mutually distinct colors different from the color of u_4 and v_1. Extend this coloring to the graphs inside the 5-cycle $u_2u_4u_3''w_2w_1$ and the 4-cycle $w_1u_3''w_2u_2$ by Proposition 11. Next, color the vertices u_1' and u_1'' with the same color and obtain a 5-coloring of G by assigning a color to the vertex v. Again, we proceed similarly if the vertices w_2 and u_4 are not adjacent. If G' contains both the edges w_1u_4 and w_2u_4, then the degree of u_4 is seven—note that all the faces incident with u_4 in the graph depicted in Figure 9 are 3-faces and G is a proper triangulation. However, this case is excluded by our assumption that G is an Eulerian triangulation.

The third and the fourth configurations depicted in Figure 8 are symmetric. Hence, we just analyze the third one. Let G' be the projective plane graph bounded by the 4-cycle $u_2u_3'wu_4$. Add an edge between the vertices u_3' and u_4. In this way, we obtain an Eulerian triangulation, a defective Eulerian triangulation or a 2-defective Eulerian triangulation of the projective plane. In all the cases, G' with the added edge is 5-colorable by Theorem 2 or Lemma 5. Next, color the vertices u_1' and u_1'' with a color different from the colors of u_3', u_2 and u_4 and extend the coloring inside the cycle $u_2u_1''u_3''u_4$ by Proposition 11. A 5-coloring of G is then obtained by

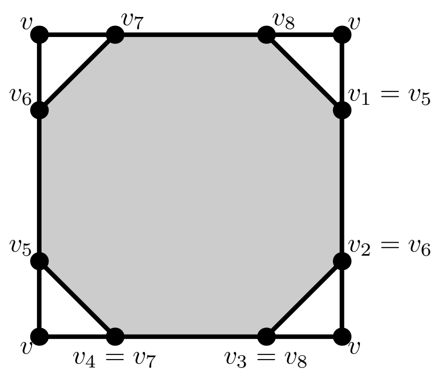


Figure 10: The resistant configuration. Note that $v_1 = v_5$, $v_2 = v_6$, $v_3 = v_8$, and $v_4 = v_7$.

assigning a color to the vertex v. □

6 Eulerian triangulations with resistant vertices

In this section, we analyze Eulerian triangulations of the Klein bottle that contain a resistant vertex. If G is such a triangulation, then it is of the form depicted in Figure 10 in which the pairs of vertices v_1 and v_5, v_2 and v_6, v_3 and v_8, and v_4 and v_7 are the same. The resistant vertex is the vertex v. Throughout this section, we refer to the configuration depicted in Figure 10 as to a *resistant configuration* and all the arithmetic with indices of the vertices v_i, $i = 1, \ldots, 8$, throughout this section, is modulo eight. When referring to adjacencies between a vertex v_i and a vertex w different from all v_i, the vertices v_i are treated as eight distinct vertices, i.e., if the configuration contains an edge wv_4 but not wv_7, we say that w is adjacent to v_4 and not adjacent to v_7.

Lemma 22. *Let G be a proper Eulerian triangulation of the Klein bottle with a resistant configuration. If G does not contain a vertex of degree four different from v, then G is either 5-colorable or it is the triangulation T_B depicted in Figure 11.*

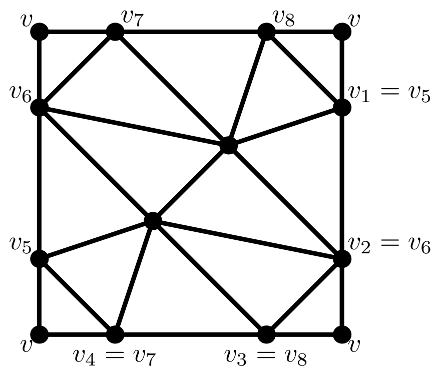


Figure 11: The Eulerian triangulation T_B of the Klein bottle.

Proof. By Theorem 15, G contains at most two vertices different from v and different from all v_i, $i = 1, \ldots, 8$. If G contains at most one such vertex, then a 5-coloring of G can be obtained as follows: first, color the neighbors of v by four mutually distinct colors and then extend the coloring to the remaining (non-adjacent) vertices of G. The same argument works if G contains two such non-adjacent vertices. Hence, we can assume that G contains adjacent vertices w_1 and w_2 that are different from v and v_i, $i = 1, \ldots, 8$.

Since the degree of both w_1 and w_2 is at least six, each of them must be adjacent to five consecutive vertices $v_i, \ldots, v_{i+4}$. There are four such triangulations G (see Figure 12): two of them are isomorphic to triangulation T_B and the remaining ones are not Eulerian. □

Next, we observe that each vertex of degree four different from v is adjacent to at least two of the vertices $v_1, \ldots, v_8$.

Lemma 23. *Let G be a proper Eulerian triangulation of the Klein bottle with the resistant configuration. If G contains a non-contractible vertex $w \neq v$, then w is adjacent to at least two of the vertices $v_1, \ldots, v_8$ and such two vertices are consecutive in the cyclic order around w.*

Proof. By Lemma 20, v and w are not adjacent. Let $u_1, \ldots, u_4$ be the neighbors of w in the cyclic order around w. Since w is resistant, all the four neighbors of w are mutually distinct and G contains non-contractible 3-cycles wu_1u_3 and wu_2u_4. If neither u_1 nor u_3 is one of the vertices $v_1, \ldots, v_8$,

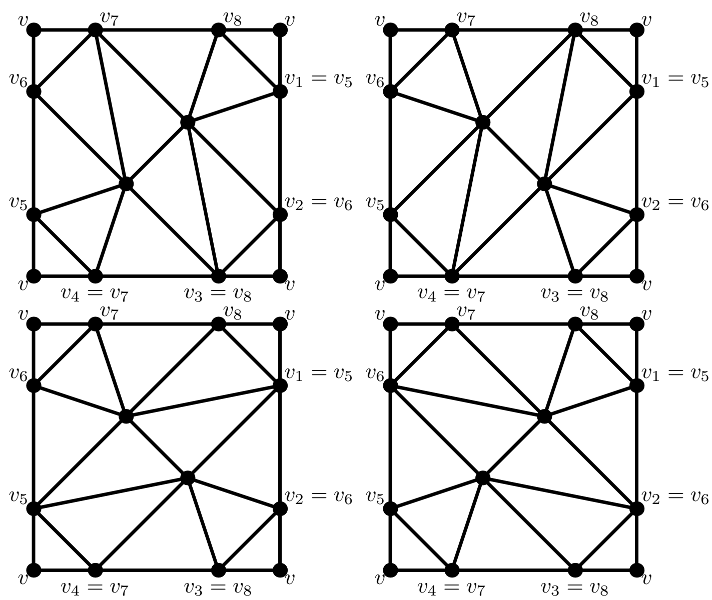


Figure 12: The four triangulations of the Klein bottle obtained in the proof of Lemma 22.

the 3-cycle wu_1u_3 must be contractible. Hence, u_1 or u_3 (or both) are one of the vertices $v_1, \ldots, v_8$. Similarly, u_2 or u_4 is one of the vertices $v_1, \ldots, v_8$. The claim now readily follows. □

Our first step towards the analysis of the triangulation with the resistant configuration is excluding the case that G has non-contractible vertex $w \neq v$ adjacent to vertices v_i and v_j with $|j-i|$ large.

Lemma 24. *Let G be a proper Eulerian triangulation of the Klein bottle with a resistant configuration. If G contains a non-contractible vertex $w \neq v$ that is adjacent to v_i and v_j such that $j \neq i-2, i-1, i+1, i+2$, then G is 5-colorable.*

Proof. By symmetry, it is enough to analyze the case when w is adjacent to one of the following pairs of vertices: v_1 and v_4, v_1 and v_5, v_1 and v_6, v_3 and v_7, and v_3 and v_8. The cases that w is adjacent to v_1 and v_5, or to v_3 and v_8 are handled by Lemma 19. In the remaining cases, let G_1 and G_2 be the near-triangulations contained in the shaded areas as in Figure 13.

We now proceed with each of the three configurations separately:

w **is adjacent to** v_1 **and** v_4**.** Consider the near-triangulation G_1, add an edge between v_6 and v_8, and identify the vertices v_1 and v_5. By Lemma 6, the resulting plane graph G_1' has a coloring with five colors that assigns the vertices v_4 and v_7 the same color. Color the vertices of G with the colors of their counterparts in G_1'. The obtained coloring can be extended to G_2 by Proposition 11 and eventually to the vertex v as well.

w **is adjacent to** v_1 **and** v_6**.** Color the vertices $v_1, \ldots, v_8$ with four mutually distinct colors in such a way that the pairs of the corresponding vertices receive the same colors. Extend the precoloring to G_1 by Proposition 14. This determines the color of w. The coloring can now be extend to G_2 by Proposition 11 and eventually to v, since the degree of v is four.

w **is adjacent to** v_3 **and** v_7**.** Precolor the vertices $v_1, \ldots, v_8$ with four mutually distinct colors in such a way that the pairs of the corresponding vertices receive the same colors. The precoloring can be extend both to G_1 and G_2 by Proposition 12. Since the degree of w in G is four, it can be recolored so that its color is the same in both the extensions of the precoloring. At the end, we color the vertex v.

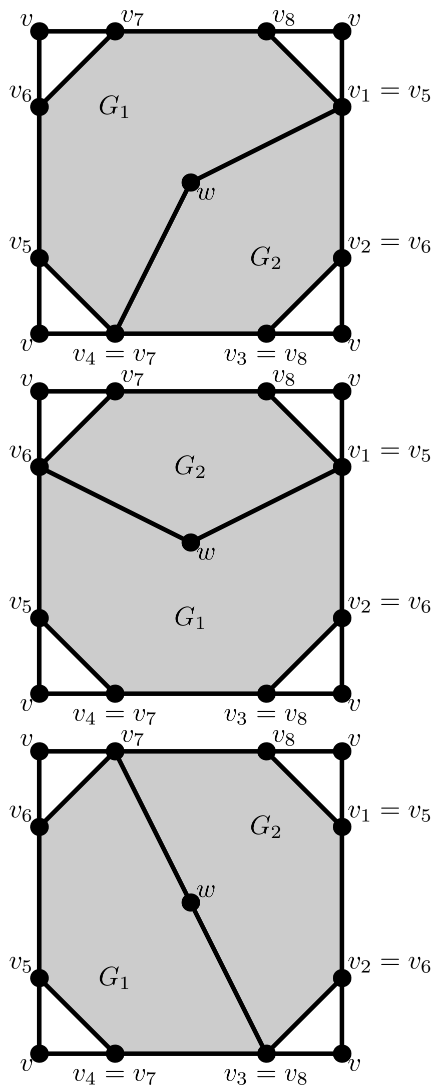


Figure 13: The three configurations analyzed in the proof of Lemma 24.

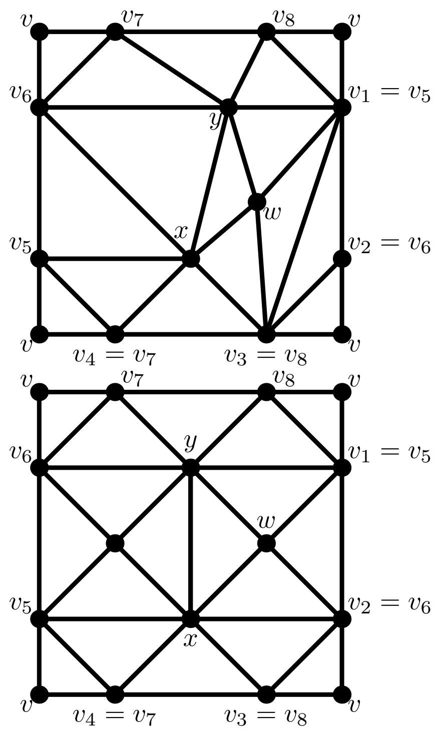


Figure 14: The Eulerian triangulations T_C and T_D of the Klein bottle.

□

Next case that we consider is when there is a non-contractible 4-vertex that has exactly two neighbors in common with v.

Lemma 25. *Let G be a proper Eulerian triangulation of the Klein bottle with the resistant configuration and without contractible vertices. If G contains a non-contractible vertex $w \neq v$ that is adjacent to exactly two of the vertices $v_1, \ldots, v_8$, then G is 5-colorable or it is the Eulerian triangulation T_C or T_D depicted in Figure 14.*

Proof. Let v_i and v_j be the neighbors of w, and let x and y be the remaining two neighbors of w. Since w is non-contractible, x is adjacent to the

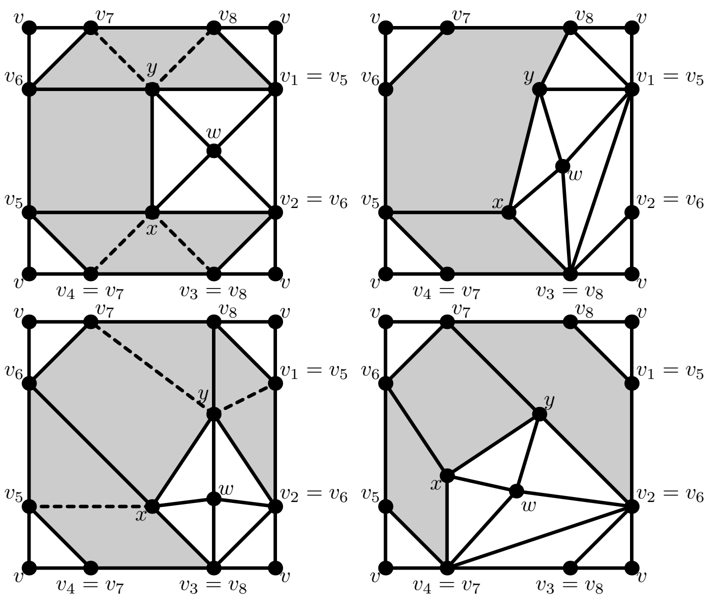


Figure 15: The four configurations considered in the proof of Lemma 25.

counterpart of v_i and y to the counterpart of v_j. If $j \notin \{i \pm 1, i \pm 2\}$, then G is 5-colorable by Lemma 24. By symmetry, we can assume that i is equal to 1, 2 or 3, and $j \in \{i+1, i+2\}$. If $i = 3$ and $j = 4$, then the vertex x cannot be adjacent to v_8 and y to v_7 (because of simple topological reasons). We conclude that G contains one of the four configurations depicted in Figure 15 (the dashed edges are drawn for future reference and are not contained in the configurations).

Let us analyze the first configuration depicted in Figure 15. Suppose first that x is not adjacent to v_3. Color the vertices $v_1, \dots, v_4$ and y with five mutually distinct colors and the vertex x with the color of v_3. The precoloring can be extended to each shaded area by Proposition 11. We proceed similarly if x is not adjacent to v_4 or y is not adjacent to v_7 and v_8. Hence, we can assume that all the four dashed edges depicted in Figure 15 are present in G. Since G is proper and Eulerian, it must be isomorphic to the triangulation T_D.

We now analyze the second configuration. Assume first that the cycle

$xv_5v_6v_7v_8y$ contains a chord. Unless the subgraph of G induced by the vertices $v_1, \ldots, v_4, x$ and y is complete, the vertices $v_1, \ldots, v_4, x, y$ can be properly colored with five colors and the coloring can be extended to the shaded areas by Proposition 11. If the subgraph induced by the vertices $v_1, \ldots, v_4, x, y$ is complete, then the vertex x is adjacent to the vertices $v_4 = v_7$ and v_6 and the vertex y to the vertices v_6 and v_7. It is easy to conclude that G must be the triangulation T_C in such a case.

Next, we assume that the cycle $xv_5v_6v_7v_8y$ is chordless. If there is no vertex of degree four inside the cycle $xv_5v_6v_7v_8y$, then by Theorem 15, the interior of the cycle $xv_5v_6v_7v_8y$ contains a single vertex of degree six adjacent to all the vertices on the boundary of the cycle and the degree of y in G is five (recall that G is proper) which contradicts our assumption that G is Eulerian. Otherwise, there is a non-contractible vertex w' inside the cycle $xv_5v_6v_7v_8y$. Since w' is non-contractible, all its four neighbors are among the vertices x, y, $v_5, \ldots, v_8$. At least two such neighbors are not consecutive on the cycle $xv_5v_6v_7v_8y$ and thus the cycle contains a chord since G is a triangulation. This violates our assumption that the cycle is chordless.

Let us now proceed with the third configuration depicted in Figure 15. If y is not adjacent to v_1, color the vertices $v_1, \ldots, v_8$ with four distinct colors, the vertex y with the color of $v_1 = v_5$ and x with the fifth color. The obtained coloring can be extended to each shaded area by Proposition 11. Next, it can be extended to v and w and we obtain a coloring of G with 5 colors. An analogous argument applies if y is not adjacent to v_7 or x is not adjacent to v_5.

Let us next assume that G contains all the edges yv_1, yv_7 and xv_5. Let G' be the near-triangulation bounded by the 4-cycle xv_6v_7y. Since G is Eulerian and proper, the degree of y in G' is even. So is the degree of v_6 by Proposition 9. However, the degree of $v_2 = v_6$ in G is then odd which is excluded by our assumption that G is Eulerian.

The last configuration depicted in Figure 15 is the easiest to analyze. First, color the vertices $v_1, \ldots, v_8$ with four distinct colors and assign the vertex y the fifth color. Next, color the vertex x with the color assigned to the vertex $v_3 = v_8$. The precoloring can be extended to each shaded area by Proposition 11 and it can eventually be extended to v and w. □

The final case is when the triangulation with the resistant configuration contains a non-contractible vertex with three neighbors in common with v.

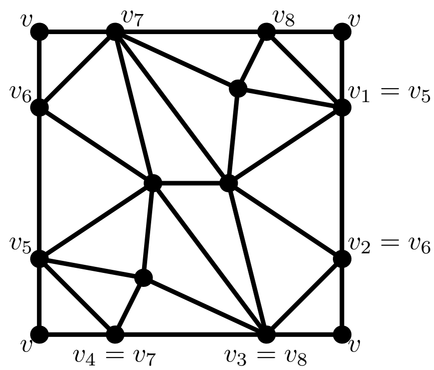


Figure 16: The Eulerian triangulations T_E of the Klein bottle.

Lemma 26. *Let G be a proper Eulerian triangulation of the Klein bottle with the resistant configuration and no contractible vertices. If G contains a non-contractible vertex $w \neq v$ that is adjacent to three of the vertices $v_1, \ldots, v_8$, then G is either 5-colorable or it is the Eulerian triangulation T_E depicted in Figure 16.*

Proof. By Lemma 24, unless G is 5-colorable, w is adjacent to vertices v_i, v_{i+1}, v_{i+2}. By symmetry, we can assume that w is adjacent to either v_1, v_2 and v_3, or to v_3, v_4 and v_5, see Figure 17. Let x be the neighbor of w that is different from the vertices $v_1, \ldots, v_8$.

Let us first consider the case that w is adjacent to v_1, v_2 and v_3. Since w is non-contractible, x must also be adjacent to v_6. Let G_1 and G_2 be the subgraphs of G as depicted in Figure 17. By Proposition 11, there exist a coloring of G_1 such that the vertices x and $v_3, \ldots, v_6$ are colored with mutually distinct colors and a coloring of G_2 such that the vertices x and $v_6, \ldots, v_1$ are colored with mutually distinct colors. Permute now the colors in such a way, that the colors v_1 and v_5, v_3 and v_8 and v_4 and v_7 are the same. Clearly, the obtained coloring can be extended to both v and w yielding a coloring of G with five colors.

Let us now consider the second configuration. If the cycle that bounds G_1 contains a chord, then we precolor the vertices x, $v_1, \ldots, v_8$ and extend the precoloring to all the subgraphs contained in the shaded areas by Proposition 11. The obtained colorings can clearly be combined and extended to

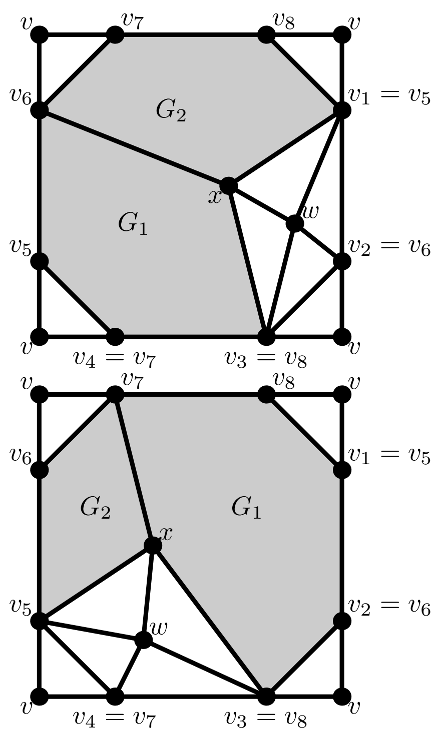


Figure 17: The two configurations that are considered in the proof of Lemma 26.

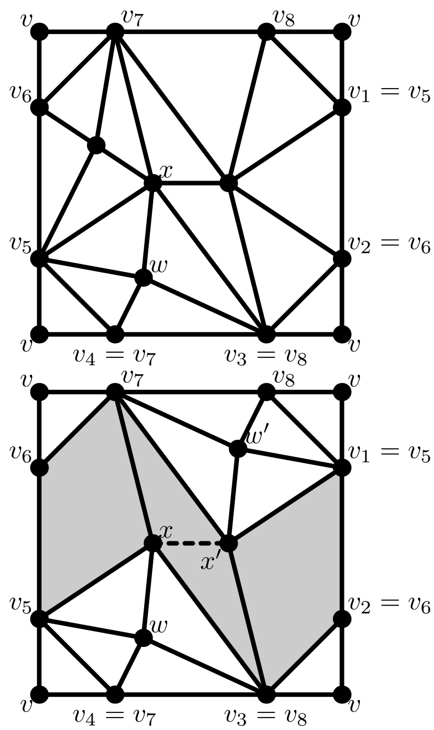


Figure 18: The final configurations considered in the proof of Lemma 26.

v and w. Hence, we assume the cycle $v_1v_2v_3xv_7v_8$ is chordless.

If G_1 does not contain an inner vertex of degree four, then it contains at most one vertex by Theorem 15. Since the cycle $v_1v_2v_3xv_7v_8$ is chordless, G must be the triangulation depicted in the left part of Figure 18 which is 5-colorable (the vertices x and $v_2 = v_6$ can have the same color). If G_1 contains a vertex w' of degree four, then w' is resistant by the assumption of the lemma. By Lemma 23, w' is adjacent to at least two of the vertices $v_1, \ldots, v_8$. Unless G is 5-colorable, w' is adjacent to three consecutive vertices $v_1, \ldots, v_8$ by Lemma 25 (note that G cannot be the triangulation T_C since T_C does not contain two vertices of degree four with three common neighbors). If w' is adjacent to v_1 and v_2 (and v_8 or v_3), we obtain a configuration analyzed in the beginning of this proof and conclude that G is

5-colorable. Hence, w' is adjacent to v_7, v_8 and v_1. The fourth neighbor x' of the vertex w' is different from x, since otherwise, $x = x'$ would be adjacent to v_1 and we have already shown that the cycle $v_1v_2v_3xv_7v_8$ is chordless. Hence, G contains the configuration depicted the right part of Figure 18. If the vertices x and x' are not adjacent, we precolor the vertices $v_1, \ldots, v_8$ with four distinct colors and the vertices x and x' with the fifth color and extend the coloring to the subgraphs contained in the shaded areas by Proposition 11. We eventually obtain a coloring of G with 5 colors. Hence, we conclude that x and x' are adjacent. It is now easy to infer from the fact that G is proper that G must be the triangulation T_E. □

7 Minimal 6-chromatic triangulations

All the 6-chromatic Eulerian triangulations of the Klein bottle with no contractible vertices that were identified in the previous sections contain the complete graph K_6 of order six as a subgraph. In fact, they contain one of the four non-isomorphic embeddings of K_6 depicted in Figure 19. We refer to these four embedding of K_6 in the Klein bottle as *bad embeddings.* Let us now state the just observed fact as a separate lemma.

Lemma 27. *Let G be a proper Eulerian triangulation of the Klein bottle with no contractible vertices. If G is not 5-colorable, then G contains a bad embedding of the complete graph of order six. Moreover, every subgraph of G isomorphic to K_6 has a bad embedding.*

Proof. By Lemma 18, G contains no separating 2- or 3-cycles. Since G is proper, its minimum degree is at least four. By Lemma 16, every Eulerian triangulation of the Klein bottle is either 6-regular, or contains a vertex of degree four adjacent to a vertex of degree at most six, or contains only vertices of degree four and eight such that no two vertices of degree four are adjacent and each vertex of degree eight is adjacent to four vertices of degree four. If G is 6-regular and it is not 5-colorable, then G is isomorphic to T_A by Lemma 17. Since G is proper, it cannot contain a vertex of degree two; otherwise, it would contain a separating contractible 2-cycle. If G contains only vertices of degree four and eight and each vertex of degree eight is adjacent to four vertices of degree four, we proceed as follows: first, we remove the vertices of degree four. The resulting 4-regular graph is clearly 5-colorable. Next, we obtain a coloring of G with five colors by extending the constructed coloring to the vertices of degree four.

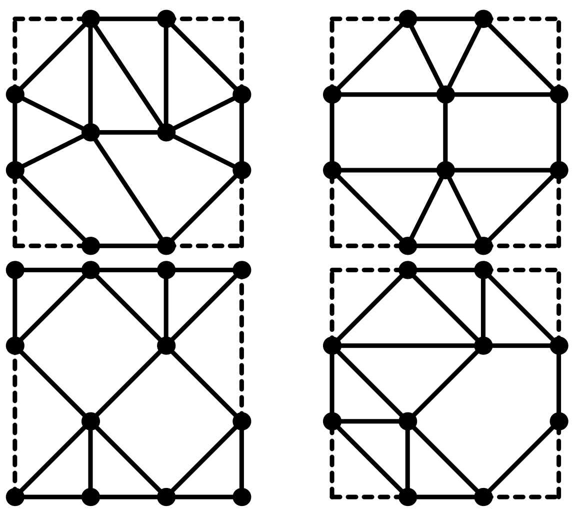


Figure 19: The four *bad embeddings* of the complete graph of order six in the Klein bottle. Note that the embeddings are distinct: only the bottom one contains a 5-face, and the top ones contains one, two and three vertices incident with three 4-faces, respectively.

Next, we analyze the case when G contains a vertex v of degree four adjacent to a vertex of degree four or six. Note that v is non-contractible by the assumption of the lemma. If v is adjacent to a vertex of degree four, then G is 5-colorable by Lemma 20. If v is adjacent to a vertex of degree six and it is not resistant, then G is 5-colorable by Lemma 21. In all other cases, G contains the resistant configuration.

If v is the only vertex of degree four of G, then G is 5-colorable by Lemma 22 unless G is the triangulation T_B. If G contains another vertex $w \neq v$ of degree four, then v and w have either two or three common neighbors by Lemmas 23 and 24. If v and w have two common neighbors, then G is 5-colorable by Lemma 25 unless G is the triangulation T_C or T_D. If v and w have three common neighbors, then G is 5-colorable by Lemma 26 unless G is the triangulation T_E.

We can now conclude that G is 5-colorable unless G is one of the five triangulations $T_A, \ldots, T_E$. We leave it to the reader to verify that each subgraph isomorphic to K_6 of each of the five triangulations $T_A, \ldots, T_E$ yields an embedding equivalent to one of the bad embeddings. □

8 Eulerian triangulations containing K_6

In this section, we prove that the only embeddings of the complete graph K_6 of order six that can be extended to an Eulerian triangulation of the Klein bottle are those that we have identified in the previous section.

Theorem 28. *If H is an embedding of K_6 in the Klein bottle that can be extended to an Eulerian triangulation, then H is bad.*

Proof. Let G be an Eulerian triangulation that contains H. We can assume without loss of generality that G is proper. If H is not a 2-cell embedding, then it is isomorphic to a unique embedding of K_6 in the projective plane with one of the faces containing a cross-cap. By Proposition 8, the three vertices of H incident only with 2-cell faces have odd degrees in G. Hence, we can assume that the embedding of H in the Klein bottle is 2-cell.

Observe next that H has no faces of size six: otherwise, by Euler's formula, all the remaining faces of H are 3-faces. Since no vertex of H can be incident with five 3-faces (its degree in G would then be five), each vertex of H is incident with the face of order six. However, inserting a new vertex in this face and joining it to all the six distinct vertices on the boundary of the face of order six, we obtain an embedding of K_7 in the Klein bottle which is impossible.

Hence, we can assume that H either contains three 4-faces or it contains a 4-face and a 5-face. Let us first analyze the latter case: the subgraph of G contained in the 5-face of H has exactly two vertices u and v of odd degree and these vertices are adjacent on its boundary by Proposition 10. Hence, we can assume that the 5-face of H contains in G two chords joining u and v to the opposite vertex. The 4-face of H then contains a single vertex w of degree of four (otherwise, G would not be both proper and Eulerian). It is now easy to infer that w is not contractible since every two opposite neighbors of w are adjacent. Hence, G is one of the triangulations described in Lemma 27. Consequently, H is one of the four bad embeddings (by Lemma 27).

The case when H has three 4-faces is similar: since G is proper, each 4-face either contains no vertices or a single vertex w of degree four. Again, w is a non-contractible vertex and thus G is one of the triangulations identified in Lemma 27. Hence, H is a bad embedding of K_6. □

9 Eulerian triangulations without K_6

In this section, we analyze graphs G that do not contain a bad embedding, but any contraction of each 4-vertex v yields a graph $G_0 = G.v_1vv_3$ that contains a bad embedding H_0. Let w be the vertex of G_0 obtained by the contraction. We say that G is obtained from G_0 by an *expansion* of w; note that G is not necessarily uniquely determined by specifying G_0 and w. Since G contains no bad embedding, H_0 contains w. Let $w_1, \ldots, w_5$ be the remaining vertices of H_0.

A 4-*expansion* is an expansion of w such that v_1 or v_3 is in G adjacent to four of the vertices $w_1, \ldots, w_5$. Otherwise, we call the expansion a 3-*expansion*. Note that neither v_1 nor v_3 can be adjacent to all the five vertices $w_1, \ldots, w_5$ since K_6 is not a subgraph of G.

Let us consider the subgraph H of G obtained from H_0 by splitting w back into v_1 and v_3 and introducing a 2-vertex (the vertex v) joined to them. The subgraph H is not necessarily an induced subgraph of G. Note that if an expansion is a 4-expansion, then we can assume that the degree of v_1 or v_3 in H is five (the choice of H_0 in G_0 can be ambiguous since the subgraph of G_0 induced by $w, w_1, \ldots, w_5$ need not to be simple, but there is always a choice of H_0 with the described property). Also note that v_1 and v_3 are not adjacent in G, but the vertex v can be adjacent in G to vertices of H distinct from v_1 and v_3. The faces of H incident with v and their counterparts in H_0 are referred to as *expanded faces*. We keep the notation introduced in this paragraph throughout this section. In particular, we always refer to the expanded vertex as w and the vertices obtained from it as v, v_1 and v_3. The subgraphs of G contained in the two expanded faces are further denoted by G_1 and G_2.

We establish in the following series of lemmas that every Eulerian triangulation of the Klein bottle with no complete graph of order six that is obtained by an expansion of a vertex of a bad embedding is 5-colorable. Before we do so, we first observe the following simple fact about a bad embedding with a 5-face:

Proposition 29. *If H_0 contains a 5-face, then its 4-face contains no chord. In particular, if two vertices of H_0 are joined by multiple edges, then one of such edges is contained in the interior of the 5-face of H_0.*

Proof. Suppose that H_0 contains a 5-face. Note that H_0 is 5-regular. We can assume without loss of generality that G_0 is proper since neither removing the interior of any 3-face nor replacing the interior of the 4-face with a chord or a single interior vertex change the parity of degrees of the vertices of G_0 by Propositions 8 and 9. Similarly, we can assume by Proposition 10 that the 5-face of H_0 contains exactly two chords. Assume now for the sake of contradiction that the 4-face of H_0 contains a chord. Hence, we are assuming that G_0 contains precisely three edges in addition to those of H_0. Since G_0 is Eulerian, the three edges form a matching. However, two of these edges (the chords in the 5-face) meet at the same vertex. This contradicts our assumption that the 4-face of H_0 contains a chord. □

We are now ready to establish the main lemma of this section.

Lemma 30. *If G is an Eulerian triangulation of the Klein bottle with no complete graph of order six that can be obtained by an expansion from an Eulerian triangulation containing a bad embedding, then G is 5-colorable.*

Proof. Let us first consider the case when G is obtained from G_0 by a 4-expansion. By symmetry, we can assume that $w_1, \dots, w_4$ are adjacent to v_1 and w_5 to v_3 in H (see Figure 20 for illustration). If neither of the expanded faces is a 5-face, color the vertices $w_1, \dots, w_4$ with four mutually distinct colors and the vertices v_1 and w_5 with the fifth color. Color next the vertex v_3 with a color distinct from all its precolored neighbors in G (note that

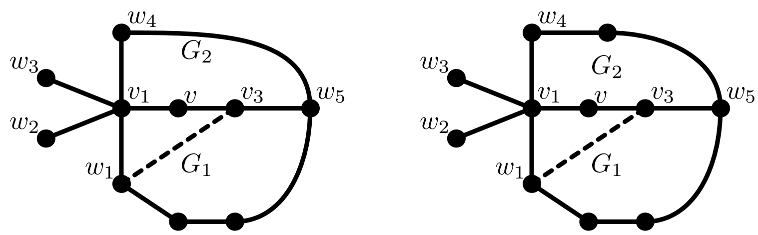


Figure 20: The configurations that can be obtained by a 4-expansion such that one of the expanded faces is a 5-face.

v_3 is adjacent to at most four precolored vertices). The precoloring can be extended to G_i, $i = 1, 2$, by Proposition 11 if G_i is bounded by a 5-cycle or by Proposition 12 if it is bounded by a 6-cycle. Since v has degree four in G, we can recolor it in such a way that the obtained coloring is proper. Finally, the coloring is extended to the subgraphs contained in the remaining faces of H by Proposition 11.

Assume now that one of the expanded faces is a 5-face, i.e., G contains one of the two configurations depicted in Figure 20. If G contains an edge between w_1 and v_3, then the edge w_1v_3 is contained in G_1 by Proposition 29. In such a case, we consider the same precoloring as in the previous paragraph and extend it separately to the two parts of G_1 delimited by the dashed edge by Proposition 11. If G does not contain the edge w_1v_3, there is a precoloring of the vertices $v_1, v_3, w_1, \ldots, w_5$ such that the vertices v_1 and w_5 have the same color and the vertices v_3 and w_1 have the same color. Such a precoloring can be extended to G_1 by Proposition 14. The remaining arguments needed to finish this case are the same as in the previous paragraph.

We now analyze the case when G is obtained from G_0 by a 3-expansion. By symmetry, we can assume that v_1 is adjacent to w_1, w_2 and w_3, and v_3 to w_4 and w_5 in H (see Figure 21). Since the expansion is a 3-expansion, the vertex v_1 is not adjacent to w_4 or w_5 in G. Similarly, v_3 cannot be adjacent to both the vertices w_1 and w_3. If v_3 is not adjacent to w_1, then color the vertices v_3 and w_1 with the same color, the vertices v_1 and w_4 with another color and the remaining vertices w_2, w_3 and w_5 with three mutually distinct colors. Note that the boundaries of both G_1 and G_2 contain two vertices with the same color. Hence, if neither of the expanded faces is a 5-face, the

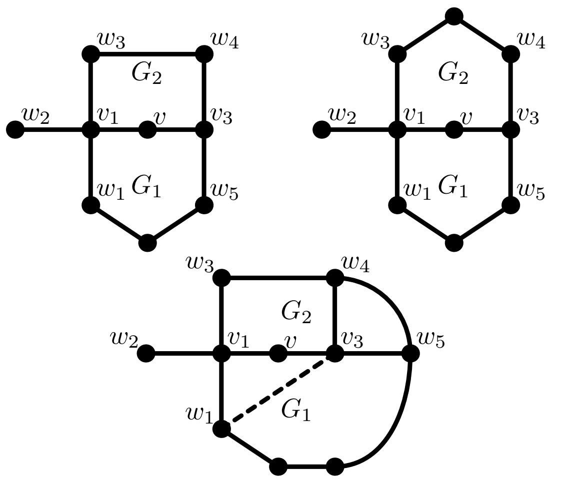


Figure 21: Some of the configurations that can be obtained by a 3-expansion.

precoloring of the vertices $v_1, v_3, w_1, \ldots, w_5$ can be extended to G_1 and G_2 by Propositions 11 and 12 and to the rest of G (after a possible recoloring of v) by Proposition 11. We conclude that G is 5-colorable in such a case.

It remains to consider the case when one of the expanded faces is a 5-face, say, the face corresponding to G_1. If G contains the edge w_1v_3, then the edge w_1v_3 is contained in G_1 by Proposition 29. We consider the precoloring of the vertices $v_1, v_3, w_1, \ldots, w_5$ as in the previous paragraph and extend it to the two parts of G_1 delimited by the edge w_1v_3 by Proposition 11. The rest of the analysis is analogous.

Hence, we can further assume that the vertices w_1 and v_3 are not adjacent in G. Color the vertices w_1 and v_3 with the same color, the vertices w_5 and v_1 with the same color and the vertices $w_2, \ldots, w_4$ with the remaining three colors. This precoloring can be extended to G_1 by Proposition 14. If G_2 is bounded by a 5-cycle, then the precoloring can be extended to it by Proposition 11. If G_2 is bounded by a 6-cycle, it has been obtained by an expansion of a 4-face. However, such a configuration can be obtained from the bad embedding containing the 5-face in a unique way (see Figure 22). In particular, the subgraph G_1 is bounded by the 7-cycle $v_1w_1w_4w_2w_5v_3v$ and G_2 by the 6-cycle $v_1w_3w_1w_4v_3v$. Hence, the precoloring can be extended to G_1 by Proposition 14 and to G_2 by Proposition 13. The rest of the analysis

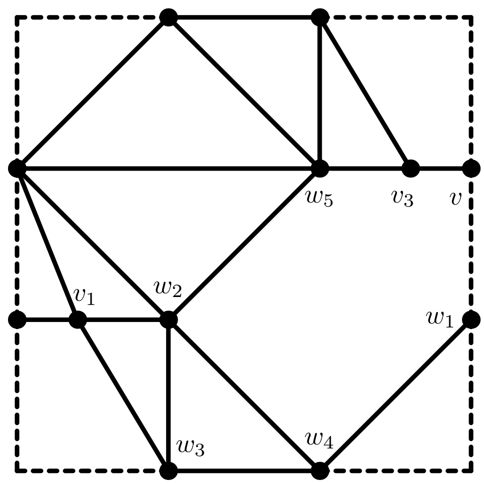


Figure 22: A unique 3-expansion such that the expanded faces are a 4-face and a 5-face.

follows the lines of the previous cases. □

10 Characterization of six-chromatic Eulerian triangulations

We now combine the results obtained in the previous sections to state and prove our main results.

Theorem 31. *An Eulerian triangulation G of the Klein bottle is 5-colorable unless it contains a complete graph of order six.*

Proof. The proof proceeds by induction on the order of G. If G is not proper, then remove the interior of a separating 3-cycle or replace the interior of a separating 4-cycle with a chord or a vertex keeping G Eulerian (note that this is always possible by Proposition 9) and color the new triangulation by induction. The obtained coloring can be extended to the original triangulation G by Proposition 11. Hence, we assume in the rest that G is proper.

If G does not contain a contractible vertex, G is 5-colorable unless it contains a complete graph of order six by Lemma 27. If G contains a contractible vertex v with neighbors v_1 and v_3, apply induction to $G.v_1vv_3$. If $G.v_1vv_3$ does not contain a complete graph of order six, then it is 5-colorable and it is easy to obtain a 5-coloring of G by assigning the color of the contracted vertex to both v_1 and v_3. If both G and $G.v_1vv_3$ contain a complete graph of order six, there is nothing to prove. Finally, if $G.v_1vv_3$ contains a complete graph of order six, but G does not, then G is 5-colorable by Lemma 30. □

An immediate corollary of Theorems 28 and 31 is the following:

Corollary 32. *Every Eulerian triangulation of the Klein bottle with face-width at least four is* 5*-colorable.*

11 Concluding remarks

We outline two directions for possible future research. Let us start with algorithmic issues. For every fixed surface Σ, Dvořák, Král' and Thomas [10] found a polynomial-time algorithm for computing the chromatic number of graphs without contractible triangles that can be embedded on Σ. In particular, the chromatic number of quadrangulations of a fixed surface can be determined in polynomial time. It would be interesting to see whether their results can be adapted to the setting of Eulerian triangulations.

Another direction for future research could be extending the results to list colorings. Let us recall that a graph G is *list k-colorable*, if for every assignments of lists of k colors to its vertices, there exists a proper coloring of the vertices with colors from their lists. Clearly, each list k-colorable graph is k-colorable but the converse is not true, e.g., there are planar graphs that are not list 5-colorable [30].

As proved in [4], every graph on the projective plane, which is not list 5-colorable, contains K_6 as a subgraph. Proposition 4 implies that no Eulerian triangulation of the projective plane contain K_6 as a subgraph. Hence, all Eulerian triangulations of the projective plane are list 5-colorable.

Our Theorem 3 asserts that Eulerian triangulations of the Klein bottle, which do not contain K_6 as a subgraph, are 5-colorable. However, it is not clear whether the same property holds for list colorings or not. One of the results in this direction is that any graph embedded on the Klein bottle (or any other fixed surface) with sufficiently large edge-width is list 5-colorable. This follows from a recent result of DeVos, Kawarabayashi, and Mohar [5].

References

[1] M. O. Albertson, J. P. Hutchinson: The three excluded cases of Dirac's map-color theorem, Ann. New York Acad. Sci. 319 (1979), 7–17.

[2] K. Appel, W. Haken: Every planar map is four colorable, Bull. Am. Math. Soc. 82 (1976), 449–456.

[3] D. Archdeacon, J. Hutchinson, A. Nakamoto, S. Negami, K. Ota: Chromatic numbers of quadrangulations on closed surfaces, J. Graph Theory 37 (2001), 100–114.

[4] T. Böhme, B. Mohar, M. Stiebitz: Dirac's map-color theorem for choosability, J. Graph Theory 32 (1999), 327–339.

[5] M. DeVos, K.-I. Kawarabayashi, B. Mohar: Locally planar graphs are 5-choosable, preprint, 2006.

[6] M. DeVos, P. D. Seymour: Extending partial 3-colourings in a planar graph, J. Comb. Theory Ser. B 88(2) (2005), 219–225.

[7] K. Diks, L. Kowalik, M. Kurowski: A new 3-color criterion for planar graphs, in: Proc. 28th Workshop Graph-Theoretic Concepts in Comp. Sci. (WG'02), LNCS vol. 2573, 2002, 138–149.

[8] G. A. Dirac: Map colour theorems related to the Heawood colour formula, J. London Math. Soc. 31 (1956), 460–471.

[9] M. DeVos, L. Goddyn, B. Mohar, D. Vertigan, X. Zhu: Coloring-flow duality of embedded graphs, Trans. Amer. Math. Soc. 357 (2005), 3993–4016.

[10] Z. Dvořák, D. Král', R. Thomas: Coloring triangle-free graphs on surfaces, in preparation.

[11] S. Fisk: Geometric coloring theory, Advances in Math. 24 (1977), 298–340.

[12] M. R. Garey, D. S. Johnson, and L. J. Stockmeyer: Some simplified NP-complete graph problems, Theor. Comput. Sci. 1 (1976), 237-267.

[13] J. Gimbel, C. Thomassen: Coloring graphs with fixed genus and girth, Trans. Am. Math. Soc. 349 (1997), 4555–4564.

[14] J. P. Hutchinson: Three-coloring graphs embedded on surfaces with all faces even-sided, J. Combin. Theory, Ser. B 65 (1995), 139–155.

[15] J. Hutchinson, R. B. Richter, P. Seymour: Colouring Eulerian triangulations, J. Combin. Theory, Ser. B 84 (2002), 225–239.

[16] M. Król: On a sufficient and necessary condition of 3-colorableness for the planar graphs. I (in Polish). Prace Nauk. Inst. Mat. Fiz. Teoret. Politechn. Wrocław Ser. Studia i Materiały, No. 6, Zagadnienia kombinatoryczne (1972), 37–40.

[17] D. Král', R. Thomas: Coloring even-faced graphs in the torus and the Klein bottle, submitted.

[18] N. Martinov: 3-colorable planar graphs (in Russian), Serdica 3 (1977), 11–16.

[19] B. Mohar: Coloring Eulerian triangulations of the projective plane, Discrete Math. 244(1–3) (2002), 339–343.

[20] B. Mohar, P. D. Seymour: Coloring locally bipartite graphs on surfaces, J. Combin. Theory Ser. B 84 (2002), 301–310.

[21] B. Mohar, C. Thomassen: Graphs on Surfaces, Johns Hopkins University Press, Baltimore, MD, 2001.

[22] A. Nakamoto: 5-chromatic even triangulations on surfaces, submitted.

[23] S. Negami: Classification of 6-regular Klein-bottlal graphs, Res. Rep. Inf. Sci. T.I.T. A-96 (1984).

[24] N. Robertson, D. P. Sanders, P. Seymour, R. Thomas, Efficiently four-coloring planar graphs, in: Proc. 28th ACM Symp. Theory Comput. (STOC), ACM Press, 1996, 571–575.

[25] N. Robertson, D. Sanders, D. Seymour, R. Thomas: The four color theorem, J. Combin. Theory Ser. B 70 (1997), 2–44.

[26] A. Sainte-Laguë: Géométrie de situation et jeux, p. 11, Gauthier-Villars, Paris, 1929; Jbuch 55, 974.

[27] N. Sasanuma: Chromatic numbers of 6-regular graphs on the Klein bottle, to appear in Discrete Mathematics.

[28] C. Thomassen: Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface, Trans. Amer. Math. Soc. 323(2) (1991), 605–635.

[29] C. Thomassen: Five-coloring maps on surfaces, J. Combin. Theory, Ser. B 59 (1993), 89–105.

[30] M. Voigt, List colouring of planar graphs, Discrete Math. 120 (1993), pp. 215–219.

[31] D. A. Youngs: 4-chromatic projective graphs, J. Graph Theory 21 (1996), 219–227.