LINEAR TIME LOW TREE-WIDTH PARTITIONS AND ALGORITHMIC CONSEQUENCES

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ABSTRACT. *Classes of graphs with bounded expansion* have been introduced in [15], [12]. They generalize both proper minor closed classes and classes with bounded degree.

For any class with bounded expansion $\mathcal{C}$ and any integer p there exists a constant $N(\mathcal{C}, p)$ so that the vertex set of any graph $G \in \mathcal{C}$ may be partitioned into at most $N(\mathcal{C}, p)$ parts, any $i \leq p$ parts of them induce a subgraph of tree-width at most $(i-1)$ [12] (actually, of *tree-depth* [16] at most i, what is sensibly stronger). Such partitions are central to the resolution of homomorphism problems like *restricted homomorphism dualities* [14].

We give here a simple algorithm to compute such partitions and prove that if we restrict the input graph to some fixed class $\mathcal{C}$ with bounded expansion, the running time of the algorithm is bounded by a linear function of the order of the graph (for fixed $\mathcal{C}$ and p).

This result is applied to get a linear time algorithm for the subgraph isomorphism problem with fixed pattern and input graphs in a fixed class with bounded expansion.

More generally, let ϕ be a first order logic sentence. We prove that any fixed graph property of type

$$\text{“ } \exists X : (|X| \leq p) \wedge (G[X] \vDash \phi) \text{ ”}$$

may be decided in linear time for input graphs in a fixed class with bounded expansion.

1. INTRODUCTION, THE MODEL AND PREVIOUS WORK

The concept of tree-width [10],[21],[27] is central to the analysis of graphs with forbidden minors of Robertson and Seymour. This concept gained much algorithmic attention thanks to the general complexity result of Courcelle about monadic second-order logic graph properties decidability for graphs with bounded tree-width [3],[4]. It appeared that many NP-complete problems may be solved in polynomial time when restricted to a class with bounded tree-width. However, bounded tree-width is quite a strong restriction, as planar graphs for instance do not have bounded tree-width.

An alternative approach consists in the partition of graphs, such that p parts induce a subgraph of tree-width at most $(p-1)$. Answering a question of Thomas [26], DeVos et al. [6] proved that for any proper minor closed class of graphs $\mathcal{C}$ – that is: any minor closed class $\mathcal{C}$ excluding at least one graph — and any integer p there exists a constant $N(\mathcal{C}, p)$ such that the vertex set of any graph $G \in \mathcal{C}$ may be partitioned into at most $N(\mathcal{C}, p)$ parts in such a way that any $j \leq p$ parts induce a subgraph of tree-width at most $(j-1)$, what the authors call a *low tree-width partition* of G. This proof, which relies on the Structural Theorem of Robertson and Seymour [22] fails to be effective from a computational point of view.

It appears that low tree-width decomposition may be established in a more general setting for classes with bounded expansion [15][12]. These results are reported here together with the algorithmic analysis. The definition of bounded expansion classes is based on a new graph invariant, the *greatest reduced average degree (grad)* with *rank* r of a graph G, $\nabla_r(G)$. This invariant is defined by $\nabla_r(G) = \max \frac{|E(H)|}{|V(H)|}$, where the maximum is taken over all the minors H of G obtained by contracting a set of vertex-disjoint subgraphs with radius at most r and then deleting any number of edges and vertices. A class of graphs $\mathcal{C}$ has *bounded expansion* if $\sup_{G\in\mathcal{C}} \nabla_r(G) < \infty$ for any integer r. Not only proper minor closed classes of graphs have bounded expansion (as then ∇_r is uniformly bounded independently of r), but so are classes with bounded degree or some usual classes arising from finite element meshes (as skeletons of d-dimensional simplicial complexes with bounded aspect ratio [11]). One of the main fundamental properties of these classes may be stated as follows: If $\mathcal{C}$ is a class with bounded expansion, then so are the following classes:

- $\mathcal{C}/\star$, the class of the graphs obtained from graphs in $\mathcal{C}$ by contracting a star forest,
- $\mathcal{C} \bullet K_2$, the class of the graphs obtained from graphs in $\mathcal{C}$ by blowing every vertex into two adjacent vertices (i.e. by taking the lexicographic product with K_2).

Note that the second property does not hold for proper minor closed classes: If $\mathcal{P}$ is the class of all planar graphs then any minor closed class including $\mathcal{P} \bullet K_2$ contains all finite graphs. It is established in [12] that for every class $\mathcal{C}$ with bounded expansion and any integer p, there exists a constant $N'(\mathcal{C}, p)$ such that any graph $G \in \mathcal{C}$ has a vertex-partition into at most

$N'(\mathcal{C}, p)$ parts such that any $j \leq p$ parts induce a subgraph of tree-width at most $(p-1)$ (actually of *tree-depth* at most p, which is quite a stronger result). For properties of tree-depth, see [16]). The strong benefit of this later proof is that it does not rely on the Structural Theorem and that it actually leads to a very simple linear time algorithm to compute such a partition [13].

Our technique facilitates the detection of local graph properties. For instance, the subgraph isomorphism problem for a fixed pattern H is known to have complexity at most $O(n^{\omega l/3})$ where l is the order of H and where ω is the exponent of square matrix fast multiplication algorithm [17] (hence $O(n^{0.792\ l})$ using the fast matrix algorithm of [2]). The particular case of subgraph isomorphism in planar graphs have been studied by Plehn and Voigt [20], Alon [1] with super-linear bounds and then by Eppstein [7][8] who gave the first linear time algorithm for fixed pattern H and G planar and then extended his result to graphs with bounded genus [9]. It appears that one of the main lemmata of [8] actually induces a linear time algorithm to solve the problem of counting all the isomorphs of H in a graph G as soon as we have a linear time algorithm allowing to compute a low-tree width partition of G.

Such complexity improvements hold for a quite general class of decisional problems, namely the problems having the form

$$\text{“ } \exists X : (|X| \leq p) \wedge (G[X] \vDash \phi) \text{ ”}$$

where ϕ is a first-order sentence, according to the general results of Courcelle [3][4]. This includes for instance, for a fixed graph H and for input graphs G belonging to a fixed class with bounded expansion, the problems

- “Does H have a homomorphism to G?”,
- “Is H isomorphic to a subgraph of G?”,
- “Is H isomorphic to an induced subgraph of G?”.

In the following, we only will be concerned with simple loopless graphs.

2. The grad of a graph and class expansion

The *distance* $d(x, y)$ between two vertices x and y of a graph is the minimum length of a path linking x and y, or ∞ if x and y do not belong to the same connected component. The *radius* $\rho(G)$ of a connected graph G is the minimum maximum distance of the vertices from a fixed vertex, that is: $\rho(G) = \min_{r \in V(G)} \max_{x \in V(G)} \mathrm{d}(r, x)$. A vertex r is a *center* of G if the maximal distance of vertices of G to r is equal to $\rho(G)$. The *radius* $\rho(G)$ of a non-connected graph G is the maximum of the radii of its components.

A (simple) graph H is a *minor* of a graph G if it may be obtained from G by contracting edges, deleting edges and deleting vertices. This is denoted by $H < G$. As edge deletion and contraction commute, we may consider contractions first and deletions next. As we only consider simple loopless graphs, each deletion is followed (if necessary) by the simplification of the graph. In other words, a minor H of a graph G is obtained by contracting some connected subset F of edges, simplifying and then taking a subgraph (i.e. $H \subseteq G/F$). Notice that the subset F is in general not uniquely determined by G and H. We denote by G_F the subgraph of G induced by the subset F of edges of G. The *depth* of a minor of a graph G is the minimum radius of the part we have to contract in G to get H. More formally:

$$\mathrm{depth}(H, G) = \min\{\rho(G_F) : H \subseteq G/F\}$$

Definition 2.1. The *greatest reduced average density* (grad) of G with *rank* r is

$$\nabla_r(G) = \max_{\substack{H<G \\ \mathrm{depth}(H,G)\leq r}} \frac{|E(H)|}{|V(H)|}$$

The first grad, ∇_0 , is closely related to the degeneracy or the maximum average degree (G is k-degenerated iff $k \geq \lfloor 2\nabla_0(G) \rfloor$; note that none of the results of this paper holds for k-degenerated graphs, the higher grads are needed). non decreasing sequence which, for every graph, starting from some index (smaller than the order of the graph).

Definition 2.2. The *expansion* of a class $\mathcal{C}$ of graphs is the function $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ defined by

$$f(r) = \sup_{G \in \mathcal{C}} \nabla_r(G)$$

A class $\mathcal{C}$ has *bounded expansion* if its expansion is finite for every $r \in \mathbb{N}$, that is:

$$\forall r \geq 0, \quad \sup_{G \in \mathcal{C}} \nabla_r(G) < \infty$$

Here are some examples of class with bounded expansion. Some are included into others (see Fig 1). However, they are examples of different expansion bounds.

- **proper minor closed classes**. Any proper minor closed class of graphs has expansion bounded by a constant function. Conversely, any class of graphs with expansion bounded by a constant is included in some proper minor closed class of graphs.

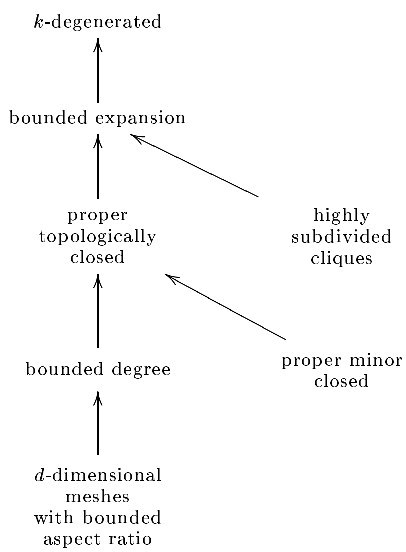

FIGURE 1. Inclusion of graph classes

- ***d*-dimensional meshes with bounded aspect ratio**. [11] introduces classes of graphs which occur naturally in finite-element and finite-difference problems. Theses classes, the classes of *d-dimensional meshes with bounded aspect ratio*, are formed by the interior skeletons of a family of d-dimensional simplicial complexes with bounded aspect ratio. As such graphs exclude K_h as a depth L minor if $h = \Omega(L^d)$ [25] they form (for each d) a class with polynomially bounded expansion. Our results (and particularly linear algorithm for low tree decompositions) present a natural link of applicable results [11].
- **bounded degree classes**. Let Δ be an integer. Then the class of graphs with maximum degree at most Δ has expansion bounded by the exponential function $f(r) = \Delta^{r+1}$.
- **proper topologically closed classes**. These classes are defined by a (possibly infinite) set $\mathcal{S}$ of forbidden configurations, in the sense of Kuratowski's configurations: a graph G belongs to the class if no subdivision of a graph in $\mathcal{S}$ is isomorphic to a subgraph of G. Such

classes have expansion bounded by a double exponential function $f(r) = 2^{r-1}(\min_{H\in\mathcal{S}} |V(H)|)^{2^{r+1}}$ (see Section 8.2).

- **highly subdivided cliques**. For any non-decreasing function $f : \mathbb{N} \to \mathbb{N} \setminus \{0,1,2\}$ we may construct a class $\mathcal{C}_f$ of graphs with expansion f by including (for each integer r) the complete graph of $2f(r)+1$ vertices whose edges are subdivided $3^r - 1$ times.

2.1. **Further Definitions and Properties Related to the grad.** We give here further definitions and simple properties related to the definition of the grad of a graph.

Definition 2.3. Let G be a graph. A *ball* of G is a subset of vertices inducing a connected subgraph. The set of all the families of balls of G is noted $\mathfrak{B}(G)$.

Let $\mathcal{P} = \{V_1, \dots, V_p\}$ be a family of balls of G.

- The *radius* $\rho(\mathcal{P})$ of $\mathcal{P}$ is $\rho(\mathcal{P}) = \max_{X\in\mathcal{P}} \rho(G[X])$
- The *complexity* of $\mathcal{P}$ is $\zeta(\mathcal{P}) = \max_{v\in V(G)} |\{i : v \in V_i\}|$.
- The *quotient* $G/\mathcal{P}$ of G by $\mathcal{P}$ is a graph with vertex set $\{1,\dots,p\}$ and edge set $E(G/\mathcal{P}) = \{\{i,j\} : (V_i \times V_j) \cap E(G) \neq \emptyset \text{ or } V_i \cap V_j \neq \emptyset\}$.

We introduce a refinement of the notion of grad:

Definition 2.4. The *greatest reduced average density* (grad) of G with *rank* r and *complexity* c is

$$\overset{c}{\nabla}_r(G) = \max_{\substack{\mathcal{P}\in\mathfrak{B}(G)\\ \rho(\mathcal{P})\le r,\zeta(\mathcal{P})\le c}} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}.$$

According to this definition, we have:

$$\nabla_r(G) = \overset{1}{\nabla}_r(G) = \max_{\substack{\mathcal{P}\in\mathfrak{B}(G)\\ \rho(\mathcal{P})\le r,\zeta(\mathcal{P})=1}} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}$$

Notice the following well known facts (usually expressed by means of the maximum average degree, see for instance [19] for a proof of Fact 2.1)

Fact 2.1. *Let G be a graph of order n and size m. Then G has an orientation such that the maximum indegree of G is at most k if and only if $k \ge \nabla_0(G)$. Moreover, there is an $O(n+m)$-time algorithm which computes an acyclic orientation of G with maximum indegree $\lfloor 2\nabla_0(G)\rfloor$.*

Fact 2.2. *Any graph G is $\lfloor 2\nabla_0(G)\rfloor$-degenerated, hence $\lfloor 2\nabla_0(G)+1\rfloor$-colorable.*

2.2. **Grad and Lexicographic Product.** Let G, H be graphs. The *lexicographic product* $G \bullet H$ is defined by $V(G \bullet H) = V(G) \times V(H)$ and $E(G \bullet H) = \{\{(x,y),(x',y') : \{x,y\} \in E(G) \text{ or } x = x' \text{ and } \{y,y'\} \in E(H)\}$.

Let us note at this place that the lexicographic product (or blowing up of vertices) is an operation which is incompatible with the minors. One can see easily that every graph is a minor of a graph of the form $G \bullet K_2$ for a planar graph G. But the lexicographic product is naturally related to the notion of complexity we have introduced for grad:

Lemma 2.1. *For any graph G and any integers c, r, we have:*

$$\overset{c}{\nabla}_r(G) = \nabla_r(G \bullet K_c)$$

Proof. Let $\mathcal{P} = \{V_1, \dots, V_p\}$ be a ball family of G with complexity $c = \zeta(\mathcal{P})$ and radius $r = \rho(\mathcal{P})$. As $\zeta(\mathcal{P}) = c$ there exists a function $f : V(G) \times \{1, \dots, p\} \to \{1, \dots, c\}$ such that if $x \in V_i \cap V_j$ then $f(x,i) \neq f(x,j)$.

For $1 \leq i' \leq p$, define $V_i' = \{(x, f(x,i)) : x \in V_i\}$. Then $\mathcal{P}' = \{V_1', \dots, V_p'\}$ has radius r and complexity 1. Moreover, $G/\mathcal{P}$ is obviously isomorphic to a subgraph of $(G \bullet K_c)/\mathcal{P}'$. It follows that $\nabla_r(G \bullet K_c) \geq \overset{c}{\nabla}_r(G)$.

Conversely, let $\mathcal{P}' = \{V_1', \dots, V_q'\}$ be a ball family of $G \bullet K_c$, define the ball family $\mathcal{P} = \{V_1, \dots, V_q\}$ of G by $x \in V_i$ if there exists $\alpha \in \{1, \dots, c\}$ such that $(x, \alpha) \in V_i'$. Then $\rho(\mathcal{P}) \leq \rho(\mathcal{P}')$ and $\zeta(\mathcal{P}) \leq c$. It follows that $\overset{c}{\nabla}_r(G) \geq \nabla_r(G \bullet K_c)$. □

The rather technical proof of the following lemma is annexed in Section 8.1 for sake of readability.

Lemma 2.2. *There exist polynomials P_i $(i \geq 0)$ such that for any graph G and integers r and c:*

$$\overset{c}{\nabla}_r(G) \leq P_r(c, \nabla_r(G)) \tag{1}$$

3. The Theory: Decompositions, Augmentations and Colorings

3.1. **Tree-width.** A *tree-decomposition* of a graph G consists in a pair (T, λ) formed by a tree T and a function λ mapping vertices of T to subsets of $V(G)$ so that for all $v \in V(G), \{x \in V(T) : v \in \lambda(x)\}$ induces a subtree of T, and such that for any edge $\{v, w\}$ of G there exists $x \in V(T)$ such that $\{v, w\} \subseteq \lambda(x)$. The *width* of a tree decomposition (T, λ) is $\max_{v \in V(G)} |\lambda(v)| - 1$. The *tree-width* of G is the minimum width of any tree-decomposition of G.

In the following, a directed graph $\vec{G}$ may not have a loop and for any two of its vertices x and y, $\vec{G}$ includes at most one arc from x to y and at most one arc from y to x (thus at most two arcs may connect x and y, one in each direction).

A class $\mathcal{C}$ has a *low tree-width coloring* if, for any integer $p \geq 1$, there exists an integer $N(p)$ such that any graph $G \in \mathcal{C}$ may be vertex-colored using $N(p)$ colors so that each of the connected components of the subgraph induced by any $i \leq p$ parts has tree-width at most $(i-1)$. DeVos et al. proved:

Theorem 3.1 ([6]). *Any minor closed class of graphs excluding at least one graph has a low tree-width coloring.*

The proof of Theorem 3.1 relies on the Structural Theorem of Robertson and Seymour [22] and fails to be applicable from a computational point of view.

3.2. **Tree-depth.** A *rooted forest* is a disjoint union of rooted trees. The *height* of a vertex x in a rooted forest F is the number of vertices of a path from the root (of the tree to which x belongs to) to x and is noted $\mathrm{height}(x, F)$. The *height* of F is the maximum height of the vertices of F. Let x, y be vertices of F. The vertex x is an *ancestor* of y in F if x belongs to the path linking y and the root of the tree of F to which y belongs to. The *closure* $\mathrm{clos}(F)$ of a rooted forest F is the graph with vertex set $V(F)$ and edge set $\{\{x, y\} : x \text{ is an ancestor of } y \text{ in } F, x \neq y\}$. A rooted forest F defines a partial order on its set of vertices: $x \leq_F y$ if x is an ancestor of y in F. The comparability graph of this partial order is obviously $\mathrm{clos}(F)$.

The *tree-depth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest F such that $G \subseteq \mathrm{clos}(F)$ [16]. This graph parameter also appeared in the literature as the *minimum height of an elimination tree* [5] and is analogous to the definition of *rank function* of a graph which has been recently used for analysis of countable graphs, see e.g. [18].

Lemma 3.2 ([16]). *Let $G = (V, E)$ be a graph and let $G_1, \dots, G_p$ be its connected components. Then:*

$$\mathrm{td}(G) = \begin{cases} 1, & \text{if } |V| = 1; \\ 1 + \min_{v \in V} \ \mathrm{td}(G - v), & \text{if } p = 1 \text{ and } |V| > 1; \\ \max_{i=1}^{p} \ \mathrm{td}(G_i), & \text{otherwise.} \end{cases}$$

As we introduced low tree-width coloring, we say that a class $\mathcal{C}$ has a *low tree-depth coloring* if, for any integer $p \geq 1$, there exists an integer $N(p)$ such that any graph $G \in \mathcal{C}$ may be vertex-colored using $N(p)$ colors so that each of the connected components of the subgraph induced by any $i \leq p$ parts has tree-depth at most i. As $\mathrm{td}(G) \geq \mathrm{tw}(G) - 1$, a class having a low-tree depth coloring has a low tree-width coloring. In [16] is proved a strengthening of Theorem 3.1:

Theorem 3.3 ([16]). *Any minor closed class of graphs excluding at least one graph has a low tree-depth coloring.*

This result has been extended in [12] where it is proved that classes with bounded expansion allows low tree-depth partitions. This last result follows another approach: instead of the Structural Theorem, the proof relies on the properties of transitive fraternal augmentations. We shall prove in this paper that this allows the partition and tree-decomposition of $i \leq p$ parts to be computed in linear time.

3.3. **Transitive Fraternal Augmentation.**

Definition 3.1. Let $\vec{G}$ be a directed graph. A 1-*transitive fraternal augmentation* of $\vec{G}$ is a directed graph $\vec{H}$ with the same vertex set, including all the arcs of $\vec{G}$ and such that, for any distinct vertices x, y, z,

- if (x, z) and (z, y) are arcs of $\vec{G}$ then (x, y) is an arc of $\vec{H}$ (*transitivity*),
- if (x, z) and (y, z) are arcs of $\vec{G}$ then (x, y) or (y, x) is an arc of $\vec{H}$ (*fraternity*).

A *transitive fraternal augmentation* of a directed graph $\vec{G}$ is a sequence $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \vec{G}_{i+1} \subseteq \cdots$, such that $\vec{G}_{i+1}$ is a 1-transitive fraternal augmentation of $\vec{G}_i$ for any $i \geq 1$.

The main key lemma here is that the notion of classes of bounded expansion is stable under 1-fraternal augmentations. More precisely:

Lemma 3.4. *Let $\vec{G}$ be a directed graph and let $\vec{H}$ be a 1-transitive fraternal augmentation of $\vec{G}$. Then*

$$\begin{aligned} \overset{c}{\nabla}_r(H) &\leq \overset{c(\Delta^-(\vec{G})+1)}{\nabla_{2r+1}}(G) \\ &\leq P_{2r+1}(c(\Delta^-(\vec{G})+1), \nabla_{2r+1}(G)). \end{aligned}$$

Proof. Consider a ball family $\mathcal{P} = \{V_1, \dots, V_p\}$ of H with radius at most r and complexity c.

Let $\mathcal{P}' = \{V_1', \dots, V_p'\}$, where

$$V_i' = V_i \cup \{z : \exists x \in V_i, (x, z) \in E(\vec{G})\}$$

Then for any $x, y \in V_i$ which are adjacent in H, either x and y are adjacent in G or there exists $z \in V_i'$ so that $\{x, z\}$ and $\{y, z\}$ are edges of G. Hence V_i' is a ball of G with radius at most $2r + 1$. Any vertex v of G belongs to a most $c + \Delta^-(\vec{G})$ balls of $\mathcal{P}'$ for v belongs to V_i' if and only if either v belongs to V_i (there are at most c such V_i) or there exists an arc from a vertex $z \in V_i$ to v in $\vec{G}$ (there are at most $\Delta^-(\vec{G})$ such z hence at most $c\Delta^-(\vec{G})$ such V_i). Hence the complexity of $\mathcal{P}'$ is at most $c(\Delta^-(\vec{G}) + 1)$. As $H/\mathcal{P}$ is isomorphic to a subgraph of $G/\mathcal{P}'$ $|E(H/\mathcal{P})| \leq |E(G/\mathcal{P}')|$ thus $\overset{c}{\nabla}_r(H) = \frac{|E(H/\mathcal{P})|}{|\mathcal{P}|} \leq \frac{|E(G/\mathcal{P}')|}{|\mathcal{P}'|} \leq \overset{c(\Delta^-(\vec{G})+1)}{\nabla}_{2r+1}(G)$. We conclude using Lemma 2.2. $\square$

Hence, by induction:

Corollary 3.5. *For any functions $f : \mathbb{N} \to \mathbb{N}$ and $F : \mathbb{N}^2 \to \mathbb{N}$ there exists functions $A_{f,F} : \mathbb{N}^2 \to \mathbb{N}$ and $B_{f,F} : \mathbb{N} \to \mathbb{N}$ with the following property:*

Assume that a transitive fraternal augmentation

$$\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \vec{G}_{i+1} \subseteq \cdots$$

of a graph G satisfies the following conditions:

$$\forall r \geq 0, \qquad \nabla_r(G) \leq f(r)$$

$$\forall i \geq 1, \qquad \Delta^-(\vec{G}_i) \leq \begin{cases} 2f(0) & \text{if } i = 1, \\ F(\Delta^-(\vec{G}_{i-1}), \nabla_0(G_i)) & \text{otherwise.} \end{cases}$$

Then for all $r \geq 0$ and $i \geq 1$ we have:

$$\nabla_r(G_i) \leq A_{f,F}(r, i),$$
$$\Delta^-(\vec{G}_i) \leq B_{f,F}(i).$$

3.4. **Centered Coloring.**

Definition 3.2. A *centered coloring* of a graph G is a coloring of the vertices such that in any connected subgraph some color appears exactly once [16]. This notion is similar to the ones of *vertex ranking* and *ordered coloring* which have been investigated in [5],[23].

As in [16], we refine this notion into a bounded version: for an integer p, a *p-centered coloring* of G is a coloring of the vertices such that in any connected subgraph either some color appears exactly once, or at least p different colors appear.

For the sake of completeness we recall some lemmas of [16]:

Lemma 3.6 ([16]). *Let G, G_0 be graphs, let $p = \mathrm{td}(G_0)$, let c be a q-centered coloring of G where $q \geq p$. Then any subgraph H of G isomorphic to G_0 gets at least p colors in the coloring of G.* □

¿From this lemma follows that p-centered colorings induce low tree-depth colorings:

Corollary 3.7. *Let p be an integer, let G be a graph and let c be a p-centered coloring of G. Then $i < p$ parts induce a subgraph of tree-depth at most i*

Proof. Let G' be any subgraph of G induced by $i < p$ parts. Assume $\mathrm{td}(G') > i$. According to Lemma 3.2, the deletion of one vertex decreases the tree-depth by at most one. Hence there exists an induced subgraph H of G' such that $\mathrm{td}(H) = i + 1 \leq p$. According to lemma 3.6 (choosing $G_0 = H$), H gets at least p colors, a contradiction. □

Lemma 3.8 ([16]). *Let p, k be integers. Then there exists an integer $N(p, k)$ such that any graph G with tree width at most k has a p-centered coloring using $N(p, k)$ colors.* □

The following lemma is proved in [16] for the particular case of proper minor closed classes of graphs and tree-width. We shall state it here in its general form.

Lemma 3.9. *Let $\mathcal{C}$ be a class of graphs. Assume that for any integer $p \geq 1$ there exists a class of graphs $\mathcal{C}_p$ such that:*

- *there exists an integer $N(\mathcal{C}_p, p)$, such that any graph $G \in \mathcal{C}_p$ has a p-centered coloring using at most $N(\mathcal{C}_p, p)$ colors,*
- *there exists an integer $C(p)$ such that any $G \in \mathcal{C}$ has a $C(p)$ vertex coloring such that p classes induce a graph in $\mathcal{C}_p$.*

Then there exists an integer $X(p)$, such that every graph in $\mathcal{C}$ has a p-centered coloring using $X(p)$ colors.

Proof. Let $G \in \mathcal{C}$. According to the assumption, there exists a vertex partition into $C(p)$ parts, such that any p parts form a graph in $\mathcal{C}_p$. This partition will be defined as a coloring $\bar{c} : V(G) \longrightarrow \{1, 2, \ldots, C(p)\}$. For any set P of p parts let G_P be the graph induced by all the parts in P. According to the assumption, each of the G_P has p-centered coloring c_P using $N(\mathcal{C}_p, p)$ colors. Consider the following ("product") coloring c defined as

$$c(v) = (\bar{c}(v), (c_P(v); |P| = p, P \subset \{1, 2, \ldots, C(p)\})).$$

This is the product of the coloring of G by $C(p)$ colors and of the colorings of the G_P. This new coloring of G (with $X(p) = C(p)N(\mathcal{C}_p, p)^{\binom{C(p)}{p}}$ colors. Let H be a connected subgraph of G. Then, either H gets at least $p+1$ colors, or $V(H)$ is included in some subgraph G_P of G induced by p parts. In the later case, some color appears exactly once in H. □

Lemma 3.10. *Let $N(p,t) = 1 + (t-1)(2 + \lceil \log_2 p \rceil)$, let $\vec{G}$ be a directed graph and let $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ be a transitive fraternal augmentation of $\vec{G}$.*

Then $\vec{G}_{N(p,\mathrm{td}(G))}$ either includes an acyclically oriented clique of size p or a rooted directed tree $\vec{Y}$ such that $G \subseteq \mathrm{clos}(Y)$ and $\mathrm{clos}(\vec{Y}) \subseteq \vec{G}_{N(p,\mathrm{td}(G))}$.

Proof. We fix the integer p and prove the lemma by induction on $t = \mathrm{td}(\vec{G})$. The base case $t = 1$ corresponds to a graph without edges, for which the property is obvious. Assume the lemma has been proved for directed graphs with tree-depth at most t and let $\vec{G}$ be a directed graph with tree-depth $t+1$. As we may consider each connected component of $\vec{G}$ independently, we may assume that $\vec{G}$ is connected. Then there exists a vertex $s \in V(\vec{G})$ such that the connected components $\vec{H}_1, \ldots, \vec{H}_k$ of $G - s$ have tree-depth at most t. $\vec{H}_i = \vec{G}_1[V(\vec{H}_i)] \subseteq \cdots \subseteq \vec{G}_j[V(\vec{H}_i)] \subseteq \cdots$ is a transitive fraternal augmentation of $\vec{H}_i$. By the induction hypothesis for each $1 \leq i \leq k$ there exists in $\vec{H}_i$ either an acyclically oriented clique of size p or a rooted tree $\vec{Y}_i$ rooted at r_i such that $H_i \subseteq \mathrm{clos}(Y_i)$ and $\mathrm{clos}(\vec{Y}_i) \subseteq \vec{G}_{N(p,\mathrm{td}(G))}[V(\vec{H}_i)]$. If the first case occurs for some i, then $\vec{G}$ includes an acyclically oriented clique of size p. Hence assume it does not. As $\vec{G}$ is connected, the vertex s has at least a neighbor x_i in $\vec{H}_i$ (for each $1 \leq i \leq k$). Let x be any neighbor of s in $\vec{H}_i$. If y is an ancestor of x in $\vec{Y}_i$, (y, x) is an arc of $\vec{G}_{N(p,t)}$ hence s and y are adjacent in $\vec{G}_{N(p,t)+1}$. Moreover, if (x, s) is an arc of $\vec{G}_{N(p,t)}$ then (y, s) is an arc of $\vec{G}_{N(p,t)+1}$. Let D_i be the subset of $V(\vec{H}_i)$ of the vertices x such that (x, s) belongs to $\vec{G}_{N(p,t)}$ and of their ancestors in $\vec{Y}_i$ and let

$D = \bigcup_{i=1}^k D_i$. Then D includes a clique in $\vec{G}_{N(p,t)+2}$. Thus there exists a directed Hamiltonian path $\vec{P}$ in $\vec{G}_{N(p,t)+2}[D]$.

Let r be the start vertex of $\vec{P}$. Define $\pi : V(G) - r \to V(G)$ as follows:

- if $x \in D$, the $\pi(x)$ is the predecessor y of x in $\vec{P}$ (the arc (y,x) belongs to $\vec{G}_{N(p,t)+2}$);
- otherwise, if $x = s$, $\pi(x)$ is the end vertex y of $\vec{P}$ (the arc (y,x) belongs to $\vec{G}_{N(p,t)+1}$);
- otherwise, if $x = r_i$ then $\pi(x) = s$ (the arc (s, r_i) belongs to $\vec{G}_{N(p,t)+2}$);
- otherwise, if the father of $x \in V(\vec{H}_i) \setminus D$ does not belong to D, then $\pi(x)$ is the father of x in $\vec{Y}_i$;
- otherwise, if no descendant of x in $\vec{Y}_i$ has an arc coming from s in $\vec{G}_{N(p,t)+1}$, $\pi(x)$ is the father of x in $\vec{Y}_i$;
- otherwise, $\pi(x) = s$ (the arc (s,x) belongs to $\vec{G}_{N(p,t)+2}$).

It is easily checked that the so defined "father mapping" π actually defines a directed rooted tree $\vec{Y}$ of $\vec{G}_{N(p,t)+2}$ with root r and that $G \subseteq \mathrm{clos}(\vec{Y})$. Moreover, either $\vec{Y}$ has height at least p and $\vec{G}_{N(p,t)+2+\lceil \log_2 p \rceil}$ includes an acyclically oriented clique of size p or $\mathrm{clos}(\vec{Y}) \subseteq \vec{G}_{N(p,t)+2+\lceil \log_2 p \rceil}$. As $N(p, t+1) = N(p,t) + 2 + \lceil \log_2 p \rceil$, the induction follows. □

Lemma 3.11. *Let p be an integer, let $\vec{G}$ be a directed graph and let $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ be a transitive fraternal augmentation of $\vec{G}$. Then either $\vec{G}_{N(p,p)}$ includes an acyclically oriented clique of size p or $\mathrm{td}(G) \le p-1$ and there exists in $\vec{G}_{N(p,p)}$ a rooted directed tree Y so that $G \subseteq \mathrm{clos}(Y)$ and $\mathrm{clos}(\vec{Y}) \subseteq \vec{G}_{N(p,p)}$.*

Proof. If $\mathrm{td}(G) > p$ we may consider a connected subgraph of H of tree-depth p. According to Lemma 3.10, there will exists in $\vec{G}_{N(p,p)}[V(H)]$ an acyclically oriented clique of size p or a rooted directed tree $\vec{Y}$ so that $H \subseteq \mathrm{clos}(Y)$ and $\mathrm{clos}(\vec{Y}) \subseteq \vec{G}_{N(p,p)}[V(H)]$. In the later case, if $\mathrm{td}(G) = p$ then the height of $\vec{Y}$ is at least $\mathrm{td}(H) = p$ and $\mathrm{clos}(\vec{Y})$ includes an acyclically oriented clique of size p. □

Corollary 3.12. *Let $R(p) = 1 + (p-1)(2 + \lceil \log_2 p \rceil) = O(p \log_2 p)$.*

For any graph G, for any transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ of G and for any integer p, any proper coloring of $\vec{G}_{R(p)}$ defines a p-centered coloring of G.

4. Application: Low Tree-width Partitions in Linear Time

Now we turn to the algorithmic aspects of the above results. We will prove that, for any class $\mathcal{C}$ with bounded expansion and any graph $G \in \mathcal{C}$

(1) the graph G has an orientation with indegree at most $c_1(\mathcal{C})$ which may be computed in linear time;
(2) the graph G has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_{R(p)}$ with $\Delta^-(\vec{G}_{R(p)}) \leq c_2(\mathcal{C}, p)$ which can be computed in linear time (for fixed p);
(3) a p-centered coloration of G may be computed from $\vec{G}_{R(p)}$ in linear time,
(4) for any $i \leq p$, a tree-decomposition of width at most $(i-1)$ of the subgraph of G induced by i colors may be computed in linear time.

4.1. Transitive Fraternal Augmentation.

Theorem 4.1. *For any class $\mathcal{C}$ with bounded expansion and any fixed integer c, there exists an algorithm which computes, given an input graph $G \in \mathcal{C}$, a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_c$ of G in time $O(n)$.*

This follows from an iterative application of a linear time 1-fraternal transitive augmentation and the application of Corollary 3.5.

In the augmentation process, we add two kind of arcs: transitivity arcs and fraternity arcs. Let us start with transitivity ones:

Require: D represents the directed graph to be augmented.
Ensure: D' represents the array of the added arcs.

```
Initialize D'.
for all v ∈ {1, ..., n} do
   for all (u, e) ∈ D[v] do
      for all (x, f) ∈ D[u] do
         m ← m + 1; append (x, m) to D'[v].
      end for
   end for
end for
```

This algorithm runs in $O(\Delta^-(\vec{G})^2 n)$ time, where $\Delta^-(\vec{G})$ is the maximum indegree of the graph to be augmented. It computes the list array D' of the transitivity arcs which are missing in $\vec{G}$, missing arcs may appear more than once in the list, but the number of added edges cannot exceed $\Delta^-(\vec{G})^2 n$.

Now, we shall consider the fraternity edges.

Require: D represents the directed graph to be augmented.
Ensure: L represents the list of edges to be added.

```
L = ().
for all v ∈ {1,...,n} do
  for all (x,e) ∈ D[v] do
    for all (y,f) ∈ D[v] do
      if x < y then
        append (x,y) to L.
      end if
    end for
  end for
end for
```

This algorithm runs in $O(\Delta^-(\vec{G})^2 n)$-time and computes the list of the fraternity edges. Edges may appear in this list more than once but the length of the list L cannot exceed $\Delta^-(\vec{G})^2 n/2$.

The simplification of L, the computation of a low indegree orientation of the edges in L and the merge/simplification with the arcs in D and D' may be achieved in linear time (precisely: in $O(\Delta^-(\vec{G})^2 n)$-time).

4.2. **Computing a p-centered Coloring.** Let G be a graph. Define $f(r) = \nabla_r(G)$ and $F(x,y) = x^2 + 2y$ and let $R(p) = 1 + (p-1)(2 + \lceil \log_2 p \rceil)$. According to Corollary 3.5, the fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_{R(p)}$ of G computed by iterating $R(p)$ times the 1-transitive fraternal augmentation algorithm is such that $G_{R(p)}$ is $(2A_{f,F}(0, R(p)))$-degenerated. Thus a proper coloring of $G_{R(p)}$ may easily be achieved in linear time. This coloring defines a p-centered coloring of G according to Corollary 3.12.

4.3. **Tree-decomposition of $i \leq p$ Parts.** To construct a tree-decomposition (T, λ) of width $(p-1)$ of a graph G from a centered coloring using p colors, we first construct a rooted forest of height p including G in its closure:

Require: c is a centered-coloring of the graph G using colors $1, \ldots, p$.
Ensure: F is a rooted forest of height p such that $G \subset \mathrm{clos}(F)$.

Set $F = \emptyset$.
Let Big[ ] be an array of size p.
for all Connected component G_i of G **do**
  Initialize Big[ ] to **false**.
  Set root_color $\leftarrow 0$.
  for all $v \in V(G_i)$ **do**
    if Big$[c[v]] =$ **false then**
      if $c[v] =$ root_color **then**
        root_color $\leftarrow 0$, Big$[c[v]] \leftarrow$ **true**.
      else
        root $\leftarrow v$; root_color $\leftarrow c[v]$.
      end if
    end if
  end for
  Recurse on $G -$root thus getting some rooted forest $F' = \{Y'_1, \dots, Y'_j\}$.
  Add to F the tree with root root and subtrees $Y_1, \dots, Y_j$, where the sons of root are the roots of $Y_1, \dots, Y_j$.
end for

This algorithms clearly runs in $O(pm)$ time. If G is connected, it returns a rooted tree Y of height at most p such that $G \subseteq \text{clos}(Y)$ (thus proving that tree depth of G is at most p).

From a rooted tree Y of height at most p such that $G \subseteq \text{clos}(Y)$ it is straightforward to construct a tree-decomposition (T, λ) of G having width at most $(p-1)$: Set $T = Y$ and define $\lambda(x) = \{v \leq_Y x\}$. Then for any v, $\{x \in V(T) : v \in \lambda(x)\} = \{x \geq_Y v\}$ induces the subtree of Y rooted at v (hence a subtree of T). Moreover, as $G \subseteq \text{clos}(Y)$, any edge $\{x, y\}$ with $x <_Y y$ is a subset of $\lambda(y)$. Hence (T, λ) is a tree-decomposition of G. As $\max_{v \in V(G)} |\lambda(v)| = \text{height}(Y) \leq p$, this tree-decomposition has width at most $(p-1)$. Last, this tree-decomposition may be obviously constructed in linear time. (All this amounts to saying that tree width does not exceeds tree depth.)

5. Subgraph Isomorphism Problem

In [8] Eppstein gives a linear time algorithm to solve the subgraph isomorphism problem for a fixed planar pattern. In this paper, he gives a linear

Subgraph isomorphism problem		
Context	*Complexity*	*Reference(s)*
General	$O(n^{0.792\ \lvert V(H)\rvert})$	[17] using [2]
Bounded tree-width	$O(n)$	[8] (also [3][4])
Planar	$O(n)$	[7][8]
Bounded genus	$O(n)$	[9]
Bounded expansion (includes the three previous classes)	$O(n)$	(this paper)

Table 1. Subgraph isomorphism problem: complexity for a fixed pattern H and for an input graph restricted to some class of graphs.

time bound for a fixed pattern and an input graph with bounded width tree decomposition:

Lemma 5.1 (Lemma 2 of [8])**.** *Assume we are given graph G with n vertices along with a tree-decomposition T of G with width w. Let S be a subset of vertices of G, and let H be a fixed graph with at most w vertices. Then in time $2^{O(w \log w)}n$ we can count all isomorphs of H in G that include some vertex in S. We can list all such isomorphs in time $2^{O(w \log w)}n + O(kw)$, where k denotes the number of isomorphs and the term kw represents the total output size.*

We deduce from this lemma and Theorem 4.1 an extension of Eppstein's result of [8][9] to classes with bounded expansion:

Theorem 5.2. *Let $\mathcal{C}$ be a class with bounded expansion and let H be a fixed graph. Then there exists a linear time algorithm which computes, from a pair (G, S) formed by a graph $G \in \mathcal{C}$ and a subset S of vertices of G, the number of isomorphs of H in G that include some vertex in S. There also exists an algorithm running in time $O(n)+O(k)$ listing all such isomorphism where k denotes the number of isomorphs (thus represents the output size).*

6. Local decidability problems

Monadic second-order logic (MSOL) is an extension of first-order logic (FOL) that includes vertex and edge sets and belonging to these sets. The following theorem of Courcelle has been applied to solve many optimization problems.

Theorem 6.1 (Courcelle [3][4]). *Let $\mathcal{K}$ be class of finite graphs $G = \langle V, E, R\rangle$ represented as τ_2-structures, that is: by two sorts of elements (vertices V and edges E) and an incidence relation R. Let ϕ be a MSOL(τ_2) sentence. If $\mathcal{K}$ has bounded tree width and $G \in \mathcal{K}$, then checking whether $G \vDash \phi$ can be done in linear time.*

Combining Theorem 6.1 with Theorem 4.1, we get:

Theorem 6.2. *Let $\mathcal{C}$ be a class with bounded expansion and let p be a fixed integer. Let ϕ be a FOL(τ_2) sentence. Then there exists a linear time algorithms to check $\exists X : (|X| \leq p) \wedge (G[X] \vDash \phi)$.*

Thus for instance:

Theorem 6.3. *Let $\mathcal{K}$ be a class with bounded expansion and let H be a fixed graph. Then, for each of the next properties there exists a linear time algorithm to decide whether a graph $G \in \mathcal{K}$ satisfies them:*

- *H has a homomorphism to G,*
- *H is a subgraph of G,*
- *H is an induced subgraph of G.*

7. Summary, concluding remarks, directions to future research

We introduced new conceptual framework (k-grad of a graph and classes with bounded expansion). This generalizes many commonly studied classes of graphs yet one can prove for these classes strong structural decomposition theorems such as low tree width (and low tree depth) decompositions. Quite surprisingly these decompositions may be obtained in linear time. This in turn solves several open problems (see e.g. [16]) and we listed several applications to subgraph testing and decision problems for satisfiability.

A natural question arises whether one can generalize low tree depth decompositions to yet larger classes of graphs. In this sense our results are optimal: It is proved in [12] that a class $\mathcal{C}$ of graphs has bounded expansion if and only if it has low tree-width (or low tree-depth) decompositions. It should be also noted that the notion of tree depth enjoys finiteness property: there are only finitely many cores of graphs of tree depth $\leq k$. This is not true even for series parallel graphs (i.e. for graphs with tree-width at most 2).

Our setting can also be generalized to oriented graphs and relational structures (see our full paper for further details and discussion). Particularly [13] contains applications to homomorphism bounds which provided the original motivation for our research.

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8. Annex

8.1. **Proof of Lemma 2.2.** Recall that Lemma 2.2 states that there exist polynomials P_i $(i \geq 0)$ such that for any graph G and integers r and c:

$$\overset{c}{\nabla}_r(G) \leq P_r(c, \nabla_r(G)).$$

In the following, a directed graph $\vec{G}$ may not have a loop and for any two of its vertices x and y, $\vec{G}$ includes at most one arc from x to y and at most one arc from y to x.

If a directed path $\vec{P}$ has starting vertex x and end vertex y, we note

$$x \overset{\vec{P}}{\rightsquigarrow} y.$$

If $x \overset{\vec{P}_1}{\rightsquigarrow} z$, $y \overset{\vec{P}_2}{\rightsquigarrow} z$ and if no internal vertex or edges of $\vec{P}_1$ belongs to $\vec{P}_2$ nor the converse, we will write symbolically

$$x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y.$$

In such a case, either $\vec{P}_1 \cup \vec{P}_2$ is a path, or $\vec{P}_1 \cup \vec{P}_2$ is a cycle and $x = y$. Moreover, if $x \neq y$, $|\vec{P}_1| \leq a$ and $|\vec{P}_2| \leq b$, we say that y is (a, b)*-reachable* from x.

Definition 8.1. Let $\vec{G}$ be a directed graph, let a, b be integers. A set $\vec{\Lambda}$ of arcs with endpoints in $V(\vec{G})$ is an (a, b)*-augmentation* of $\vec{G}$ if, for any $x, y \in V(\vec{G})$ with y (a, b)-reachable from x, either (x, y) or (y, x) belongs to $\vec{\Lambda}$.

The *maximum indegree* of $\vec{\Lambda}$ is

$$\Delta^-(\vec{\Lambda}) = \max_{y \in V(\vec{G})} |\{x \in V(\vec{G}) : (x, y) \in \vec{\Lambda}\}|.$$

Notice that if a or b is at least 1, $E(\vec{G})$ is obviously included in any (a, b)-augmentation of $\vec{G}$.

Lemma 8.1. *Let $\vec{G}$ be a directed graph, let a, b be integers and let $\vec{\Lambda}$ be an (a, b)-augmentation of $\vec{G}$. Then there exists a vertex coloring $\gamma_{\vec{\Lambda}}$ using at most $2\Delta^-(\vec{\Lambda}) + 1$ colors such that for any vertex x, $\gamma_{\vec{\Lambda}}(y) \neq \gamma_{\vec{\Lambda}}(x)$ for any vertex y (a, b)-reachable from x.*

Proof. Let $\vec{H}$ be the directed graph with vertex set $\vec{G}$ and arc set $\vec{\Lambda}$. If y is (a, b)-reachable from x in $\vec{G}$ then (x, y) or (y, x) belongs to $E(\vec{H})$. As $\vec{H}$

has maximum indegree $\Delta^-(\vec{\Lambda})$, it is $(2\Delta^-(\vec{\Lambda})+1)$-choosable. Any proper coloration of $\vec{H}$ will do. $\square$

Lemma 8.2. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let a, b be integers and let $\vec{\Lambda}$ be an (a,b)-augmentation of $\vec{G}$. Then there exists an edge coloring $\Upsilon_{\vec{\Lambda}}$ using at most $(2\Delta^-(\vec{\Lambda})+1)\Delta^-(\vec{G})$ colors such that for any $x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$ with $|\vec{P}_1| \leq a+1$ and $|\vec{P}_2| \leq b+1$, all the edges of $\vec{P}_1 \cup \vec{P}_2$ get different colors.*

Proof. Consider an edge coloring c_0 such that two edges having the same end vertex have different colors (this is achieved with $\Delta^-(\vec{G})$ colors) and the vertex coloring $\gamma_{\vec{\Lambda}}$ defined in Lemma 8.1. Then for any arc $e = (x, y)$ define $\Upsilon_{\vec{\Lambda}}(e) = (c_0(e), \gamma_{\vec{\Lambda}}(y))$. Then if $e = (x, y)$ and $f = (x', y')$ are two different arcs in $\vec{P}_1 \cup \vec{P}_2$ where either $y \neq y'$ thus y' is (a,b)-reachable from y or y is (a,b)-reachable from y' hence $\gamma_{\vec{\Lambda}}(y') \neq \gamma_{\vec{\Lambda}}(y)$, or $y = y'$ hence $c_0(e) \neq c_0(f)$. $\square$

Notation 8.2. Let Υ be an arc-coloring of a directed graph $\vec{G}$ and let $\vec{P}$ be a directed path of $\vec{G}$ of length l. We note $\Upsilon(\vec{P}) = \vec{\alpha} = (\alpha_1, \ldots, \alpha_l)$ the sequence of the colors $\Upsilon(e)$ of the arcs of $\vec{P}$, taken in the order in which they appear on $\vec{P}$.

Lemma 8.3. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let a, b be integers and let $\vec{\Lambda}$ be an (a,b)-augmentation of $\vec{G}$. Let $\Upsilon_{\vec{\Lambda}}$ be the edge coloring defined in Lemma 8.2.*

Let $\vec{P}_1, \vec{P}_2$ be two directed paths of length $l \leq \max(a,b)+1$, such that the initial vertex of one of them is different from the end vertex of the other one. If $\Upsilon_{\vec{\Lambda}}(\vec{P}_1) = \Upsilon_{\vec{\Lambda}}(\vec{P}_2)$ then either $\vec{P}_1$ and $\vec{P}_2$ do not intersect, or they share the same initial vertex and there exists $0 \leq a \leq l$ such that $\vec{P}_1$ and $\vec{P}_2$ share their a first edges and do not intersect thereafter.

Proof. Without loss of generality, we may assume $a \geq b$. Let $\vec{\alpha} = \Upsilon_{\vec{\Lambda}}(\vec{P}_1)$. Assume there exists a vertex v having one incoming edge in $\vec{P}_1$ (the ith of $\vec{P}_1$, hence colored α_i) and one (different) incoming edge in $\vec{P}_2$ (the jth of $\vec{P}_2$, hence colored α_j). Without loss of generality, we may assume $i \geq j$. Then the $(j+1)$th vertex u of $\vec{P}_1$ has an incoming edge in $\vec{P}_1$ colored α_j and belong to the initial subpath of $\vec{P}_1$ ending at v. It follows that v is $(a,0)$ reachable from u. Hence an incoming edge of u may not have the same color of an incoming edge of v, contradiction. Similarly, the initial vertex of one of the path may not be internal to the second one. As the case where the initial vertex of one of the path is the end vertex of the other one, we conclude that either the two paths do not intersect or they share their a first edges. $\square$

Lemma 8.4. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let a, b be integers and let $\vec{\Lambda}$ be an (a,b)-augmentation of $\vec{G}$. Let $\Upsilon_{\vec{\Lambda}}$ be the edge coloring defined in Lemma 8.2. Let $\vec{\alpha}$ be a sequence of $l \leq \max(a,b)+1$ distinct edge colors. Then the union $T_{\vec{\Lambda}}(\vec{\alpha})$ of all the directed paths $\vec{P}$ such that $\Upsilon_{\vec{\Lambda}}(\vec{P}) = \vec{\alpha}$ is a directed rooted forest.*

Proof. This is a direct consequence of Lemma 8.3. $\square$

Lemma 8.5. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let $a \geq b$ be integers and let $\vec{\Lambda}$ be an (a,b)-augmentation of $\vec{G}$. Let $\Upsilon_{\vec{\Lambda}}$ be the edge coloring defined in Lemma 8.2. Let $\vec{\alpha}$ and $\vec{\beta}$ be sequences of respective lengths $p \leq a+1$ and $q \leq b+1$. Let $\Pi_{\vec{\Lambda}}(\vec{\alpha},\vec{\beta})$ be the union of all the $\vec{P}_1 \cup \vec{P}_2$ where $\Upsilon_{\vec{\Lambda}}(\vec{P}_1) = \vec{\alpha}$, $\Upsilon_{\vec{\Lambda}}(\vec{P}_2) = \vec{\beta}$ and there exists three distinct vertices x, y, z so that $x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$.*

Then a directed tree Y_1 in $\Pi_{\vec{\Lambda}}(\vec{\alpha},\vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\alpha})$ and a directed tree Y_2 in $\Pi_{\vec{\Lambda}}(\vec{\alpha},\vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\beta})$ with different roots may only intersect at a leaf of both of them.

Proof. Let r_1, r_2 be the roots of Y_1 and Y_2. If Y_1 and Y_2 intersects, there exists vertices z, z' so that:

- $r_1 \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$, $\quad x' \overset{\vec{P}_1'}{\rightsquigarrow} z' \overset{\vec{P}_2'}{\leftsquigarrow} r_2$,
- $\Upsilon_{\vec{\Lambda}}(\vec{P}_1) = \Upsilon_{\vec{\Lambda}}(\vec{P}_1') = \vec{\alpha}$, $\quad \Upsilon_{\vec{\Lambda}}(\vec{P}_2) = \Upsilon_{\vec{\Lambda}}(\vec{P}_2') = \vec{\beta}$,
- $\vec{P}_2'$ intersects $\vec{P}_1$ at a vertex v (up to an exchange of Y_1 and Y_2).

As $r_1 \neq r_2$, v has in $\vec{P}_2$ an incoming edge e of color β_i for some $1 \leq i \leq b+1$. Let w be the vertex of $\vec{P}_2$ having in $\vec{P}_2$ an incoming edge of color β_i. If $w \neq v$, we are led to a contradiction, according to Lemma 8.2, as w is then (p,q)-reachable from v. Hence $v = w$ and v is the end vertex of $\vec{P}_1$ and $\vec{P}_2$. Thus v is also the end vertex of $\vec{P}_1'$ and $\vec{P}_2'$. It follows that v is a leaf of both Y_1 and Y_2. $\square$

Lemma 8.6. *Let $\vec{G}$ be a directed graph with maximum indegree $\Delta^-(\vec{G})$, let r be an integer and let $\vec{\Lambda}$ be an $(r, r-1)$-augmentation of $\vec{G}$. Then $\vec{\Lambda}$ may* *be extended into an $(r+1,r)$-augmentation $\vec{\Lambda}'$ such that:*

$$\Delta^-(\vec{\Lambda}') \leq \Delta^-(\vec{\Lambda}) + ((2\Delta^-(\vec{\Lambda})+1)\Delta^-(\vec{G}))^{2r+1}\nabla_r(G)$$

Proof. Let $\Upsilon_{\vec{\Lambda}}$ be the edge coloring defined in Lemma 8.2. For two sequences $\vec{\alpha}$ and $\vec{\beta}$ of respective lengths $p \leq r+1$ and $q \leq r$, let $\Pi_{\vec{\Lambda}}(\vec{\alpha},\vec{\beta})$ be the union of all the $\vec{P}_1 \cup \vec{P}_2$ where $\Upsilon_{\vec{\Lambda}}(\vec{P}_1) = \vec{\alpha}$, $\Upsilon_{\vec{\Lambda}}(\vec{P}_2) = \vec{\beta}$ and there exists three distinct vertices x, y, z so that $x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$. Also, let $G_{\vec{\alpha},\vec{\beta}}$ be the graph obtained from G by contracting all the edges of $\Pi_{\vec{\Lambda}}(\vec{\alpha},\vec{\beta})$ but those colored α_p.

Let x, y be vertices of G so that y is $(r+1,r)$-reachable from x, as witnessed by $x \overset{\vec{P}_1}{\rightsquigarrow} z \overset{\vec{P}_2}{\leftsquigarrow} y$. Let $\vec{\alpha} = \Upsilon_{\vec{\Lambda}}(\vec{P}_1)$ and $\vec{\beta} = \Upsilon_{\vec{\Lambda}}(\vec{P}_2)$. The vertices x, y are the roots of directed trees in $\Pi_{\vec{\Lambda}}(\vec{\alpha},\vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\alpha})$ and $\Pi_{\vec{\Lambda}}(\vec{\alpha},\vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\beta})$, respectively, hence to two adjacent distinct vertices in $G_{\vec{\alpha},\vec{\beta}}$. Similarly, two distinct vertices of $G_{\vec{\alpha},\vec{\beta}}$ adjacent by an edge of color α_p (where $p = |\vec{\alpha}|$) correspond uniquely to the roots of a tree in $\Pi_{\vec{\Lambda}}(\vec{\alpha},\vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\alpha})$ and $\Pi_{\vec{\Lambda}}(\vec{\alpha},\vec{\beta}) \cap T_{\vec{\Lambda}}(\vec{\beta})$, respectively.

It follows that there exists an $(r+1,r)$-augmentation $\vec{\Lambda}'$ of $\vec{G}$ extending $\vec{\Lambda}$ such that

$$\begin{aligned}\Delta^-(\vec{\Lambda}') - \Delta^-(\vec{\Lambda}) &\leq \sum_{\substack{|\vec{\alpha}|\leq r+1\\ |\vec{\beta}|\leq r}} \nabla_0(G_{\vec{\alpha},\vec{\beta}})\\ &\leq ((2\Delta^-(\vec{\Lambda})+1)\Delta^-(\vec{G}))^{2r+1}\nabla_r(G)\end{aligned}$$

□

Corollary 8.7. *For any integer r, there exists a polynomial Φ_r such that any directed graph $\vec{G}$ has a $(r+1,r)$-augmentation $\vec{\Lambda}$, where $\Delta^-(\vec{\Lambda}) \leq \Phi_r(\Delta^-(\vec{G}), \nabla_r(G))$, where G is the underlying simple graph of $\vec{G}$.*

of Lemma 2.2. Define $P_r(x,y) = \Phi_r(x+y,y)$. Consider a family $\mathcal{P}$ of balls of G with radius at most r and complexity at most c. We construct a directed graph $\vec{G}$ with underlying undirected graph G. Recall that $\vec{G}$ may have, for each edge of G, one arc in each direction. First we orient the edges of G with indegree $\nabla_0(G)$ (thus obtaining one arc per edge). For each $X \in \mathcal{P}$, let v be the center of $G[X]$. Let Y be a minimum distance tree of $G[X]$ with root v. If $\vec{G}$ does not include the arcs corresponding to an orientation of Y from its root v, we add the missing arcs. We also add if necessary all the arcs going from a leaf of Y to a vertex out of X.

Notice that the vertices of $\vec{G}$ have indegree at most $\nabla_0(G)+c$. Moreover, if r_1, r_2 are the roots of the trees Y_1 and Y_2 corresponding to some parts $X_1, X_2 \in \mathcal{P}$ which are adjacent in $G/\mathcal{P}$ then r_2 is $(r+1,r)$-reachable from r_1 in $\vec{G}$ (by a directed path of length at most r in Y_1, possibly followed by an arc between the parts and a directed path of length at most r in Y_2 in opposite direction). Hence r_1 and r_2 are adjacent in any $(r+1,r)$-augmentation of $\vec{G}$. According to Corollary 8.7, there exists such an augmentation $\vec{\Lambda}$ with $\Delta^-(\vec{\Lambda}) \leq \Phi_r(\nabla_0(G)+c, \nabla_r(G))$. As $G/\mathcal{P}$ is isomorphic to a subgraph of the graph with vertex set $V(G)$ and edge set $\vec{\Lambda}$. As this subgraph has an orientation with indegree at most $\Delta^-(\vec{\Lambda})$ we have, according to Fact 2.1 and Corollary 8.7:

$$\begin{aligned}\overset{c}{\nabla}_r(G) = \nabla_0(G/\mathcal{P}) &\leq \Delta^-(\vec{\Lambda}) \\ &\leq \Phi_r(\nabla_0(G)+c, \nabla_r(G)) \\ &\leq P_r(c, \nabla_r(G)).\end{aligned}$$

□

8.2. **Proof that topologically closed classes have bounded expansion.** We recall the following result of Komlós and Szemerédi [24]: If a simple graph on n vertices has at least $\frac{1}{2}p^2 n$ edges, then it has a K_p-subdivision. Hence a graph G with no K_p-subdivision is such that $\nabla_0(G) < \frac{p^2}{2}$. Inequalities for the grads of further ranks are inductively deduced using the following lemma (which yields $\nabla_r(G) < 2^{r-1}p^{2^{r+1}}$):

Lemma 8.8. *Let H be a minor of depth 1 of a graph G. Assume H includes a subdivision of $K_{p'}$. Then G includes a subdivision of K_p if $p' \geq 2p^2 - 6p + 8$.*

Proof. If $p = 1, 2$ or 3 the result is obvious as $p' \geq p$ and G will obviously include a vertex, an edge or a cycle (respectively). Thus we may assume $p \geq 4$ hence $p' - p(p-1) \geq \max(p, (p-2)(p-3)+2)$.

By considering a subgraph of G if necessary, we may assume that $V(G)$ is partitioned into $A_1, \dots, A_i, \dots, A_{p'}$, $L_{1,1}, \dots, L_{i,j}, \dots, L_{p',p'}$ where:

- for $1 \leq i \leq p'$, $G[A_i]$ is a star (possibly reduced to a single vertex or a single edge);
- for $1 \leq i < j \leq p'$, there exists $v_{i,j} \in A_i$ and $v_{j,i} \in A_j$ such that $G[L_{i,j} \cup \{v_{i,j}, v_{j,i}\}]$ is a path with endpoint $v_{i,j}$ and $v_{j,i}$.

For sake of simplicity, we define $L_{j,i} = L_{i,j}$ and $L_{i,i} = \emptyset$. For a subset Y of $\{1,\dots,p\}$ we also define G_Y has the subgraph of G induced by $\bigcup_{i\in Y} A_i \cup \bigcup_{i,j\in Y} L_{i,j}$.

We first claim the following result: Let N be a positive integer and let X be a subset of $\{1,\dots,p'\}$ of cardinality at least $\max(N,(N-2)(N-3)+2)$. Then there exists a subset $X' = \{k_{a,1},\dots,k_{a,N}\}$ of X of cardinality $(N-1)$ such that there exists in $G_{X'}$ a spider (that is: a subdivision of a star) with center $r_a \in A_{k_{a,a}}$ and leaves $l_{a,1},\dots,l_{a,a-1},l_{a,a+1},\dots,l_{a,N}$ with $l_{a,i} \in L_{a,k_{a,i}}$. This claim is easily proved as follows: Assume no vertex of $A_{k_{a,a}}$ has degree at least $(N-1)$ in G_X. Then $|X|-1 \leq (N-2)(N-3)$, a contradiction. Choose for r_a any vertex of $A_{k_{a,a}}$ with degree at least $(N-1)$ in G_X. Then there exists in G_X a spider with center r_a and at least $(N-1)$ leaves belonging to different $A_{k_{a,i}}$.

Assume $p' - N(N-1) \geq (N-2)(N-3)+2$, i.e. $p' \geq 2N^2 - 6N + 8$. Using the previous claim, we inductively define $Z_1,\dots,Z_N$ with $Z_i = \{k_{i,1},\dots,k_{i,N}\}$ such that G_{Z_i} contains a spider with center $r_i \in A_{k_{i,i}}$ and leaves $l_{i,j} \in A_{k_{i,j}}$: to construct Z_i, we consider $X = \{1,\dots,p'\} \setminus \bigcup_{1\leq j<i} Z_j$. Then G includes a subdivision of K_N with principal vertices $r_1,\dots,r_N$ as the union of all the spiders (and connections within the $L_{i,j}$ if necessary). □

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