On the Bounded Degree Restriction of Constraint Satisfaction Problems

Mark H. Siggers
Department of Applied Mathematics
and Institute for Theoretical Computer Science (ITI)
Charles University Malostranské nám. 25,
11800 Praha 1 Czech Republic.
email: `mhsiggers@gmail.com`

Abstract

Given a graph H, let $b(H)$ be the minimum integer b, if it exists, for which H-colouring is NP-complete when restricted to instances with degree bounded by b. We show for any loopless non-bipartite graphs H that $b(H)$ is bounded above by a function of the size of H. Futhermore, we get tight upper bounds on $b(H)$ for various H. For example, we show that $b(H) = 3$ for any graph H with girth at least 7 in which every edge lies in a g-cycle, where g is the odd-girth of H.

Extending our purview to CSPs, we show for relational systems $(\mathcal{H})$ with projective cores $\mathcal{C}$, that the analagously defined function $b(\mathcal{H})$ is bounded above by a polynomial function of the maximum degree of $\mathcal{C}$.

1 Introduction

Definitions for graphs are given in Subsection 1.3, and definitions for relational systems and general CSPs are given in Subsection 5.1.

1.1 Background

In the last few decades there has been much interest in the computational complexity of the H-colouring problem, its natural generalisation to CSPs, and various restricted versions of these problems. A good introduction to such problems is [8].

In [7] it was shown that the question of H-colourability (CSP(H)) for graphs H, has the following dichotomy. If H is bipartite, then CSP(H) is polynomial time solvable; otherwise, CSP(H) is NP-complete. Thus CSP(K_k) is NP-complete for any $k \geq 3$.

On the other hand, by a well known theorem of Brooks, [1], CSP(K_k) is polynomial time solvable, for any $k \geq 3$, when restricted to instances of degree bounded by k.

It was observed in [5] that CSP(K_3) is NP-complete when restricted to instances of degree bounded by 4. This fact follows from a result of Holyer showing that the problem of deciding if there exists 3-edge colouring of a 3-regular graph, is NP-complete.

For any relational system $\mathcal{H}$, we will denote the restriction of the $\mathcal{H}$-colouring problem CSP($\mathcal{H}$) to instances of maximum degree b by CSP$(\mathcal{H})_b$. Thus by the examples above, CSP$(K_3)_b$ is polynomial time solvable for any $b \leq 3$ and NP-complete for any $b \geq 4$.

Let $b(\mathcal{H})$ be the minimum b such that CSP$(\mathcal{H})_b$ is NP-complete. (If there is no such b, then let $b(\mathcal{H}) = \infty$.) Thus we have that $b(K_3) = 4$. Though in [6] it was shown that $b(\mathcal{H})$ can be arbitrarily large, even when $\mathcal{H}$ is a graph, the following is conjectured in [3].

Conjecture 1.1. *For any relational system $\mathcal{H}$ for which* CSP($\mathcal{H}$) *is NP-complete, $b(\mathcal{H})$ is finite.*

In [5], a stronger version of a particular graph case of Conjecture 1.1 is investigated. It is shown that the graphs H constructed in [6], which show that $b(H)$ can be arbitrarily large, all have chromatic number greater than three. It is then conjectured that

Conjecture 1.2. *[5] For any triangle-free graph H with chromatic number three, $b(H) = 3$.*

In the same paper, the following, which, in particular, settles their conjecture in the case of odd-cycles, is also proved.

Theorem 1.3. *[5] For a graph H of odd girth $g \geq 5$, $b(H) = 3$ if*

i. No two g-cycles of H share more than an edge.

ii. Every vertex of H lies in a g-cycle.

1.2 Results and Outline of Paper

In Section 2, we recall the Indicator and Sub-Indicator constructions of [7], and then introduce an Alternate Sub-Indicator construction. We informally observe that when the Alternate Sub-Indicator construction is used in place of the Sub-Indicator construction in the proof of the CSP(H) dichotomy from [7], we get a verification of Conjecture 1.1 for graphs. In Section 3 we use the CSP(H) dichotomy, and many ideas of its proof to formally verify Conjecture 1.1, for graphs. We do this with the following result.

Theorem 1.4. *For any loopless non-bipartite graph H,*

$$b(H) \leq \beta(H) := (2\Delta(H)+1)2^{|E(\overline{H})|},$$

where $\Delta(H)$ is the maximum degree of H.

This upper bound for $b(H)$ is far from tight. In Section 4 we show the following tight bound for $b(H)$, which overlaps the scope of Theorem 1.3, but neither implies nor is implied by it.

Theorem 1.5. *For a graph H of odd girth $g \geq 5$, $b(H) = 3$ if*

i. H has no 4 or 6-cycles.

ii. Every vertex of H lies in a g-cycle.

In particular, this gives that $b(H) = 3$ for any graph H with girth at least 7 and odd girth g, in which every vertex lies in a g-cycle.

Weakening condition (i), we get

Theorem 1.6. *For a graph H of odd girth $g \geq 3$, $b(H) \leq 4$ if*

i. H has no 4-cycles.

ii. Every vertex of H lies in a g-cycle.

This gives us in particular that $b(H) \leq 4$ for the Petersen graph.

A simple construction then allows us to get rid of condition (ii) in these theorems at the price or doubling the bound. In particular, we get the following.

Corollary 1.7. *For any non-bipartite graph H, $b(H) \leq 8$ if H is C_4-free, and $b(H) \leq 6$ if H has girth at least 7.*

In Section 5 we improve the bound from Theorem 1.4 for projective graph cores H. Indeed we get a bound that is polynomial in $\Delta(H)$. Moreover, the bound works for all relational systems that are projective cores. We show that

Theorem 1.8. *For any relational system $\mathcal{H}$ which is a projective core,*

$$b(\mathcal{H}) \leq 4 \cdot \Delta(\mathcal{H})^6.$$

It was shown in [9] that, asymptotically, almost all relational systems are projective. Thus this generally stronger bound applies for most graphs and relational systems.

1.3 Graph Definitions and Conventions

All graphs are assumed to be simple, undirected and loopless. Let G and H be graphs. Then an *H-colouring* of G, is a mapping χ from the vertices $V(G)$ of G to the vertices $V(H)$ of H, for which $\chi(u)\chi(v)$ is an edge of G if uv is an edge of H. Given a graph H, CSP(H) is the problem of deciding whether an given graph, or instance, admits a homomorphism to H, $\Delta(H)$ is the maximum degree of H, and $\overline{H}$ is the complement of H. A graph H is a *(graph) core* if its only monomorphisms are automorphisms.

It is well known (see [8]) that any graph H has a unique core H' and the problem CSP(H) is polynomially equivalent to the problem CSP(H'). Thus when considering CSP(H), we will always assume H to be a core.

The necessary definitions for general CSPs are not needed until Section 5, so will be given there.

2 Indicator-type Constructions

In their seminal paper [7], Hell and Nešetřil showed that CSP(H) is NP-complete for a graph H if and only if H is not bipartite. Their proof contained several applications of the so-called Indicator and Sub-Indicator constructions, specific cases of which had been used in [10, 12]. In this section, we recall these constructions and provide an Alternate Sub-Indicator construction. The Indicator Construction and the Alternate Sub-Indicator Construction will be used in Section 3 to prove Theorem 1.4.

Construction 2.1 (Indicator Construction [7]). *Let H be a fixed core, and let I be a fixed graph with specified vertices i and j such that there is* *an automorphism of I switching i and j. For any graph G, construct ${}^*G$ as follows. For any edge uv of G, replace uv with a copy I_e of I by identifying the copies of i and j in I_e with u and v respectively.*

Let H^* be the graph on $V(H)$ whose edges uv are those pairs of vertices u and v for which there exists an H-colouring χ of I with $\chi(i) = u$ and $\chi(j) = v$. Then

$$^*G \to H \iff G \to H^*.$$

This construction, is used in [7] for the implications

- CSP(H^*) is NP-complete ⇒ CSP(H) is NP-complete.
- CSP(H) is polynomial time solvable ⇒ CSP(H^*) is polynomial time solvable .

Observing that $\Delta({}^*G) = c\Delta(G)$, where c is the maximum degree of the vertices x and y in I, we can replace these implications with the following inequality.

$$b(H) \leq c \cdot b(H^*). \tag{1}$$

Construction 2.2 (Sub-indicator Construction [7]). *Let H be a fixed core with specified vertices $x_1, \ldots, x_t \in V(H)$, and let J be a fixed graph with specified vertices $k_1, \ldots, k_t$, and $j \in V(J)$. For any graph G, construct ${}^+G$ as follows.*

i. Begin with the disjoint union of G and H.

ii. For every vertex v_α of G, let J_α be a new copy of J.

iii. Identify the copy of j in J_α with v_α, and for $i = 1, \ldots, t$, identify the copy of k_i in J_α with x_i of H.

Let ${}^+G$ be the union of G, H, and J_α for $\alpha = 1, \ldots, n$.

Given a mapping ϕ taking k_i of J to x_i of H for all $i = 1, \ldots, t$, let V be the set of all vertices of H that are the image of j under some H-colouring of J that extends ϕ. Let H^+ be the subgraph of H induced by V. Then

$$^+G \to H \iff G \to H^+.$$

This construction is again used for the implications

- CSP(H^+) is NP-complete ⇒ CSP(H) is NP-complete, and

- CSP(H) is polynomial time solvable $\Rightarrow$ CSP(H^+) is polynomial time solvable,

but does not lead immediately to a statement analogous to inequality (1). This is because $\Delta({}^+G)$ depends on $|V(G)|$, as opposed to just on $\Delta(G)$.

We present an alternate version of Construction 2.2 in which $\Delta({}^+G)$ depends on $\Delta(G)$ and $\Delta(H)$.

The only vertices in ${}^+G$ of Construction 2.2 that have degree growing with $n = |V(G)|$, are the vertices of $x_1, \ldots, x_t$ in the copy of H. In Construction 2.3, we repeat Construction 2.2, but replace the copy of H with a new graph $H'(n)$. This graph will allow us to 'share' the neighbours of a vertex x_i in H among n different 'copies' of x_i. This will be enough to reduce maximum degree of the graph ${}^+G$.

Construction 2.3 (Alternate Sub-indicator Construction.). *Let H be a fixed core with specified vertices $x_1, \ldots, x_t \in V(H)$.*
The Graph $H'(n)$. *For $i = 1, \ldots, t$, let H_i be a copy of the graph H with an extra vertex y_i which has the same neighbourhood as x_i. Given an integer n, we now define the graph $H' = H'(n)$ from H.*

For $i = 1, \ldots, t$, do the following.

- *For $\alpha = 1, \ldots, n$, let H_i^α be a copy of H_i, and let x_i^α and y_i^α be the copies of x_i and y_i, respectively, in H_i^α*
- *For $\alpha = 1, \ldots, n-1$, identify the copy of y_i in H_i^α with the copy of x_i in $H_i^{\alpha+1}$.*
- *Identify the copy of y_i in H_i^n with the vertex x_i of H.*

Let H' be the union of H, and H_i^α for $i = 1, \ldots, t$ and $\alpha = 1, \ldots, n$.

Observe that in any H-colouring of H_i, H being a core implies that x_i and y_i are mapped to the same vertex. Thus in any H-colouring χ of H', we have for $i = 1, \ldots, k$ that

$$\chi(x_i^1) = \chi(x_i^2) = \cdots = \chi(x_i^n) = \chi(x_i). \tag{2}$$

Let J be a fixed graph with specified vertices $k_1, \ldots, k_t$, and $j \in V(J)$.

The Graph ${}^{++}G$.

Given a graph G with $V(G) = \{v_1, \ldots, v_n\}$, construct ${}^{++}G$ as follows.

i. Begin with the disjoint union of G and $H' = H'(n)$.

ii. For every vertex v_α of G, let J_α be a new copy of J.

iii. Identify the copy of j in J_α with v_α, and for $i = 1, \dots, t$, identify the copy of k_i in J_α with x_i^α of H'.

Let ${}^{++}G$ be the union of G, H', and J_α for $\alpha = 1, \dots, n$.

Let H^+ be as in Construction 2.2. Observe that the only difference in the construction of ${}^+G$ in Construction 2.2, and the construction of ${}^{++}G$, is in step (iii). *It then follows from this, and from from the statement* (2), *that*

$$^{++}G \to H \iff {}^+G \to H \iff G \to H^+.$$

One observes that with this construction,

$$\Delta({}^{++}G) \leq \max\big(\Delta(G) + \deg_J(j), 2\Delta(H) + 1\big).$$

Since J and H are independent of G, $\Delta({}^{++}G)$ is bounded if $\Delta(G)$ is. Indeed, we get the implication

$$b(H) \leq \max\big(b(H^+) + \deg_J(j), 2\Delta(H) + 1\big). \tag{3}$$

Remark 2.4. *In the same way, one could construct an alternate version of the Edge-Sub-Indicator construction of [7] which would give an inequality analagous to (3). With this we could now prove that Conjecture 1.1 is true for graphs. Indeed, inequality (1) implies the following implications:*

- $\mathrm{CSP}(H^*)_b$ *is NP-complete for some $b \Rightarrow$* $\mathrm{CSP}(H)_{b'}$ *is NP-complete for some b'.*
- $\mathrm{CSP}(H)_b$ *is polynomianl time solvable for all $b \Rightarrow$* $\mathrm{CSP}(H^*)_b$ *is polynomial time solvalbe for all b .*

We get similar statements for the Alternate Sub-Indicator and Alternate Edge-Sub-Indicator constructions. Using these Alternate constructions in the proof from [7] that $\mathrm{CSP}(H)$ *is NP-complete for any non-bipartite graph H, one could actually prove the stronger statement that for all non-bipartite graphs H, there exists some b for which* $\mathrm{CSP}(H)_b$ *is NP-complete. This implies that Conjecture 1.1 is true for graphs.*

3 Proof of Theorem 1.4

In this section, we essentially follow the outline of Remark 2.4 to prove Theorem 1.4; however, to get the bound given by the theorem, we must take some care. On the other hand, we save ourselves some work by assuming the result of [7].

We will prove Theorem 1.4 by contradiction, following much the same strategy used in [7]. Assume that H is a non-bipartite graph core which is a counter-example to Theorem 1.4. This assumption is equivalent to the assumption that

$$b(H) > \beta(H) := (2\Delta(H) + 1)2^{|E(\overline{H})|}.$$

Further assume that

a) H is a counter-example that minimises $|V(H)|$, and

b) subject to condition a), H is a counter-example that minimises $|E(\overline{H})|$.

We now show that we can make several assumptions about H.

Claim 3.1. *H may be assumed to contain a triangle.*

Proof. Since H is non-bipartite, it contains an odd cycle. Assume it does not contain a triangle. It is shown in [7] that letting I be a path on four vertices, with endpoints i and j, the Indicator Construction on such an H yields an H^* with $V(H^*) = V(H)$ and $|E(\overline{H^*})| < |E(\overline{H})|$.

Using equation (1) with the fact that $c = \max(\deg_I(i), \deg_I(j)) = 1$ gives that $b(H^*) > b(H)$. We have $b(H) > \beta(H)$ by assumption. Thus we have just to show that $\beta(H) > \beta(H^*)$ to get

$$b(H^*) > \beta(H^*),$$

which would show that H^* is a counterexample contradicting b).

Observe the following obvious inequalities.

- for $d \geq 1$, $\frac{2(\Delta(H)+d)+1}{2\Delta(H)+1} < 2^d$
- $d := |E(\overline{H})| - |E(\overline{H^*})| \geq \Delta(H^*) - \Delta(H)$

Using these inequalities, we get

$$\begin{aligned}\beta(H) &= (2\Delta(H)+1)2^{|E(\overline{H})|} \\ &> (2(\Delta(H)+d)+1)2^{|E(\overline{H})|-d} \\ &\geq (2\Delta(H^*)+1)2^{|E(\overline{H^*})|} \\ &= \beta(H^*).\end{aligned}$$

This is what remained to be shown, so completes the proof of the claim. □

Claim 3.2. *It may be assumed that every vertex in H is in a triangle.*

Proof. Given a graph G, construct ${}^{++}G$ as follows. Attach a new triangle to each vertex v of G by identifying v with one vertex of the triangle. This can be seen as a simplified (Alternate) Sub-Indicator construction with $J = K_3$, and $t = 0$. It is observed in [7] that this yields an induced subgraph graph H^+ of H consisting of all vertices that are in a triangle. Since $\deg_J(j) = 2$ equation (3) gives us that either $2\Delta(H) + 1 \geq b(H)$, or $b(H^+) \geq b(H) - 2$. Since the former is false by the assumption $b(H) > \beta(H)$, the latter is true. Again, to show that

$$b(H^+) > \beta(H^+),$$

which would contradict a), we must only show that $\beta(H) - 2 > \beta(H^+)$.

Because H^+ is an induced subgraph of H, we have that $\Delta(H^+) \leq \Delta(H)$, and $E(\overline{H^+}) \subseteq E(\overline{H})$. In fact $E(\overline{H^+}) \subsetneq E(\overline{H})$. To see this, observe that there is some vertex v in H that is not in a triangle, so not in H^+ (otherwise the claim is proved). Since v is not in a triangle in H, and $E(H^+)$ is non-empty by Claim 3.1, there is some u in H^+ such that uv is not in H. That is, uv is in $E(\overline{H})$ but not in $E(\overline{H^+})$. Using $\Delta(H^+) \leq \Delta(H) \geq 2$ and $|E(\overline{H^+})| < |E(\overline{H})|$, we get

$$\begin{aligned}\beta(H^+) &= (2\Delta(H^+)+1)2^{|E(\overline{H^+})|-1} \\ &< (2\Delta(H)+1)2^{|E(\overline{H})|-1} \\ &= \beta(H)/2 \\ &\leq \beta(H) - 2.\end{aligned}$$

Thus H^+ is another counterexample to the theorem, contradicting a). This proves the claim. □

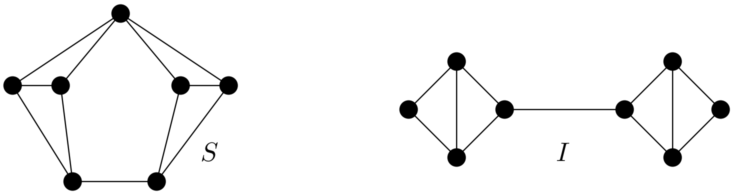

Figure 1: S from Claim 3.4, and I from Claim 3.5.

Claim 3.3. *H can be assumed to contain no K_4.*

Proof. Assume that H contains a K_4 containing the vertex v. Let J be an edge with endpoints j and k_1. Then the Alternate Sub-Indicator Construction on H, where $v = x_1$, returns the subgraph H^+ of H induced by the neighbourhood of v. Since H^+ contains a triangle, it is non-bipartite.

Now $deg_J(j) = 1$, so using equation (3) the same way as we did in Claim 3.2, we get that $b(H^+) > \beta(H) - 1$. We thus have to show that $\beta(H) \geq \beta(H^+) + 1$ to imply that $b(H^+) > \beta(H^+)$, and contradict a).

Since H^+ is an induced subgraph of H, we have that

$$|E(\overline{H^+})| \leq |E(\overline{H})|. \tag{4}$$

Since H^+ is induced by the neighbours of v, we have that

$$\Delta(H^+) \leq \deg_{H^+}(v) \leq \deg_H(v) \leq \Delta(H).$$

If middle inequality is equality, then v has a non-neighbour outside of $V(H^+)$, and so the inequality in equation (4) is strict. Either way, we have the strict inequality $\beta(H) > \beta(H^+)$, as needed.

□

Now let S be the Penny graph shown in Figure 1.

Claim 3.4. *H can be assumed to contain no homomorphic image of S.*

Proof. The proof used the fact that H contains no copy of K_4, which we can assume by Claim 3.3. Let J be a 3-star having leaves j, k_1, and k_2. It is shown in [7] that if H contains no K_4 and contains a homomorphic image of S then the Sub-Indicator Construction with J yields an induced subgraph

H^+ of H, with fewer vertices than H, but still containing a triangle. The same is true using the Alternate Sub-Indicator Construction.

Since $\deg_J(j) = 1$, we can use equation (3) as we did in Claim 3.2 to assert that $b(H^+) > \beta(H) - 1$. We must again show that $\beta(H) \geq \beta(H^+) + 1$.

Since H^+ is an induced subgraph of H we again have the inequalities $|E(\overline{H^+})| \leq |E(\overline{H})|$ and $\Delta(H^+) \leq \Delta(H)$. Moreover, $|V(H^+)| < |V(H)|$, so there is some vertex v in $V(H) - V(H^+)$. Because H^+ contains a triangle, and H is K_4-free, v has a non-neighbour in the triangle. Thus $|E(\overline{H^+})| < |E(\overline{H})|$, and so $\beta(H) > \beta(H^+)$, as needed.

□

Let K_4^- be the graph constructed by removing an edge from K_4.

Claim 3.5. *H can be assumed to contain no K_4^-.*

Proof. Let I be the graph show in Figure 1. In [7] it is shown that if H is a core containing a homomorphic image of S, in which every vertex is in a triangle, the Indicator Construction with I returns a graph H^* on the same vertices as H and containing all the edges of H. By Claims 3.2 and 3.4, we make make these assumptions on H. They then show that if H contains a K_4^-, then H^* has some edge not in H. Thus $|E(\overline{H^+})| < |E(\overline{H})|$.

Using equation (1) with the fact that $\max(\deg_I(i), \deg_I(j)) = 2$, and the assumption that $b(H) > \beta(H)$, we get that $b(H^*) > \beta(H)/2$. Using the fact that $|E(\overline{H}) - E(\overline{H^*})| \geq \Delta(H^*) - \Delta(H)$ we can show, as we did in Claim 3.1, that $\beta(H)/2 \geq \beta(H^*)$. Thus $b(H^*) > \beta(H^*)$, and H^* is another counter-example to the theorem, contradicting b).

□

Using the assumptions that H is a K_4^--free graph in which every vertex is in a triangle, the following lemma finishes of the proof of the theorem.

Lemma 3.6. *Let H be any graph core for which*

i. H is K_4^--free, and

ii. every vertex is in a K_3,

then $b(H) \leq 5$.

Proof. Given any integer $d \geq 5$, let $I(d)$ be the graph shown in Figure 2, which is made from $d-1$ copies of K_4^-, and contains the vertices $a_1, \ldots, a_d$.

Given a graph G with vertices $v_1, \ldots, v_n$, let G' be the graph constructed algorithmically as follows.

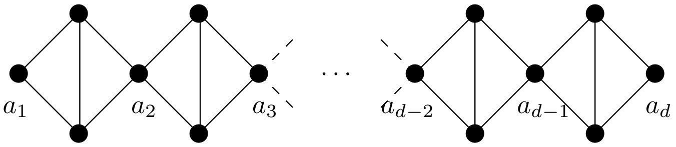

Figure 2: $I(d)$ from Lemma 3.6.

- Let $G_0 = G$.
- For $i = 1, \ldots, n$, if the vertex v_i has degree $d > 5$, then let G_i be constructed from G_{i-1} as follows.
  - Let I^i be a new copy of $I(d)$, and for $j = 1, \ldots, d$ let a^i_j be the copy of a_j in I^i.
  - Let M be a matching between the neighbourhood of v_i in G_{i-1}, and the set $\{a^i_1, \ldots, a^i_d\}$ in I^i.
  - Let $V(G_i) = V(G_{i-1} - v_i) \cup V(I^i)$.
  - Let $E(G_i) = E(G_{i-1} - v_i) \cup E(I^i) \cup M$.

  Otherwise, let $G_i = G_{i-1}$.
- Let $G' = G_n$.

Because H is K_4^- free, $\chi(a_1) = \cdots = \chi(a_d)$ is forced in any H-colouring χ of $I(d)$. Furthermore, for any vertex h of H, h is in a triangle, so there is an H-colouring χ of $I(d)$, for any d, in which $\chi(a_1) = \cdots = \chi(a_d) = h$. It is thus not hard to see that

$$G' \to H \iff G \to H.$$

Indeed, let χ be an H-colouring of G. Setting $\chi'(a^i_j) = \chi(v_i)$, for $i = 1, \ldots, n$ and $j = 1, \ldots, \deg(v_i)$, defines an H-colouring χ' of G'. On the other hand, let χ' be an H-colouring of G'. Then setting $\chi(v_i)$ equal to $\chi(a^i_1) = \cdots = \chi(a^i_{\deg(v_i)})$ for $i = 1, \ldots, n$ defines an H-colouring of G.

Since G' has maximum degree 5, this gives a reduction of $\mathrm{CSP}(H)$ to $\mathrm{CSP}(H)_5$. But H contains a triangle, so $\mathrm{CSP}(H)$ is NP-complete by [7]. Thus $\mathrm{CSP}(H)_5$ is also NP-complete, and so $b(H) \leq 5$. $\square$

Starting with the assumption that H was a minimal counter-example to Theorem 1.4, we proved several claims about H, much like Hell and Nešetřil do in their dichotomy proof [7], and use these claims in Lemma 3.6 to show that H is not a counter-example to the theorem. Thus by contradiction, the theorem is proved.

4 Proof of Theorems 1.5 and 1.6

In this section we prove Theorem 1.6, then Theorem 1.5 and finally Corollary 1.7.

Proof of Theorem 1.6. Let H be any graph of odd girth $g \geq 3$ which is 4-cycle free, and in which every vertex is in a g-cycle. We present a polynomial time construction that provides, for any graph G, a graph G' with maximum degree 4 such that

$$G' \to H \iff G \to H.$$

Since H-COL is NP-complete, this will imply that H-COL$_4$ is NP-complete. The consturction is very similar to that Lemma 3.6.

Let a be a vertex in the g-cycle C_g. Let C_g^3 be the graph constructed from C_g by adding two new vertices x and y and giving them the same neighbours as a. For any integer d, let $I(d, g)$ be the graph shown in Figure 3, which is made from d copies of C_g^3 and contains the vertices $a_1, \ldots, a_d$.

Given a graph G, construct G' from G as in the proof of Lemma 3.6, except use copies of $I(d, g)$ in place of copies of $I(d)$, and replace vertices of degree greater than 4, instead of just those of degree greater than 5.

Because H is C_4 free, $\chi(a_1) = \cdots = \chi(a_d)$ is forced in any H-colouring χ of $I(d, g)$. Furthermore, because every vertex h of H is in a g-cycle, there is an H-colouring χ of $I(d, g)$, for any d, in which $\chi(a_1) = \cdots = \chi(a_d) = h$. Now G' has maximum degree 4, and it is not hard to see that

$$G' \to H \iff G \to H.$$

The rest of proof is almost identical to the proof of Lemma 3.6, so we omit it.

□

Remark 4.1. *Observe that in Theorem 1.6, we could have forbidden the graph C_g^2, which is C_g^3 less the vertex x, instead of forbidding C_4, and still*

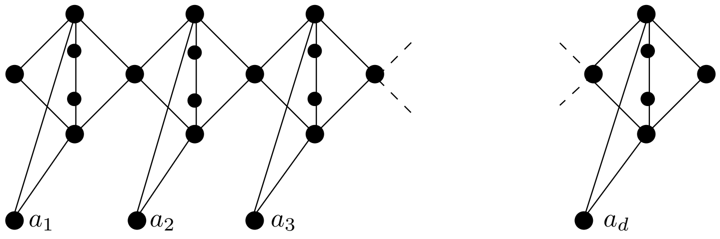

Figure 3: $I(d, 5)$ from the proof of Theorem 1.6.

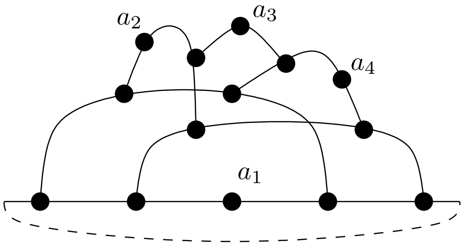

Figure 4: $\Gamma(g)$ from the proof of Theorem 1.5.

been able to get $b(H) \leq 4$. *When* $g = 3$, C_g^2 *is* K_4^-, *so this gives a slightly stronger result than Lemma 3.6.*

Proof of Theorem 1.5. Let H be a graph of odd girth $g \geq 5$, with no 4 or 6-cycles, for which every vertex is in a g-cycle. By Theorem 1.6, we have that $\mathrm{CSP}(H)_4$ is NP-complete.

We present a polynomial time construction which provides for any graph G of maximum degree 4, a graph G' of maximum degree 3 for which

$$G' \to H \iff G \to H.$$

This will prove that $\mathrm{CSP}(H)_3$ is NP-complete, as claimed be the theorem.

Let $\Gamma(g)$ be the graph in Figure 4 where the half dotted cycle on the bottom is a g-cycle. Given a graph G of maximum degree 4, construct G' from G as in the proof of Lemma 3.6, except replace vertices of degree greater than 3 (i.e. vertices of degree exactly 4) instead of vertices of degree greater than 4, and use copies of $\Gamma(g)$ instead of copies of $I(4)$.

One can check that for any H-colouring of $\Gamma(g)$, the vertices a_1, a_2, a_3 and a_4 map to the same vertex. Indeed, begin by mapping the bottom cycle to a g-cycle. Observe that because H contains no 6-cycle, any 6-cycle of $\Gamma(g)$ in which 4 consecutive vertices already have distinct colours, has a unique H-colouring. Using this and the fact that H also has no 4-cycles, one sees that there is a unique way to extend the colouring of the g-cycle to an H-colouring of $\Gamma(g)$.

Moreover, every vertex u of H is in a g-cycle, so one can similarily check that there is an H-colouring of $\Gamma(g)$ in which a_1, a_2, a_3 and a_4 all get mapped to u.

Thus $G' \to H \iff G \to H$, and the theorem follows. $\square$

We now observe how Corollary 1.7 follows by the following construction, which can be seen as a special case of the Edge-Sub-Indicator construction from [7].

Given any graph G and any odd integer g, let ${}^*G$ be the graph constructed from G as follows. Replace every edge of G with a new g-cycle by removing it and identifying its endpoints with some adjacent pair of vertices in the g-cycle. For any graph H of odd girth g, let H^* be the subgraph of H consisting of all edges that lie in a g-cycle of H. Then

$$ {}^*G \to H \iff G \to H^*. $$

Clearly $\Delta({}^*G) = 2\Delta(G)$, so $b(H) \leq 2b(H^*)$. Thus for Theorems 1.3, 1.6, and 1.5, we can remove restriction (ii) at the price of multiplying our bound on $b(H)$ by 2. Corollary 1.7 thus follows from Theorems 1.6, and 1.5.

5 The Proof of Theorem 1.8

In this section we widen our scope, and look at CSPs in general. We begin with some necessary definitions.

5.1 Definitions for Relational Systems and CSPs

A *vocabulary* is a vector $\mathcal{K} = (k_i)_{i \in I}$ of positive integers, called *arities.* A *relational system*, $\mathcal{H}$, with vocabulary $\mathcal{K}$, consists of a finite vertex set $V = V(\mathcal{H})$, and a k_i-ary relation $R_i = R_i(\mathcal{H})$ on V, for each $i \in I$. Observe that a (di)graph is just a relational system with vocabulary $\mathcal{K} = (2)$. The *degree* of a vertex v in $\mathcal{H}$ is the number of tuples it occurs in in $\bigcup R_i$. Given two relational systems $\mathcal{G}$ and $\mathcal{H}$ with the same vocabulary, an $\mathcal{H}$-*colouring of* $\mathcal{G}$ is a map $\chi : V(\mathcal{G}) \to V(\mathcal{H})$ such that for all $i \in I$ and every k_i-tuple $(v_1, \ldots, v_{k_i}) \in R_i(\mathcal{G})$, $(\chi(v_1), \ldots, \chi(v_{k_i}))$ is in $R_i(\mathcal{H})$. The notation $\Delta(\mathcal{H})$, $\mathcal{G} \to \mathcal{H}$, $\mathrm{CSP}(\mathcal{H})$ and $\mathrm{CSP}(\mathcal{H})_b$ is analogous to the graph case.

Given an relational system $\mathcal{H}$, and a positive integer d, $\mathcal{H}^d$ is the relational system with the same vocabulary as $\mathcal{H}$, defined as follows.

- $V(\mathcal{H}^d) = \{(v_1, \ldots, v_d) \mid v_1, \ldots, v_d \in V(\mathcal{H})\}$.
- For $i \in I$, $((v_{1,1}, v_{1,2}, \ldots, v_{1,d}), \ldots, (v_{k_i,1}, \ldots, v_{k_i,d}))$ is in $R_i(\mathcal{H}^d)$ if and only if all of $(v_{1,1}, v_{2,1}, \ldots, v_{k_i,1}), \ldots, (v_{1,d}, \ldots, v_{k_i,d})$ are in $R_i(\mathcal{H})$.

A relational system $\mathcal{H}$ is a *core* if its only $\mathcal{H}$-colourings are automorphisms. The set of automorphisms of $\mathcal{H}$ is denoted $\mathrm{Aut}(\mathcal{H})$. An $\mathcal{H}$-colouring χ of $\mathcal{H}^d$ is a *projection* if there is some $i \in 1, \ldots, d$ such that $\chi((v_1, \ldots, v_d)) = v_i$ for every vertex $(v_1, \ldots, v_d) \in V(\mathcal{H}^d)$. The set of projections of $\mathcal{H}^d$ is denoted $\mathrm{Proj}(\mathcal{H}^d)$. A core is *projective* if every $\mathcal{H}$-colouring of $\mathcal{H}^d$, for all d, is a projection composed with an automorphism of $\mathcal{H}$.

5.2 A Indicator-type Construction for Projective Cores

In this subsection, we provide a construction $G \mapsto \mathcal{M}_{\mathcal{C}}(G)$ for any projective core $\mathcal{C}$, such that for any graph G,

$$G \to K_3 \iff \mathcal{M}_{\mathcal{C}}(G) \to \mathcal{C}.$$

This reduces $\mathrm{CSP}(K_3)$ to $\mathrm{CSP}(\mathcal{C})$. A variation of this construction was used in [13].

The following is based on a simple but effective observation that Müller made about complete graphs in [11].

Lemma 5.1. *Let $\mathcal{C}$ be a projective core, and W be a set. Let $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ be a set of maps from W to $V(\mathcal{C})$, with the following property (*).*

> *For any pair $w \neq w' \in W$, there exists some $i \in 1, \ldots, d$ for which $\gamma_i(w) \neq \gamma_i(w')$.*

Then there exists a relational system $\mathcal{M}$, isomorphic to $\mathcal{C}^d$, with $W \subset V(\mathcal{M})$, such that the set of $\mathcal{C}$-colourings of $\mathcal{M}$, when restricted to W, is exactly

$$\{\alpha \circ \gamma \mid \alpha \in \mathrm{Aut}(\mathcal{C}), \gamma \in \Gamma\}.$$

Proof. Let $\mathcal{M}$ be the graph $\mathcal{C}^d$ and for each $w \in W$, identify w with the vertex $(\gamma_1(w), \ldots, \gamma_d(w))$ of $\mathcal{M}$. By (*), these are distinct elements of $V(\mathcal{M})$.

By the projectivity of $\mathcal{C}$, the only $\mathcal{C}$-colourings of $\mathcal{M} = \mathcal{C}^d$ are $\alpha \circ \pi$ where $\alpha \in \text{Aut}(\mathcal{C})$, and $\pi \in \text{Proj}(\mathcal{C}^d)$. But the projections in $\text{Proj}(\mathcal{C}^d)$ restrict on W to the maps of Γ, so the lemma follows. □

In the remainder of this section we will have many copies of the three element set $W' = \{w'_1, w'_2, w'_3\}$. These copies will have various superscripts. We will take the following definition to apply to any such copy of W'.

Definition 5.2. *Let χ be a map defined on a set $W' = \{w'_1, w'_2, w'_3\}$ such that two elements of W' are mapped to one image, and the other, w'_i is mapped to a different image. Let $\chi^*(W') = i$.*

We will now use Lemma 5.1 to construct a graph $\mathcal{M}_{\mathcal{C}}$ which will be used in our construction $G \mapsto \mathcal{M}_{\mathcal{C}}(G)$.

Construction 5.3. *Let $\mathcal{C}$ be a projective core containing vertices 0 and 1. Let $W^a = \{w^a_1, w^a_2, w^a_3\}, W^b = \{w^b_1, w^b_2, w^b_3\}$, and $W = W^a \cup W^b$. For distinct i and j in $\{1, 2, 3\}$, define $\gamma_{i,j} : W \to V(\mathcal{C})$ as follows. For $w \in W$, let $\gamma_{i,j}(w) = 1$ if $w = w^a_i$ or $w = w^b_j$, and let $\gamma_{i,j}(w) = 0$ otherwise. Let Γ be the set of these six maps $\gamma_{i,j}$. Let $\mathcal{M}_{\mathcal{C}}$ be the graph $\mathcal{M}$ returned by Lemma 5.1 for this choice of $\mathcal{C}, W$, and Γ.*

Observe that the set W is an independent set in $\mathcal{M}_{\mathcal{C}}$. Indeed, for any pair of vertices in W, there is some $\gamma \in \Gamma$ which maps them both to 0, so there can be no edge between them. Thus $\mathcal{M}_{\mathcal{C}}$ has the following properties.

- $V(\mathcal{M}_{\mathcal{C}})$ contains the sets $W^a = \{w^a_1, w^a_2, w^a_3\}$ and $W^b = \{w^b_1, w^b_2, w^b_3\}$ of three independent vertices each.
- For any $\mathcal{C}$-colouring χ of $\mathcal{M}_{\mathcal{C}}$, $\chi^*(W^a), \chi^*(W^b) \in \{1, 2, 3\}$, and $\chi^*(W^a) \neq \chi^*(W^b)$.

We are now ready to present the construction the $G \mapsto \mathcal{M}_{\mathcal{C}}(G)$.

Construction 5.4. *Given a graph G, and a projective core $\mathcal{C}$ containing the vertices 0 and 1, define the graph $\mathcal{M}_{\mathcal{C}}(G)$ as follows.*

i. For every vertex $u \in V(G)$, let $W^u = \{w^u_1, w^u_2, w^u_3\}$ be a set of 3 independent vertices.

ii. *For every edge uu' in G let $\mathcal{M}_{uu'}$ be a copy of the graph $\mathcal{M}_{\mathcal{C}}$ from Construction 5.3. Let the vertices of all the sets W^u and all the copies of $\mathcal{M}_{\mathcal{C}}$ be distinct unless explicitly identified below.*

iii. *For every edge uu' in G, identify the copy of W^a in $\mathcal{M}_{uu'}$ with W^u and the copy of W^b with $W^{u'}$ (or vice-versa) by identifing vertices with the same indices.*

Let $\mathcal{M}_{\mathcal{C}}(G)$ be the graph thus constructed.

Lemma 5.5. *Given a graph G and a projective core $\mathcal{C}$ containing the vertices 0 and 1, the graph $\mathcal{M}_{\mathcal{C}}(G)$ of Construction 5.4 has the following property:*

$$G \to K_3 \iff \mathcal{M}_{\mathcal{C}}(G) \to \mathcal{C}.$$

Proof. To simplify notation, we assume that G is connected. Clearly G has an K_3-colouring if and only if every component does.

We begin with the forward implication. Assume that χ is a K_3-colouring of G, and define the map $\chi' : V(\mathcal{M}_{\mathcal{C}}(G)) \to V(\mathcal{C})$ as follows. For every vertex u in G, let

$$\chi'(w_i^u) = \begin{cases} 1 & \text{if } \chi(u) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Now for any edge uu' of G, χ' restricts, on the copy $W^u \cup W^{u'}$ of W in $\mathcal{M}_{uu'}$, to the map $\gamma_{\chi(u),\chi(u')} \in \Gamma$. Thus it be extended to a $\mathcal{C}$-colouring of $\mathcal{M}_{uu'}$. Since this was for an arbitrary edge $uu' \in G$, χ' can be extended to a $\mathcal{C}$-colouring of $\mathcal{M}_{\mathcal{C}}(G)$. This completes the forward implication.

Assume now that χ is a $\mathcal{C}$-colouring of $\mathcal{M}_{\mathcal{C}}(G)$. For any $u \in V(G)$, W^u is identified with the copy of W^a or W^b from at least one copy of $\mathcal{M}_{\mathcal{C}}(G)$. Thus $\chi^*(W^u)$ (see Def 5.2) is well defined. We may thus define the map $\chi' : V(G) \to \{1, 2, 3\}$ of G as follows. For all $u \in V(G)$, let

$$\chi'(u) = \chi^*(W^u).$$

For any edge uu' of G, the sets W^u and $W^{u'}$ are identified with the copies of W^a and W^b in $\mathcal{M}_{uu'}$, so $\chi'(u) = \chi^*(W^u) \neq \chi^*(W^{u'}) = \chi'(u')$. Thus χ' is a K_3-colouring of G. This gives us the backwards implication, so completes the proof of the lemma. □

Now the proof of Theorem 1.8 follows immediatley from from Construction 5.4.

Proof of Theorem 1.8. We have to show that for any projective core $\mathcal{C}$, $b(\mathcal{C}) \leq 4\Delta(\mathcal{C})^6$. Since $\mathcal{M}_{\mathcal{C}} = \mathcal{C}^{|\Gamma|} = \mathcal{C}^6$, $\Delta(\mathcal{M}_{\mathcal{C}}) = \Delta(\mathcal{C})^6$. Thus for any graph G of degree at most 4, $\Delta(\mathcal{M}_{\mathcal{C}}(G)) \leq 4\Delta(\mathcal{C})^6$. Since $b(K_3) = 4$, Lemma 5.5 implies that $b(\mathcal{C}) \leq 4\Delta(\mathcal{C})^6$. $\square$

6 Concluding Remarks

Construction 5.4 can easily be extended to a construction $\mathcal{G} \mapsto \mathcal{M}^{\mathcal{H}}_{\mathcal{C}}(\mathcal{G})$ which reduces CSP($\mathcal{H}$) to CSP($\mathcal{C}$) for any relational system $\mathcal{H}$. However, because the construction reduces CSP($\mathcal{H}$) to CSP($\mathcal{C}$), and not the other way, it lends more to showing that restrictions of CSP($\mathcal{H}$) are polynomial time solvable when the same restriction of CSP($\mathcal{C}$) has already been established to be polynomial time solvable.

For example, in [14], (see also [4], [2]) it is shown that CSP($\mathcal{H}$) is polynomial time solvable for any relational system $\mathcal{H}$, when restricted to instances of bounded tree width.

Using the fact that $\mathcal{M}^{\mathcal{H}}_{K_3}(\mathcal{G})$ has bounded tree width if $\mathcal{G}$ does, it would be enough to have proved the above result for CSP(K_3). The construction $\mathcal{G} \mapsto \mathcal{M}^{\mathcal{H}}_{K_3}(\mathcal{G})$ would then give an immediate extension to CSP($\mathcal{H}$).

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