Separation of two convex polyhedral sets with parameters in one row of the constraint matrix

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Abstract

In [6, 7] we were concerned with the separation properties of two convex polyhedral sets, which have parameters situated in the right-hand side and in one column of the constraint matrix, respectively. In this paper we study the separation properties of two convex polyhedral sets for the case there are parameters in one row of the constraint matrix. Especially, we deal with the existence, description and stability properties of the separating hyperplanes of such convex polyhedral sets. Briefly, we are also interested in supporting separation (separating hyperplanes supports convex polyhedral sets at given faces).

Keywords: *separating hyperplane, supporting hyperplane, parameters, convex polyhedra, solution set, stability set.*

1 Introduction

Separation is a famous and important principle useful not only in optimization theory, but for various applications as well. There are several kinds of separability of convex sets (cf. [9]). For the purpose of this paper we introduce the following one.

Definition 1. Sets $X, Y \subset \mathbb{R}^n$ are called *strongly separable* if $dim\, X = dim\, Y = n$ and there exists a hyperplane $\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} = s\}$ such that $X \subseteq \overline{\mathcal{R}^-} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \leq s\}$, and $Y \subseteq \overline{\mathcal{R}^+} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \geq s\}$ hold. $\mathcal{R}$ is called *the separating hyperplane* of the sets X, Y.

We will use the following well known separation theorem (see e.g. [3, 8]):

Theorem 1. *Convex sets X, $Y \subset \mathbb{R}^n$ are strongly separable if and only if $dim\, X = dim\, Y = n$, and $int\, X \cap int\, Y = \emptyset$.*

In this article we deal with two convex polyhedral sets ($\widetilde{\mathbf{A}} \in \mathbb{R}^{m\times n}$, $\widetilde{\mathbf{C}} \in \mathbb{R}^{l\times n}$, $\widetilde{\mathbf{b}} \in \mathbb{R}^m$, $\widetilde{\mathbf{d}} \in \mathbb{R}^l$):

$$\mathcal{M}_1 \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \widetilde{\mathbf{A}}\mathbf{x} \leq \widetilde{\mathbf{b}}\}, \tag{1}$$

$$\mathcal{M}_2 \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \widetilde{\mathbf{C}}\mathbf{x} \leq \widetilde{\mathbf{d}}\}, \tag{2}$$

In [6] we derived the basic separation properties of the sets (1), (2) with parameters in the right-hand side of inequalities. Some of the results obtained there, which we need for the purpose of this paper, will be presented at the end og this section. In next sections we study separation for the cases there are parameters in one row of the matrix $\widetilde{\mathbf{A}}$. In dealing with parameters, we were inspired by [1, 2, 10]. We will define so called solution set and in the sequel so called stability sets. For definition of stability sets we use the explicit description of all separating hyperplanes of two fixed convex polyhedral sets from [4, 5].

Let us introduce some notation. Given a matrix $\mathbf{M}$, the expressions $\mathbf{M}_{i,\cdot}$, $\mathbf{M}_{\cdot,j}$ denote i-th row and j-th column of the matrix $\mathbf{M}$, respectively. For given vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$, the expression $\mathbf{a} < \mathbf{b}$ means $a_i < b_i\ \forall i$. For any set $\mathcal{X}$ let us denote by $\overline{\mathcal{X}}$, $int\, \mathcal{X}$, $dim\, \mathcal{X}$, $conv\, \mathcal{X}$ and $\mathcal{X}^\perp$ the closer, the interior, the dimension, the convex hull, and the orthogonal complement of $\mathcal{X}$, respectively.

Definition 2. The *basis* of a convex polyhedral set described by $\mathbf{Mx} = \mathbf{v}$, $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{M} \in \mathbb{R}^{m\times n}$, $\mathbf{v} \in \mathbb{R}^m$, $m \leq n$) is any vector $B \in \{1, \ldots, n\}^m$ for which $rank(\mathbf{M}_B) = m$ (where $\mathbf{M}_B$ means the restriction of the matrix $\mathbf{M}$ to the basic columns). A basis B is *feasible*, if $\mathbf{M}_B^{-1}\mathbf{v} \geq \mathbf{0}$.

The sub-basis of the convex polyhedral set described by $\mathbf{Mx} \leq \mathbf{v}$ ($\mathbf{M} \in \mathbb{R}^{m\times n}$, $\mathbf{v} \in \mathbb{R}^m$) is any vector $S \in \{1, \ldots, m\}^k$, $1 \leq k \leq n$, for which $rank(\mathbf{M}_S) = k$ holds (where $\mathbf{M}_S$ in this case means the restriction of the matrix $\mathbf{M}$ to the sub-basic rows). The sub-basis S is called *feasible*, if

$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{M}_S\mathbf{x} = \mathbf{v}_S,\ \mathbf{M}_N\mathbf{x} \leq \mathbf{v}_N\} \neq \emptyset$ for $N = \{1, \ldots, m\} \setminus S$. The basis of $\mathbf{Mx} \leq \mathbf{v}$ is any n-elemental sub-basis.

Let us introduce

$$\mathcal{Q}^* \equiv \Big\{ (\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathbb{R}^{m+l+1} \mid \begin{pmatrix} \widetilde{\mathbf{A}}^T & \widetilde{\mathbf{C}}^T & \mathbf{0} \\ \widetilde{\mathbf{b}}^T & \widetilde{\mathbf{d}}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix}, \\ (\mathbf{u}, \mathbf{v}, v_{l+1}) \geq \mathbf{0} \Big\}. \tag{3}$$

The set $\mathcal{Q}^*$ enables us to describe all separating hyperplanes of $\mathcal{M}_1,\ \mathcal{M}_2$ from (1), (2). Theorem 2 comes from [6], Theorem 3 is a direct consequence of theorems from [5], [6].

Theorem 2. *Suppose that* $dim\, \mathcal{M}_1 = dim\, \mathcal{M}_2 = n,\ int\, \mathcal{M}_1 \cap int\, \mathcal{M}_2 = \emptyset$. *Let* $(\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}^*,\ \mathbf{u}^T\widetilde{\mathbf{A}} \neq \mathbf{0}^T$, *and* $\eta \in \langle 0, v_{l+1} \rangle$ *is arbitrary. Then*

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T(\widetilde{\mathbf{A}}\mathbf{x} - \widetilde{\mathbf{b}}) = \eta\} \tag{4}$$

represents a separating hyperplane of the convex polyhedral sets $\mathcal{M}_1,\ \mathcal{M}_2$. *Conversely, any separating hyperplane* $\mathcal{R}$ *of* $\mathcal{M}_1,\ \mathcal{M}_2$ *we can express in the form of* (4) *for a certain* $(\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}^*,\ \mathbf{u}^T\widetilde{\mathbf{A}} \neq \mathbf{0}^T$, *and* $\eta \in \langle 0, v_{l+1} \rangle$.

Theorem 3. *Let* $dim\, \mathcal{M}_1 = dim\, \mathcal{M}_2 = n$. *Then the convex sets* $\mathcal{M}_1$, $\mathcal{M}_2$ *are strongly separable if and only if* $\mathcal{Q}^* \neq \emptyset$.

2 Solution set

In this paper we are concerned with the situation when the fixed values of one row of the matrix $\widetilde{\mathbf{A}}$ from (1) (and the corresponding right-hand side) are replaced by parameters. We consider the following family of convex polyhedral sets

$$M_1(\boldsymbol{\lambda}, \mu) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \leq \mathbf{b},\ \boldsymbol{\lambda}^T\mathbf{x} \leq \nu\} \tag{5}$$

and the convex polyhedral set

$$M_2 \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Cx} \leq \mathbf{d}\}, \tag{6}$$

where $\mathbf{A} \in \mathbb{R}^{(m-1)\times n}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$, $\mathbf{b} \in \mathbb{R}^{m-1}$, $\mathbf{d} \in \mathbb{R}^{l}$, $m > 1$, $l > 0$ are fixed and $\boldsymbol{\lambda} \in \mathbb{R}^n$, $\nu \in \mathbb{R}$ are parameters. Let us introduce

$$M_1 \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \le \mathbf{b}\}. \tag{7}$$

Definition 3. *The solution set* (for the strong separability of the convex polyhedral set $M_1(\boldsymbol{\lambda}, \nu)$ from (5) and M_2 from (6)) is the set of all values of parameters $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ for which the convex polyhedral sets $M_1(\boldsymbol{\lambda}, \nu)$, M_2 are strongly separable.

From now on we suppose that $dim\, M_1 = dim\, M_2 = n$. Otherwise the solution set is empty.

Theorem 4. *Denote by $\mathbf{g}_k$, $k \in L$, any basis of the lineality space $\mathcal{L} \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$, denote by $\mathbf{x}_i$, $i \in V$, all vertices and $\mathbf{h}_j$, $j \in H$, all extremal directions[*] of the convex polyhedral set $M_1 \cap \mathcal{L}^{\perp}$. Then the set of all $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ satisfying $M_1(\boldsymbol{\lambda}, \nu) \ne \emptyset$, has the description*

$$\mathbb{R}^{n+1} \setminus \mathcal{P},$$

where $\mathcal{P}$ is the convex cone with description

$$\begin{aligned} \mathcal{P} = \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1} \mid & \mathbf{x}_i^T \boldsymbol{\lambda} > \nu \ \forall i \in V, \quad \mathbf{h}_j^T \boldsymbol{\lambda} \ge 0 \ \forall j \in H, \\ & \mathbf{g}_k^T \boldsymbol{\lambda} = 0 \ \forall k \in L\}. \end{aligned} \tag{8}$$

Proof. Let $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ be given arbitrarily. Each point $\mathbf{x} \in M_1$ can be (according to [11, Assertions 7.2c, 7.3d]) expressed as follows

$$\mathbf{x} = \sum_{i \in V} \alpha_i \mathbf{x}_i + \sum_{j \in H} \beta_j \mathbf{h}_j + \sum_{k \in L} \gamma_k \mathbf{g}_k,$$

where $\sum_{i \in V} \alpha_i = 1$, $\alpha_i \ge 0$, $i \in V$, $\beta_j \ge 0$, $j \in H$, $\gamma_k \in \mathbb{R}$, $k \in L$, and $V \ne \emptyset$ (from the assumption $dim\, M_1 = n$). If $(\boldsymbol{\lambda}, \nu) \in \mathcal{P}$, then

$$\mathbf{x}^T \boldsymbol{\lambda} = \sum_{i \in V} \alpha_i \mathbf{x}_i^T \boldsymbol{\lambda} + \sum_{j \in H} \beta_j \mathbf{h}_j^T \boldsymbol{\lambda} + \sum_{k \in L} \gamma_k \mathbf{g}_k^T \boldsymbol{\lambda} > \sum_{i \in V} \alpha_i \nu = \nu$$

and $M_1(\boldsymbol{\lambda}, \nu) = \emptyset$. Conversely, if $(\boldsymbol{\lambda}, \nu) \notin \mathcal{P}$, then it would occur one of the following three possibilities. Either $\mathbf{x}_i^T \boldsymbol{\lambda} \le \nu$ holds for a certain $i \in V$, and therefore $\mathbf{x}_i \in M_1(\boldsymbol{\lambda}, \nu) \ne \emptyset$. Or $\mathbf{h}_j^T \boldsymbol{\lambda} < 0$ holds for a certain $j \in H$.

[*] vectors in directions of unbounded edges

Consider a point $\mathbf{x}_c \equiv \mathbf{x}_1 + c.\mathbf{h}_j,\ c \geq 0$, where $\mathbf{x}_1$ is any vertex of $M_1 \cap \mathcal{L}^\perp$. Then for an arbitrary $c \geq \max\left\{\frac{\nu - \mathbf{x}_1^T\boldsymbol{\lambda}}{\mathbf{h}_j^T\boldsymbol{\lambda}}, 0\right\}$ we have $\mathbf{x}_c^T\boldsymbol{\lambda} = \mathbf{x}_1^T\boldsymbol{\lambda} + c\mathbf{h}_j^T\boldsymbol{\lambda} \leq \nu$, and thus $\mathbf{x_c} \in M_1(\boldsymbol{\lambda}, \nu) \neq \emptyset$. The third possibility is that $\mathbf{g}_k^T\boldsymbol{\lambda} \neq 0$ holds for a certain $k \in L$. Without the loss of generality assume that $\mathbf{g}_k^T\boldsymbol{\lambda} < 0$. Denote by $\mathbf{x}_1$ any vertex of $M_1 \cap \mathcal{L}^\perp$. Then the vector $\mathbf{x}_c \equiv \mathbf{x}_1 + c\mathbf{g}_k$ belongs to M_1 for all $c \in \mathbb{R}$ and for any $c \geq \frac{\nu - \mathbf{x}_1^T\boldsymbol{\lambda}}{\mathbf{g}_k^T\boldsymbol{\lambda}}$ we have $\mathbf{x}_c \in M_1(\boldsymbol{\lambda}, \nu) \neq \emptyset$. □

Theorem 5. *Let $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ be given arbitrarily. The set of all values of parameters $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ satisfying dim $M_1(\boldsymbol{\lambda}, \nu) = n$, is equal to*

$$\mathbb{R}^{n+1} \setminus \overline{\mathcal{P}},$$

where $\overline{\mathcal{P}}$ is the closer of the convex cone from (8) *described by*

$$\overline{\mathcal{P}} = \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1} \mid \mathbf{x}_i^T\boldsymbol{\lambda} \geq \nu\ \forall i \in V,\ \ \mathbf{h}_j^T\boldsymbol{\lambda} \geq 0\ \forall j \in H,\ \ \mathbf{g}_k^T\boldsymbol{\lambda} = 0\ \forall k \in L\}. \tag{9}$$

Proof. Let $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ be given arbitrarily. Each point $\mathbf{x} \in M_1$ can be (according to [11]) expressed as follows

$$\mathbf{x} = \sum_{i \in V} \alpha_i \mathbf{x}_i + \sum_{j \in H} \beta_j \mathbf{h}_j + \sum_{k \in L} \gamma_k \mathbf{g}_k,$$

where $\sum_{i \in V} \alpha_i = 1,\ \alpha_i \geq 0,\ i \in V,\ \beta_j \geq 0,\ j \in H,\ \gamma_k \in \mathbb{R}\ \forall k \in L$, and $V \neq \emptyset$ (from the assumption *dim* $M_1 = n$). If $(\boldsymbol{\lambda}, \nu) \in \overline{\mathcal{P}}$, then

$$\mathbf{x}^T\boldsymbol{\lambda} = \sum_{i \in V} \alpha_i \mathbf{x}_i^T\boldsymbol{\lambda} + \sum_{j \in H} \beta_j \mathbf{h}_j^T\boldsymbol{\lambda} + \sum_{k \in L} \gamma_k \mathbf{g}_k^T\boldsymbol{\lambda} \geq \sum_{i \in V} \alpha_i \nu = \nu$$

and according to the description (5) of $M_1(\boldsymbol{\lambda}, \nu)$ we have $\mathbf{x}^T\boldsymbol{\lambda} = \nu$. Hence the relation *dim* $M_1(\boldsymbol{\lambda}, \nu) < n$ holds. Conversely, if $(\boldsymbol{\lambda}, \nu) \notin \overline{\mathcal{P}}$, then it would occur one of the following three possibilities. Either $\mathbf{x}_i^T\boldsymbol{\lambda} < \nu$ holds for a certain $i \in V$, and therefore *dim* $M_1(\boldsymbol{\lambda}, \nu) = n$. Or $\mathbf{h}_j^T\boldsymbol{\lambda} < 0$ holds for a certain $j \in H$. Consider the point $\mathbf{x}_c \equiv \mathbf{x}_1 + c.\mathbf{h}_j,\ c > 0$, where $\mathbf{x}_1$ is any vertex of $M_1 \cap \mathcal{L}^\perp$. Then for an arbitrary $c > \max\left\{\frac{\nu - \mathbf{x}_1^T\boldsymbol{\lambda}}{\mathbf{h}_j^T\boldsymbol{\lambda}}, 0\right\}$ we have $\mathbf{x}_c^T\boldsymbol{\lambda} = \mathbf{x}_1^T\boldsymbol{\lambda} + c\mathbf{h}_j^T\boldsymbol{\lambda} < \nu$ and arbitrarily close to this point there is an interior point of $M_1(\boldsymbol{\lambda}, \nu)$. Hence *dim* $M_1(\boldsymbol{\lambda}, \nu) = n$. The third possibility is that $\mathbf{g}_k^T\boldsymbol{\lambda} \neq 0$ holds for a certain $k \in L$. Without the loss of generality assume that $\mathbf{g}_k^T\boldsymbol{\lambda} < 0$. If $\mathbf{x}_1$ is any vertex of $M_1 \cap \mathcal{L}^\perp$, then the vector

$\mathbf{x}_c \equiv \mathbf{x}_1 + c\mathbf{g}_k$ belongs to M_1 for all $c \in \mathbb{R}$. For any $c > \frac{\nu - \mathbf{x}_1^T \boldsymbol{\lambda}}{\mathbf{g}_k^T \boldsymbol{\lambda}}$ we have $\mathbf{x}_c \in M_1(\boldsymbol{\lambda}, \nu)$ and arbitrarily close to this point there is an interior point of $M_1(\boldsymbol{\lambda}, \nu)$. □

Remark 1. (The description of the solution set) Let us introduce

$$\mathcal{W} \equiv \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1} \mid int\, M_1(\boldsymbol{\lambda}, \nu) \cap int\, M_2 = \emptyset\}. \tag{10}$$

Then according to Theorem 1 and 5 the solution set is formed by the following (generally nonconvex) polyhedral set

$$(\mathbb{R}^{n+1} \setminus \overline{\mathcal{P}}) \cap \mathcal{W} = \mathcal{W} \setminus \overline{\mathcal{P}}. \tag{11}$$

If $int\, M_1 \cap int\, M_2 = \emptyset$, then for all $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ we have $int\, M_1(\boldsymbol{\lambda}, \nu) \cap int\, M_2 = \emptyset$, $\mathcal{W} = \mathbb{R}^{n+1}$ and the solution set is formed by $\mathbb{R}^{n+1} \setminus \overline{\mathcal{P}}$. Hence the convex polyhedral sets $M_1(\boldsymbol{\lambda}, \nu)$, M_2 are strongly separable whenever $dim\, M_1(\boldsymbol{\lambda}, \nu) = n$. Consider the situation that $int\, M_1 \cap int\, M_2 \neq \emptyset$. Since $int\, M_1 \cap int\, M_2 \neq \emptyset$ holds if and only if $dim\,(M_1 \cap M_2) = n$, we can apply the statement of Theorem 5 to the set

$$M_1 \cap M_2 = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{C}\mathbf{x} \leq \mathbf{d}\}$$

with the added inequality $\boldsymbol{\lambda}^T \mathbf{x} \leq \nu$. If we denote $\mathcal{L}_{12} \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0},\ \mathbf{C}\mathbf{x} = \mathbf{0}\}$ and by $\mathbf{g}_k$, $k \in L'$ any basis of the lineality space $\mathcal{L}_{12}$, we obtain the following relation

$$\mathcal{W} = \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1} \mid \mathbf{x}_i^T \boldsymbol{\lambda} \geq \nu\ \forall i \in V',\ \ \mathbf{h}_j^T \boldsymbol{\lambda} \geq 0\ \forall j \in H', \\ \mathbf{g}_k^T \boldsymbol{\lambda} = 0\ \forall k \in L'\}, \tag{12}$$

where $\mathbf{x}_i$, $i \in V'$, are all vertices and $\mathbf{h}_j$, $j \in H'$ all extremal directions of the convex polyhedral set $M_1 \cap M_2 \cap \mathcal{L}_{12}^{\perp}$.

Note that the description (12) for the set $\mathcal{W}$ cannot be used in the case that $0 < dim\,(M_1 \cap M_2) < n$ holds.

Theorem 6. *For the sets* $\overline{\mathcal{P}}$ *from* (9) *and* $\mathcal{W}$ *from* (10) *the inclusion* $\overline{\mathcal{P}} \subseteq \mathcal{W}$ *holds.*

Proof. If $(\boldsymbol{\lambda}^0, \mu^0) \notin \mathcal{U}$, then $int\ M_1(\boldsymbol{\lambda}^0, \mu^0) \neq \emptyset$ a hence $dim\ M_1(\boldsymbol{\lambda}^0, \mu^0) = n$. Therefore $(\boldsymbol{\lambda}^0, \mu^0) \notin \overline{\mathcal{P}}$. □

3 Special cases

Let us consider some special cases of the family of convex polyhedral sets $M_1(\boldsymbol{\lambda}, \nu)$ from (5).

3.1 One parameter

Let us consider only one parameter in the description (5) of the convex polyhedral set $M_1(\boldsymbol{\lambda}, \nu)$, without the loss of generality at the first position. Thus

$$
\begin{aligned}
M_1(\lambda) &\equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ (\lambda, \mathbf{r}^T)\mathbf{x} \leq s\},\\
M_2 &\equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \leq \mathbf{d}\},
\end{aligned}
$$

where $\mathbf{A} \in \mathbb{R}^{(m-1)\times n}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$, $\mathbf{b} \in \mathbb{R}^{m-1}$, $\mathbf{d} \in \mathbb{R}^{l}$, $\mathbf{r} \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$ are known values and $\lambda \in \mathbb{R}$ is a parameter. Then sets $\mathcal{W}$ from (12) and $\overline{\mathcal{P}}$ from (9) are described as follows

$$
\begin{aligned}
\mathcal{W} = \{\lambda \in \mathbb{R}\ |\mathbf{x}_i^T \begin{pmatrix} \lambda \\ \mathbf{r} \end{pmatrix} &\geq s\ \forall i \in V',\ \ \mathbf{h}_j^T \begin{pmatrix} \lambda \\ \mathbf{r} \end{pmatrix} \geq 0\ \forall j \in H',\\
\mathbf{g}_k^T \begin{pmatrix} \lambda \\ \mathbf{r} \end{pmatrix} &= 0\ \forall k \in L'\},\\
\overline{\mathcal{P}} = \{\lambda \in \mathbb{R}\ |\mathbf{x}_i^T \begin{pmatrix} \lambda \\ \mathbf{r} \end{pmatrix} &\geq s\ \forall i \in V,\ \ \mathbf{h}_j^T \begin{pmatrix} \lambda \\ \mathbf{r} \end{pmatrix} \geq 0\ \forall j \in H,\\
\mathbf{g}_k^T \begin{pmatrix} \lambda \\ \mathbf{r} \end{pmatrix} &= 0\ \forall k \in L\}.
\end{aligned}
$$

The sets $\mathcal{W}$, $\overline{\mathcal{P}}$ represent real intervals (not necessary bounded), hence the solution set $\mathcal{W} \setminus \overline{\mathcal{P}}$ represents a union of at most two real intervals. Therefore the solution set for this case is not generally convex.

3.2 Linear structure with one parameter

Let us consider one parameter with a fixed direction in this way

$$
\begin{aligned}
M_1(\lambda) &\equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ (\mathbf{r} + \lambda\mathbf{r}')^T\mathbf{x} \leq s + \lambda s'\},\\
M_2 &\equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \leq \mathbf{d}\},
\end{aligned}
$$

where $\mathbf{A} \in \mathbb{R}^{(m-1)\times n}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$, $\mathbf{b} \in \mathbb{R}^{m-1}$, $\mathbf{d} \in \mathbb{R}^{l}$, $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^n$, $s, s' \in \mathbb{R}$ are known values and $\lambda \in \mathbb{R}$ a parameter. Then sets $\mathcal{W}$ from (12) and $\overline{\mathcal{P}}$ from (9) has the following description

$$
\begin{aligned}
\mathcal{W} = \{\lambda \in \mathbb{R} \,|\mathbf{x}_i^T(\mathbf{r}+\lambda\mathbf{r}') \geq s + \lambda s' \ \forall i \in V', \ \ \mathbf{h}_j^T(\mathbf{r}+\lambda\mathbf{r}') \geq 0 \ \forall j \in H', \\
\mathbf{g}_k^T(\mathbf{r}+\lambda\mathbf{r}') = 0 \ \forall k \in L'\}, \\
\overline{\mathcal{P}} = \{\lambda \in \mathbb{R} \,|\mathbf{x}_i^T(\mathbf{r}+\lambda\mathbf{r}') \geq s + \lambda s' \ \forall i \in V, \ \ \mathbf{h}_j^T(\mathbf{r}+\lambda\mathbf{r}') \geq 0 \ \ \forall j \in H, \\
\mathbf{g}_k^T(\mathbf{r}+\lambda\mathbf{r}') = 0 \ \forall k \in L\}.
\end{aligned}
$$

The sets $\mathcal{W}$, $\overline{\mathcal{P}}$ represent real intervals (not necessary bounded), hence the solution set $\mathcal{W} \setminus \overline{\mathcal{P}}$ represents a union of at most two real intervals. Therefore the solution set also for this case is not generally convex.

3.3 A permanent separating hyperplane

Consider $M_1(\boldsymbol{\lambda}, \mu)$ from (5) with the property $(\boldsymbol{\lambda}, \mu) \in \mathcal{Z}$, where $\mathcal{Z} \subset \mathbb{R}^{n+1}$ is a convex polytope. Without the loss of generality assume that $\mathcal{Z} \cap \overline{\mathcal{P}} = \emptyset$ (otherwise we restrict the set to $\mathcal{Z} \setminus \overline{\mathcal{P}}$). The question is whether there exists a separating hyperplane $\mathcal{R}$ such that:

$$M_1(\boldsymbol{\lambda}, \mu) \subseteq \overline{\mathcal{R}^-} \ \forall (\boldsymbol{\lambda}, \mu) \in \mathcal{Z}, \quad M_2 \subseteq \overline{\mathcal{R}^+}.$$

Such a separating hyperplane $\mathcal{R}$ is called *permanent*. We check the existence of a permanent separating hyperplane by the following process: Compute the convex hull $conv\left(\cup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}} M_1(\boldsymbol{\lambda}, \mu)\right)$ and check separability of this convex hull and the set M_2. Proposition 1 says how to compute $\cup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}} M_1(\boldsymbol{\lambda}, \mu)$. This set represents a polyhedral set, but generally not convex – see Example 1.

Assertion 1. *Let $(\mathbf{r}^i, s^i)$, $i \in V$ be all vertices of convex polytope $\mathcal{Z}$. Then*

$$\bigcup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}} M_1(\boldsymbol{\lambda}, \mu) = \bigcup_{i \in V} M_1(\mathbf{r}^i, s^i).$$

Proof. Let $(\boldsymbol{\lambda}_1, \mu_1), (\boldsymbol{\lambda}_2, \mu_2) \in \mathcal{Z}$. It is sufficient to prove that for a convex combination $(\boldsymbol{\lambda}_c, \mu_c) \equiv q(\boldsymbol{\lambda}_1, \mu_1) + (1-q)(\boldsymbol{\lambda}_2, \mu_2)$, $q \in (0,1)$, the inclusion

$$M_1(\boldsymbol{\lambda}_c, \mu_c) \subseteq \left(M_1(\boldsymbol{\lambda}_1, \mu_1) \cup M_1(\boldsymbol{\lambda}_2, \mu_2)\right).$$

holds. To prove this inclusion it is sufficient to prove the following relation

$$\{\mathbf{x} \in \mathbb{R}^n \mid \boldsymbol{\lambda}_c^T \mathbf{x} \le \mu_c\} \subseteq \big(\{\mathbf{x} \in \mathbb{R}^n \mid \boldsymbol{\lambda}_1^T \mathbf{x} \le \mu_1\} \cup \{\mathbf{x} \in \mathbb{R}^n \mid \boldsymbol{\lambda}_2^T \mathbf{x} \le \mu_2\}\big).$$

This relation we prove by contradiction. Suppose that for a certain point $\mathbf{x}^0 \in \mathbb{R}^n$

$$\boldsymbol{\lambda}_1^T \mathbf{x}^0 > \mu_1, \ \boldsymbol{\lambda}_2^T \mathbf{x}^0 > \mu_2, \text{ and } \boldsymbol{\lambda}_c^T \mathbf{x}^0 \le \mu_c$$

hold. If we multiply the first inequality by a number $q > 0$ and the second inequality by a number $1 - q > 0$, we obtain

$$q\boldsymbol{\lambda}_1^T \mathbf{x}^0 + (1-q)\boldsymbol{\lambda}_2^T \mathbf{x}^0 > q\mu_1 + (1-q)\mu,$$

i.e. $\boldsymbol{\lambda}_c^T \mathbf{x}^0 > \mu_c$, which contradicts our assumption. □

Example 1. Given

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -5 & 2 \\ -1 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 28 \end{pmatrix},$$

$$\mathcal{Z} = \{(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^3 \mid \lambda_1 = 1, \ |\lambda_2| \le 1, \ \mu = 2\lambda_2 + 4\}.$$

We compute the set $\cup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}} M_1(\boldsymbol{\lambda}, \mu)$. The convex polytope $\mathcal{Z}$ contains two vertices, $(\mathbf{r}^1, s^1) = (1, 1, 6)$, $(\mathbf{r}^2, s^2) = (1, -1, 2)$. Hence the convex hull $conv\big(\cup_{(\boldsymbol{\lambda},\mu)\in\mathcal{Z}} M_1(\boldsymbol{\lambda}, \mu)\big)$ we obtain as $conv\big(M_1(\mathbf{r}^1, s^1) \cup M_1(\mathbf{r}^2, s^2)\big)$ (for an explicit description of convex hull see [5]), which represents a convex polytope with vertices $(0, 0)$, $(6, 0)$, $(8, 6)$, $(2, 5)$ and is described by the following system of inequalities

$$\begin{pmatrix} 0 & -1 \\ -5 & 2 \\ -1 & 6 \\ 3 & -1 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} 0 \\ 0 \\ 28 \\ 18 \end{pmatrix}.$$

4 Description of separating hyperplanes

Let us introduce

$$\mathcal{Q}^*(\boldsymbol{\lambda}, \nu) \equiv \Big\{(\mathbf{u}, u_m, \mathbf{v}, v_{l+1}) \in \mathbb{R}^{m+l+1} \mid \mathbf{Z}(\boldsymbol{\lambda}, \nu) \begin{pmatrix} \mathbf{u} \\ u_m \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \mathbf{z}, \quad (\mathbf{u}, u_m, \mathbf{v}, v_{l+1}) \ge \mathbf{0}\Big\}, \tag{13}$$

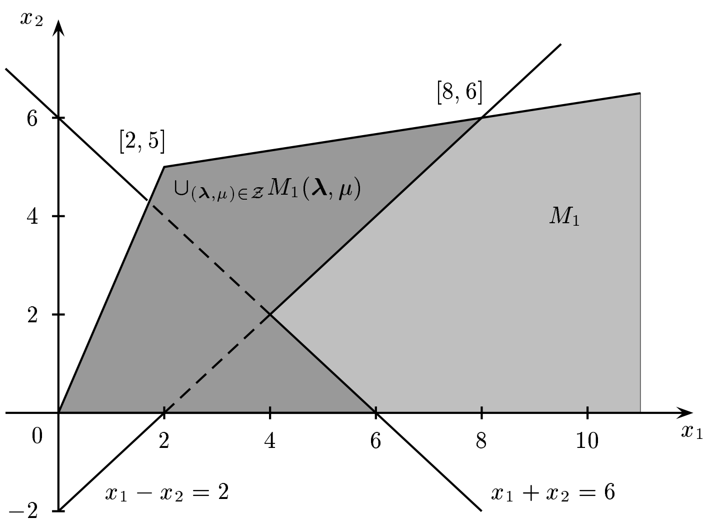

Figure 1: Illustration to Example 1.

where

$$\mathbf{Z}(\boldsymbol{\lambda},\nu) \equiv \begin{pmatrix} \mathbf{A}^T & \boldsymbol{\lambda} & \mathbf{C}^T & \mathbf{0} \\ \mathbf{b}^T & \nu & \mathbf{d}^T & 1 \\ \mathbf{1}^T & 1 & \mathbf{1}^T & 0 \end{pmatrix}, \quad \mathbf{z} \equiv \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix}. \tag{14}$$

From Theorem 2 we obtain the explicit description of all separating hyperplanes of the convex polyheral sets $M_1(\boldsymbol{\lambda},\nu)$, M_2 for $(\boldsymbol{\lambda},\nu)$ from the solution set $\mathcal{W} \setminus \overline{\mathcal{P}}$.

Assertion 2. *Let $(\boldsymbol{\lambda},\nu) \in \mathcal{W} \setminus \overline{\mathcal{P}}$ and $(\mathbf{u}, u_m, \mathbf{v}, v_{l+1}) \in \mathcal{Q}^*(\boldsymbol{\lambda},\nu)$. Suppose that $\mathbf{u}^T\mathbf{A} + u_m\boldsymbol{\lambda}^T \neq \mathbf{0}^T$, and $\eta \in \langle 0, v_{l+1}\rangle$ is aritrary. Then*

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{u}^T\mathbf{A} + u_m\boldsymbol{\lambda}^T)\mathbf{x} - (\mathbf{u}^T\mathbf{b} + u_m\nu) = \eta\} \tag{15}$$

represents a separating hyperplane of the convex polyhedral sets $M_1(\boldsymbol{\lambda},\nu)$, M_2. Conversely, any separating hyperplane $\mathcal{R}$ of the convex polyhedral sets $M_1(\boldsymbol{\lambda},\nu)$, M_2 can be expressed in the form (15) for a certain $(\mathbf{u}, u_m, \mathbf{v}, v_{l+1}) \in \mathcal{Q}^*(\boldsymbol{\lambda},\nu)$, $\mathbf{u}^T\mathbf{A} + u_m\boldsymbol{\lambda}^T \neq \mathbf{0}^T$ and $\eta \in \langle 0, v_{l+1}\rangle$.

5 Stability sets

Stability sets in the situation when are parameters in one row of the constraint matrix from the description of the convex polyhedral set $M_1(\boldsymbol{\lambda}, \nu)$ from (5) will be defined in the same way as stability set in [6, 7]. I.e., on stability sets there will be preserved all feasible bases of $\mathcal{Q}^*(\boldsymbol{\lambda}, \nu)$.

Definition 4. Let $(\mathbf{r}, s) \in \mathcal{W} \setminus \overline{\mathcal{P}}$ be arbitrary and let us denote by $\mathcal{S}$ the system of all feasible bases of the convex polyhedral set $\mathcal{Q}^*(\mathbf{r}, s)$. Then *the stability set*, corresponding to the system $\mathcal{S}$, is the intersection of the solution set $\mathcal{W} \setminus \overline{\mathcal{P}}$ and the closer of the set of all $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ for which $\mathcal{S}$ represents the system of all feasible bases of the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{\lambda}, \nu)$.

Note that we require stability sets to be closed only for the sake of simplicity of their description.

Now we derive a description of stability sets. Let $(\mathbf{r}, s) \in \mathcal{W} \setminus \overline{\mathcal{P}}$ and B arbitrary basis of the convex polyhedral set $\mathcal{Q}^*(\mathbf{r}, s)$. If $m \notin B$, then the basis B remain feasible for all $(\boldsymbol{\lambda}, \nu) \in \mathcal{W} \setminus \overline{\mathcal{P}}$. Thus consider the case, when $m \in B$, i.e. $m = B_k$ for a certain $k \in \{1, \ldots, n+2\}$. Let us introduce $\mathbf{D}(\boldsymbol{\lambda}, \nu) \equiv \mathbf{Z}_B(\boldsymbol{\lambda}, \nu)$, $\mathbf{D} \equiv \mathbf{D}(\mathbf{r}, s)$. The basis B remain feasible for all $(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^{n+1}$ satisfying

$$\mathbf{D}^{-1}(\boldsymbol{\lambda}, \nu)\mathbf{z} \geq \mathbf{0}. \tag{16}$$

Moreover, let us define vectors $\mathbf{p}, \widetilde{\mathbf{q}}, \mathbf{q} \in \mathbb{R}^{n+2}$ in this way

$$\mathbf{p} \equiv \mathbf{e}_k \text{ (unit vector)}, \quad \widetilde{\mathbf{q}} \equiv \begin{pmatrix} \boldsymbol{\lambda} - \mathbf{r} \\ \nu - s \\ 0 \end{pmatrix}, \quad \mathbf{q} \equiv \begin{pmatrix} \boldsymbol{\lambda} \\ \nu \\ 0 \end{pmatrix}. \tag{17}$$

Under the assumption that $1 + \mathbf{p}^T\mathbf{D}^{-1}\widetilde{\mathbf{q}} \neq 0$ we have according to the familiar Sherman–Morrison formula

$$\mathbf{D}^{-1}(\boldsymbol{\lambda}, \nu) = (\mathbf{D} + \widetilde{\mathbf{q}}\mathbf{p}^T)^{-1} = \mathbf{D}^{-1} - \frac{\mathbf{D}^{-1}\widetilde{\mathbf{q}}\mathbf{p}^T\mathbf{D}^{-1}}{1 + \mathbf{p}^T\mathbf{D}^{-1}\widetilde{\mathbf{q}}}.$$

For the choice $\boldsymbol{\lambda} = \mathbf{r}$, $\nu = s$ is the denominator $1 + \mathbf{p}^T\mathbf{D}^{-1}\widetilde{\mathbf{q}} = 1$ positive number, thus we will consider in addition the condition

$$1 + \mathbf{p}^T\mathbf{D}^{-1}\widetilde{\mathbf{q}} > 0. \tag{18}$$

Let us rearrange the expression (16)

$$\begin{aligned}\mathbf{D}^{-1}(\boldsymbol{\lambda},\nu)\mathbf{z} &\geq \mathbf{0},\\ \left(\mathbf{D}^{-1}-\frac{\mathbf{D}^{-1}\widetilde{\mathbf{q}}\mathbf{e}_k^T\mathbf{D}^{-1}}{1+\mathbf{e}_k^T\mathbf{D}^{-1}\widetilde{\mathbf{q}}}\right)\mathbf{e}_{n+2} &\geq \mathbf{0},\\ \mathbf{D}^{-1}_{\cdot,n+2}-\frac{\mathbf{D}^{-1}\widetilde{\mathbf{q}}\mathbf{D}^{-1}_{k,n+2}}{1+\mathbf{D}^{-1}_{k,\cdot}\widetilde{\mathbf{q}}} &\geq \mathbf{0}.\end{aligned}$$

Under the assumption (18) is this inequality equivalent with

$$\mathbf{D}^{-1}_{\cdot,n+2}+(\mathbf{D}^{-1}_{k,\cdot}\widetilde{\mathbf{q}})\mathbf{D}^{-1}_{\cdot,n+2}-\mathbf{D}^{-1}_{k,n+2}(\mathbf{D}^{-1}\widetilde{\mathbf{q}})\geq \mathbf{0}. \tag{19}$$

For the vector $\widetilde{\mathbf{q}}$ the relation $\widetilde{\mathbf{q}}=\mathbf{q}-\mathbf{D}_{\cdot,k}+\mathbf{e}_{n+2}$ holds. Thus the absolute term of the expression (19) is equal to

$$\begin{aligned}&\mathbf{D}^{-1}_{\cdot,n+2}+\left(\mathbf{D}^{-1}_{k,\cdot}(\mathbf{e}_{n+2}-\mathbf{D}_{\cdot,k})\right)\mathbf{D}^{-1}_{\cdot,n+2}-\mathbf{D}^{-1}_{k,n+2}\left(\mathbf{D}^{-1}(\mathbf{e}_{n+2}-\mathbf{D}_{\cdot,k})\right)=\\ &=\mathbf{D}^{-1}_{\cdot,n+2}+\mathbf{D}^{-1}_{k,n+2}\mathbf{D}^{-1}_{\cdot,n+2}-\mathbf{D}^{-1}_{\cdot,n+2}-\mathbf{D}^{-1}_{k,n+2}\mathbf{D}^{-1}_{\cdot,n+2}+\mathbf{D}^{-1}_{k,n+2}\mathbf{e}_k=\\ &=\mathbf{D}^{-1}_{k,n+2}\mathbf{e}_k=\begin{cases}0 & \text{for any row } \neq k,\\ \mathbf{D}^{-1}_{k,n+2} & \text{for the row } k.\end{cases}\end{aligned}$$

The remaining terms of the expression (19) are the following

$$(\mathbf{D}^{-1}_{k,\cdot}\mathbf{q})\mathbf{D}^{-1}_{\cdot,n+2}-\mathbf{D}^{-1}_{k,n+2}.\mathbf{D}^{-1}\mathbf{q},$$

especially for k-th row we have

$$(\mathbf{D}^{-1}_{k,\cdot}\mathbf{q})\mathbf{D}^{-1}_{k,n+2}-\mathbf{D}^{-1}_{k,n+2}.\mathbf{D}^{-1}_{k,\cdot}\mathbf{q}=0.$$

Since the basis B is feasible for the convex polyhedral set $\mathcal{Q}^*(\mathbf{r},s)$, it $\mathbf{D}^{-1}_{k,n+2}\geq 0$ holds. Therefore k-th inequality of (19) is redundant and the resulting system of inequalities has the description (without k-th inequality)

$$(\mathbf{D}^{-1}_{k,\cdot}\mathbf{q})\mathbf{D}^{-1}_{\cdot,n+2}-\mathbf{D}^{-1}_{k,n+2}.\mathbf{D}^{-1}\mathbf{q}\geq \mathbf{0}. \tag{20}$$

Let us investigate the expression (18). It is equivalent to

$$\begin{aligned}1+\mathbf{e}_k^T\mathbf{D}^{-1}(\mathbf{q}+\mathbf{e}_{n+2}-\mathbf{D}_{\cdot,k}) &> 0,\\ 1+\mathbf{D}^{-1}_{k,\cdot}(\mathbf{q}+\mathbf{e}_{n+2}-\mathbf{D}_{\cdot,k}) &> 0,\\ \mathbf{D}^{-1}_{k,\cdot}\mathbf{q}+\mathbf{D}^{-1}_{k,n+2} &> 0.\end{aligned}$$

Let us multiply the system (20) by the vector $\mathbf{D}_{n+2,\cdot} \geq \mathbf{0}$. We obtain

$$\begin{aligned} \mathbf{D}^{-1}_{k,\cdot}\mathbf{q} - \mathbf{D}^{-1}_{k,n+2}.\mathbf{e}^T_{n+2}\mathbf{q} &\geq 0, \\ \mathbf{D}^{-1}_{k,\cdot}\mathbf{q} &\geq 0. \end{aligned}$$

Hence the condition (18) is redundant (stability set is closed from its definition). The following remark summarizes the description of stability sets.

Remark 2. (The description of stability sets) Let $(\mathbf{r}, s) \in \mathcal{W} \setminus \overline{\mathcal{P}}$ and B a basis of the convex polyhedral set $\mathcal{Q}^*(\mathbf{r}, s)$ such that $B_k = m$ for a certain $k \in \{1, \ldots, n+2\}$. Then the system of inequalities corresponding to the basis B has the description (without k-th inequality, which is redundant)

$$(\mathbf{D}^{-1}_{\cdot,n+2}\mathbf{D}^{-1}_{k,\cdot})\mathbf{q} - \mathbf{D}^{-1}_{k,n+2}.\mathbf{D}^{-1}\mathbf{q} \geq \mathbf{0}. \tag{21}$$

The corresponding stability set is described by a union of all these systems of inequalities (21) for all feasible bases of $\mathcal{Q}^*(\mathbf{r}, s)$ containing the element m. To be the stability set a subset of the solution set, we must to the descritpion of the stability set add constraints from $\mathcal{W} \setminus \overline{\mathcal{P}}$ as well.

Note that there is always a finite number of stability sets.

Example 2. Given convex polyhedral sets

$$M_1 = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

$$M_2 = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} \right\}$$

and the family of convex polyhedral sets

$$M_1(\boldsymbol{\lambda}, \nu) = \{\mathbf{x} \in M_1 \mid \boldsymbol{\lambda}^T\mathbf{x} \leq \nu\}, \quad (\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3.$$

We will compute the solution set and all stability sets. Since $int\, M_1 \cap int\, M_2 \neq \emptyset$, we use the following proceeding.

The convex polyhedral set M_1 contains only one vertex $(0,0)^T$ and extremal directions of M_1 are $(-1,0)^T$ and $(0,-1)^T$. Lineality space $\mathcal{L} = \{\mathbf{0}\}$. Hence $\overline{\mathcal{P}}$ from (9) has the description

$$\overline{\mathcal{P}} = \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \mid 0 \geq \nu,\ -\lambda_1 \geq 0,\ -\lambda_2 \geq 0\}.$$

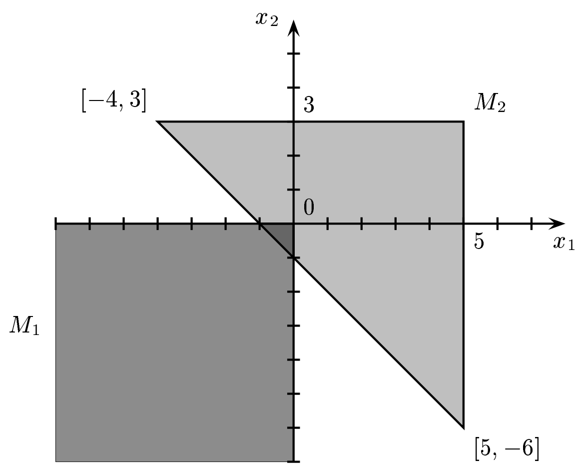

Figure 2: Illustration to Example 2.

The convex polyhedral set $M_1 \cap M_2$ contains three vertices $(0,0)^T$, $(-1,0)^T$, $(0,-1)^T$, but no unbounded edge. Lineality space $\mathcal{L}_{12} = \{\mathbf{0}\}$. Hence the set $\mathcal{W}$ from (12) is described as follows

$$\mathcal{W} = \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \mid 0 \geq \nu,\ -\lambda_1 \geq \nu,\ -\lambda_2 \geq \nu\}.$$

The solution set has according to (11) the description

$$\begin{aligned}
\mathcal{W} \setminus \overline{\mathcal{P}} =& \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \mid 0 \geq \nu,\ -\lambda_1 \geq \nu,\ -\lambda_2 \geq \nu\} \setminus \\
& \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \mid 0 \geq \nu,\ -\lambda_1 \geq 0,\ -\lambda_2 \geq 0\} \\
=& \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \mid 0 \geq \nu,\ -\lambda_1 \geq \nu,\ -\lambda_2 \geq \nu,\ \lambda_1 > 0\} \cup \\
& \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \mid 0 \geq \nu,\ -\lambda_1 \geq \nu,\ -\lambda_2 \geq \nu,\ \lambda_2 > 0\} \\
=& \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \mid -\nu \geq \lambda_1 > 0,\ -\nu \geq \lambda_2\} \cup \\
& \{(\boldsymbol{\lambda}, \nu) \in \mathbb{R}^3 \mid -\nu \geq \lambda_1,\ -\nu \geq \lambda_2 > 0\}.
\end{aligned}$$

Now we compute stability sets according to Remark 2.

1. Choose $(\lambda_1^1, \lambda_2^1, \nu^1) \in \mathcal{W} \setminus \overline{\mathcal{P}}$ for instance $(\lambda_1^1, \lambda_2^1, \nu^1) = (1, 2, -4)$. All feasible bases of the convex polyhedral set $\mathcal{Q}^*(\lambda_1^1, \lambda_2^1, \nu^1)$ that contains the index $m = 3$ are the following:

The basis $(1,2,3,6)$, the corresponding system of inequalities:

$$-\lambda_1 - \nu \geq 0,\ -\lambda_2 - \nu \geq 0,\ -\nu \geq 0.$$

The basis $(1,3,4,6)$, the corresponding system of inequalities:

$$-\lambda_2 - \nu \geq 0,\ \lambda_2 \geq 0,\ -5\lambda_1 + 6\lambda_2 + \nu \geq 0.$$

The basis $(1,3,5,6)$, the corresponding system of inequalities:

$$-4\lambda_1 + 3\lambda_2 - \nu \geq 0,\ -\lambda_2 - \nu \geq 0,\ 3\lambda_2 - \nu \geq 0.$$

The basis $(1,3,6,7)$, the corresponding system of inequalities:

$$-\lambda_1 + \lambda_2 \geq 0,\ -\lambda_2 - \nu \geq 0,\ \lambda_2 \geq 0.$$

The stability set for $(\lambda_1^1, \lambda_2^1, \nu^1)$ is described by the final system of inequalities

$$-\lambda_2 - \nu \geq 0,\ \lambda_2 > 0,\ -5\lambda_1 + 6\lambda_2 + \nu \geq 0.$$

2. Choose $(\lambda_1^2, \lambda_2^2, \nu^2) \in \mathcal{W} \setminus \overline{\mathcal{P}}$, but not from the first stability set. For instance $(\lambda_1^2, \lambda_2^2, \nu^2) = (1, 2, -9)$. The stability set for $(\lambda_1^2, \lambda_2^2, \nu^2)$ has the description

$$-\lambda_1 + \lambda_2 \geq 0,\ \lambda_2 > 0,\ 5\lambda_1 - 6\lambda_2 - \nu \geq 0.$$

3. Choose $(\lambda_1^3, \lambda_2^3, \nu^3) \in \mathcal{W} \setminus \overline{\mathcal{P}}$, but not from the first or second stability set. For instance $(\lambda_1^3, \lambda_2^3, \nu^3) = (1, -1, -9)$. The stability set for $(\lambda_1^3, \lambda_2^3, \nu^3)$ has the description

$$\lambda_1 - \lambda_2 \geq 0,\ \lambda_1 > 0,\ -4\lambda_1 + 3\lambda_2 - \nu \geq 0.$$

4. Choose $(\lambda_1^4, \lambda_2^4, \nu^4) \in \mathcal{W} \setminus \overline{\mathcal{P}}$, but not from the previous stability sets. For instance $(\lambda_1^4, \lambda_2^4, \nu^4) = (1, -1, -2)$. The stability set for $(\lambda_1^4, \lambda_2^4, \nu^4)$ has the description

$$-\lambda_1 - \nu \geq 0,\ \lambda_1 > 0,\ 4\lambda_1 - 3\lambda_2 + \nu \geq 0.$$

The union of the above stability sets forms the whole solution set.

Tables 1 – 2 contain another examples. For the pseudorandomly generated input matrices $\mathbf{A}$, $\mathbf{C}$, and vectors $\mathbf{b}$, $\mathbf{d}$ the tables involve the corresponding number of stability sets and the computing time. Our source code was written in MATLAB 6.5. The results were carried out on `PC (x86), Pentium 4, 2.6 GHz, 512 MB RAM, Gentoo Linux`.

Table 1: Examples in $\mathbb{R}^2$.

matrix $\mathbf{A}$	vector $\mathbf{b}$	matrix $\mathbf{C}$	vector $\mathbf{d}$	number of stability sets	computing time
$\begin{pmatrix} -2 & 5 \\ -2 & 8 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ -8 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -7 & -1 \\ 4 & 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix}$	19	10 s
$\begin{pmatrix} 0 & 5 \\ 8 & -4 \\ 9 & -3 \\ 7 & -4 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 3 \\ -3 \\ -6 \end{pmatrix}$	$\begin{pmatrix} 3 & 7 \\ -3 & -3 \\ 8 & -3 \\ -9 & 1 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 1 \\ 3 \\ 7 \end{pmatrix}$	33	37 s
$\begin{pmatrix} 6 & 1 \\ -1 & -1 \\ 9 & -7 \\ 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 7 \\ 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 8 & -9 \\ 9 & 4 \\ -7 & -6 \\ -5 & 6 \\ 7 & -1 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 4 \\ 6 \\ 6 \\ -1 \end{pmatrix}$	65	2 min 39 s
$\begin{pmatrix} 4 & -2 \\ 7 & -7 \\ 1 & -5 \\ 3 & -9 \\ 0 & 6 \\ -7 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 6 \\ 14 \\ 10 \\ 2 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ -3 & -2 \\ -4 & 0 \\ -6 & 8 \\ -1 & -2 \\ -7 & 1 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 7 \\ -1 \\ 7 \\ 2 \\ 14 \end{pmatrix}$	93	17 min 49 s

6 Separating supporting hyperplanes

In this section we derive the description of the set of all parameters $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$ for which there exists a separating hyperplane of the convex polyhedral sets $M_1(\boldsymbol{\lambda}, \mu)$, M_2 supporting faces of $M_1(\boldsymbol{\lambda}, \mu)$, M_2 determined by given sub-basis of $M_1(\boldsymbol{\lambda}, \mu)$, M_2.

Let us introduce $\mathcal{S}^1_{B^1}$ as a set of $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$ for which the sub-basis B^1 of the convex polyhedral set $M_1(\boldsymbol{\lambda}, \mu)$ is feasible. The description of the set $\mathcal{S}^1_{B^1}$ follows from Lemma 1 and 2.

Lemma 1. *Let B^1 be a sub-basis of the convex polyhedral set $M_1(\boldsymbol{\lambda}, \mu)$ and suppose $m \notin B^1$. Denote $N^1 \equiv \{1, \ldots, m-1\} \setminus B^1$. Let $\mathbf{g}^k$, $k \in L$ be a basis of the lineality space $\mathcal{L} \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$. Denote by $\mathbf{x}^i$, $i \in V$, all vertices and $\mathbf{h}^j$, $j \in H$, all extremal directions of the convex polyhedral set*

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}_{B^1}\mathbf{x} = \mathbf{b}_{B^1},\ \mathbf{A}_{N^1}\mathbf{x} \leq \mathbf{b}_{N^1}\} \cap \mathcal{L}^{\perp}.$$

Table 2: Examples in $\mathbb{R}^3$, $\mathbb{R}^4$.

matrix $\mathbf{A}$	vector $\mathbf{b}$	matrix $\mathbf{C}$	vector $\mathbf{d}$	number of stability sets	computing time
$\begin{pmatrix} -3 & 5 & 1 \\ 5 & 1 & 2 \\ -6 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 13 \\ 13 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -7 & 6 & -7 \\ -6 & -8 & -5 \\ 7 & -5 & 8 \end{pmatrix}$	$\begin{pmatrix} 14 \\ 12 \\ 14 \end{pmatrix}$	37	18 s
$\begin{pmatrix} 1 & -9 & -4 \\ -8 & 0 & -6 \\ 7 & 3 & -9 \\ -6 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 10 \\ -1 \\ 4 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 2 & 7 & -6 \\ 0 & 4 & -4 \\ 2 & 4 & 8 \\ -5 & -8 & 7 \end{pmatrix}$	$\begin{pmatrix} 12 \\ 13 \\ 5 \\ 3 \end{pmatrix}$	263	3 min 28 s
$\begin{pmatrix} 4 & 6 & -3 \\ -4 & 4 & -4 \\ 9 & -3 & -5 \\ -7 & -9 & -7 \end{pmatrix}$	$\begin{pmatrix} 7 \\ 0 \\ 3 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 7 & -3 & -6 \\ 6 & -7 & 1 \\ -5 & 2 & -1 \\ 1 & 5 & 3 \\ -1 & -5 & -9 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 11 \\ 0 \\ -3 \\ 5 \end{pmatrix}$	569	22 min 25 s
$\begin{pmatrix} -4 & 5 & -1 & 0 \\ 7 & 7 & -7 & -1 \\ -3 & -8 & 3 & -3 \\ 2 & 4 & -7 & 6 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 4 \\ 4 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 7 & -2 & 7 & -6 \\ 3 & -1 & -5 & 4 \\ -7 & 6 & 6 & -1 \\ 0 & -8 & -5 & 9 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 9 \\ 10 \\ 4 \end{pmatrix}$	882	28 min 13 s

Then we have

$$\mathcal{S}^1_{B^1} = \mathbb{R}^{n+1} \setminus \mathcal{P}_{B^1},$$

where $\mathcal{P}_{B^1}$ *forms a convex cone with description*

$$\mathcal{P}_{B^1} = \{(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1} \mid \boldsymbol{\lambda}^T \mathbf{x}^i > \mu \ \forall i \in V, \quad \boldsymbol{\lambda}^T \mathbf{h}^j \geq 0 \ \forall j \in H, \\ \boldsymbol{\lambda}^T \mathbf{g}^k = 0 \ \forall k \in L\}.$$

Proof. The sub-basis B^1 of the convex polyhedral set $M_1(\boldsymbol{\lambda}, \mu)$ is feasible for all $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$, such that the set $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}_{B^1}\mathbf{x} = \mathbf{b}_{B^1},\ \mathbf{A}_{N^1}\mathbf{x} \leq \mathbf{b}_{N^1},\ \boldsymbol{\lambda}^T\mathbf{x} \leq \mu\}$ is nonempty. Now we can just apply the statement of Theorem 4. □

Lemma 2. *Let B^1 be a sub-basis of the convex polyhedral set $M_1(\boldsymbol{\lambda}, \mu)$ and suppose $m \notin B^1$. Denote $N^1 \equiv \{1, \ldots, m-1\} \setminus B^1$. Let $\mathbf{g}^k$, $k \in L$ be a basis of the lineality space $\mathcal{L} \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$. Denote by $\mathbf{x}^i$, $i \in V$, all vertices and $\mathbf{h}^j$, $j \in H$, all extremal diractions of the convex polyhedral set*

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}_{B^1}\mathbf{x} = \mathbf{b}_{B^1},\ \mathbf{A}_{N^1}\mathbf{x} \leq \mathbf{b}_{N^1}\} \cap \mathcal{L}^{\perp}.$$

Then we have

$$\mathcal{S}^1_{B^1 \cup \{m\}} = \mathbb{R}^{n+1} \setminus (\mathcal{P}_{B^1} \cup -\mathcal{P}_{B^1}),$$

where $\mathcal{P}_{B^1}$ *is the convex cone from Lemma 1.*

Proof. The sub-basis $B^1 \cup \{m\}$ of the convex polyhedral set $M_1(\boldsymbol{\lambda}, \mu)$ is feasible for all $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$, for which the set $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}_{B^1}\mathbf{x} = \mathbf{b}_{B^1},\ \boldsymbol{\lambda}^T\mathbf{x} = \mu,\ \mathbf{A}_{N^1}\mathbf{x} \le \mathbf{b}_{N^1}\}$ is not empty. In other words, two set $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}_{B^1}\mathbf{x} = \mathbf{b}_{B^1},\ \boldsymbol{\lambda}^T\mathbf{x} \le \mu,\ \mathbf{A}_{N^1}\mathbf{x} \le \mathbf{b}_{N^1}\}$ and $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}_{B^1}\mathbf{x} = \mathbf{b}_{B^1},\ -\boldsymbol{\lambda}^T\mathbf{x} \le -\mu,\ \mathbf{A}_{N^1}\mathbf{x} \le \mathbf{b}_{N^1}\}$ must be simultaneously nonempty. From Lemma 1 we obtain the description of the set $\mathcal{S}^1_{B^1\cup\{m\}}$ as $\left(\mathbb{R}^{n+1} \setminus \mathcal{P}_{B^1}\right) \cap \left(\mathbb{R}^{n+1} \setminus -\mathcal{P}_{B^1}\right) = \mathbb{R}^{n+1} \setminus \left(\mathcal{P}_{B^1} \cup -\mathcal{P}_{B^1}\right).$ ☐

Assertion 3. *Let us consider the family of convex polyhedral sets*

$$\mathcal{M}(\boldsymbol{\xi}) \equiv \{(\mathbf{x}, x_n) \in \mathbb{R}^n \mid \mathbf{M}\mathbf{x} + \boldsymbol{\xi} x_n = \mathbf{v},\ \mathbf{x} \ge \mathbf{0},\ x_n \ge 0\},$$

where $\mathbf{M} \in \mathbb{R}^{m\times(n-1)}$, $\mathbf{v} \in \mathbb{R}^m$ *are fixed and* $\boldsymbol{\xi}$ *is* m*-elemental vector of parameters. Denote by* $\mathbf{h}_k,\ k \in L$ *any basis of the lineality space* $\mathcal{L} \equiv \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{M}^T\mathbf{y} = \mathbf{0},\ \mathbf{v}^T\mathbf{y} = \mathbf{0}\}$. *For the convex polyhedral cone*

$$\{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{M}^T\mathbf{y} \le \mathbf{0},\ \mathbf{v}^T\mathbf{y} \ge \mathbf{0}\} \cap \mathcal{L}^\perp$$

we denote $\mathbf{g}_i,\ i \in I_1$, *its extremal directions with the property* $\mathbf{g}_i^T\mathbf{v} > 0$ *and by* $\mathbf{h}_j,\ j \in I_2$, *its extremal directions with the property* $\mathbf{h}_j^T\mathbf{v} = 0$. *Then the set* $\mathcal{S}^{\mathcal{M}}$ *of all* $\boldsymbol{\xi} \in \mathbb{R}^m$ *for which the set* $\mathcal{M}(\boldsymbol{\xi})$ *is nonempty, is described as follows: If* $I_1 = \emptyset$, *then* $\mathcal{S}^{\mathcal{M}} = \mathbb{R}^m$. *Otherwise*

$$\mathcal{S}^{\mathcal{M}} = \{\boldsymbol{\xi} \in \mathbb{R}^m \mid \mathbf{g}_i^T\boldsymbol{\xi} > 0\ \forall i \in I_1,\ \ \mathbf{h}_j^T\boldsymbol{\xi} \ge 0\ \forall j \in I_2,\ \ \mathbf{h}_k^T\boldsymbol{\xi} = 0\ \forall k \in L\}. \tag{22}$$

Proof. $\mathcal{S}^{\mathcal{M}}$ is the set of all $\boldsymbol{\xi} \in \mathbb{R}^m$ for which $\mathcal{M}(\boldsymbol{\xi}) \ne \emptyset$, i.e. the problem

$$\min\{\mathbf{0}^T\mathbf{x} + 0x_n \mid \mathbf{M}\mathbf{x} + \boldsymbol{\xi}x_n = \mathbf{v},\ \mathbf{x} \ge \mathbf{0},\ x_n \ge 0\}$$

has an optimal solution. From the duality in linear programming this is true if and only iff the problem

$$\max\{\mathbf{v}^T\mathbf{y} \mid \mathbf{M}^T\mathbf{y} \le \mathbf{0},\ \boldsymbol{\xi}^T\mathbf{y} \le 0\} \tag{23}$$

has an optimal solution. Since the set of feasible solutions to the problem (23) represents a convex polyhedral cone (with one vertex in the origin), we can this situation formulate as

$$\{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{M}^T\mathbf{y} \le \mathbf{0},\ \boldsymbol{\xi}^T\mathbf{y} \le 0,\ \mathbf{v}^T\mathbf{y} > 0\} = \emptyset. \tag{24}$$

If $I_1 = \emptyset$, then obviously $\mathcal{S}^{\mathcal{M}} = \mathbb{R}^m$. Suppose that $I_1 \neq \emptyset$.

Suppose that $\boldsymbol{\xi}^0 \in \mathbb{R}^m$ satisfies $\mathbf{g}_i^T \boldsymbol{\xi}^0 > 0 \ \forall i \in I_1$, $\mathbf{h}_j^T \boldsymbol{\xi}^0 \geq 0 \ \forall j \in I_2$ and $\mathbf{h}_k^T \boldsymbol{\xi}^0 = 0 \ \forall k \in L$. Each point $\mathbf{y} \in \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{M}^T \mathbf{y} \leq \mathbf{0},\ \mathbf{v}^T \mathbf{y} > 0\}$ can be written as a linear combination

$$\mathbf{y} = \sum_{i \in I_1} \alpha_i \mathbf{g}_i + \sum_{j \in I_2} \beta_j \mathbf{h}_j + \sum_{k \in L} \gamma_k \mathbf{h}_k$$

for certain $\alpha_i, \beta_j \geq 0$, $\sum_{i \in I_1} \alpha_i > 0$ and $\gamma_k \in \mathbb{R}$. Then

$$\mathbf{y}^T \boldsymbol{\xi}^0 = \sum_{i \in I_1} \alpha_i \mathbf{g}_i^T \boldsymbol{\xi}^0 + \sum_{j \in I_2} \beta_j \mathbf{h}_j^T \boldsymbol{\xi}^0 + \sum_{k \in L} \gamma_k \mathbf{h}_k^T \boldsymbol{\xi}^0 > 0.$$

Therefore the condition (24) holds.

Conversely, suppose $\boldsymbol{\xi}^0 \in \mathbb{R}^m$ and (24) holds. Then for all $\mathbf{y} \in \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{M}^T \mathbf{y} \leq \mathbf{0},\ \mathbf{v}^T \mathbf{y} > 0\}$ we have $\mathbf{y}^T \boldsymbol{\xi}^0 > 0$. Specially, $\mathbf{g}_i^T \boldsymbol{\xi}^0 > 0 \ \forall i \in I_1$. For infinitesimal $\varepsilon > 0$ also $(1-\varepsilon)\mathbf{h}_j^T \boldsymbol{\xi}^0 + \frac{\varepsilon}{|I_1|} \sum_{i \in I_1} \mathbf{g}_i^T \boldsymbol{\xi}^0 > 0 \ \forall j \in I_2$. Hence $(1-\varepsilon)\mathbf{h}_j^T \boldsymbol{\xi}^0 \geq 0$ and therefore $\mathbf{h}_j^T \boldsymbol{\xi}^0 \geq 0 \ \forall j \in I_2$. Analogically we can prove $\mathbf{h}_k^T \boldsymbol{\xi}^0 = 0 \ \forall k \in L$. Hence $\boldsymbol{\xi}^0$ belongs to the set from (22). □

Remark 3. (Description of the set in question) Denote as $\mathcal{S}_{B^1 B^2}$ the set of $(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^{n+1}$ for which there exists a separating hyperplane of the convex poluhedral sets $M_1(\boldsymbol{\lambda}, \mu)$, M_2, which supports the faces of $M_1(\boldsymbol{\lambda}, \mu)$, M_2 determined by sub-bases B^1 and B^2. We derive the description of the set $\mathcal{S}_{B^1 B^2}$. Let us consider the system

$$\mathbf{Z}(\boldsymbol{\lambda}, \mu)_B \mathbf{w} = \mathbf{z}, \quad \mathbf{w} \geq \mathbf{0}, \tag{25}$$

where B is a sub-basis of the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{\lambda}, \mu)$ from (13). It can occur one of the following possibilities.

1. Let $m \notin B^1$. If for $B \equiv B^1 \cup \left(B^2 + m\right)$ the system (25) has no solution, then $\mathcal{S}_{B^1 B^2} = \emptyset$. Otherwise, we have

$$\mathcal{S}_{B^1 B^2} = \mathcal{S}_{B^1},$$

where $\mathcal{S}_{B^1}$ is described according to Lemma 1.

2. Let $m \in B^1$, i.e. $B^1 = B_r^1 \cup \{m\}$ for a certain sub-basis B_r^1. Then

$$\mathcal{S}_{B^1 B^2} = \mathcal{S}_{B^1} \cap \mathcal{S}_{B^1 B^2}^{\mathcal{Q}}.$$

The sense of the set $\mathcal{S}_{B^1}$ is to preserve feasibility of the sub-basis B^1 for the convex polyhedral set $M_1(\boldsymbol{\lambda}, \mu)$. The description of $\mathcal{S}_{B^1}$ we obtain from

Lemma 2. The sense of the set $\mathcal{S}^{\mathcal{Q}}_{B^1B^2}$ is to ensure the existence of separating supporting hyperplane of $M_1(\boldsymbol{\lambda},\mu)$, M_2 (supproting the given faces). We define $\mathcal{S}^{\mathcal{Q}}_{B^1B^2}$ as the set of $(\boldsymbol{\lambda},\mu) \in \mathbb{R}^{n+1}$, for which the system (25) with $B \equiv B^1 \cup \big(B^2 + m\big)$ is solvable.

The description of the set $\mathcal{S}^{\mathcal{Q}}_{B^1B^2}$ follows from Proposition 3, when we for $B_r \equiv B^1_r \cup \big(B^2 + m\big)$ assign

$$\mathbf{M} = \mathbf{Z}(\boldsymbol{\lambda},\mu)_{B_r}, \quad \mathbf{v} = \mathbf{z}, \quad \boldsymbol{\xi} = (\boldsymbol{\lambda}^T,\mu,1)^T.$$

Example 3. Given convex polyhedral sets

$$M_1 = \left\{\mathbf{x} \in \mathbb{R}^2 \mid \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} \le \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}\right\},$$

$$M_2 = \left\{\mathbf{x} \in \mathbb{R}^2 \mid \begin{pmatrix} 0 & 1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}\mathbf{x} \le \begin{pmatrix} 2 \\ -8 \\ -8 \end{pmatrix}\right\},$$

family of convex polyhedral sets

$$M_1(\boldsymbol{\lambda},\mu) = \{\mathbf{x} \in M_1 \mid \boldsymbol{\lambda}^T\mathbf{x} \le \mu\}, \quad (\boldsymbol{\lambda},\mu) \in \mathbb{R}^3$$

and bases $B^1 = (2,4)$, $B^2 = (1,2)$ of convex polyhedral sets $M_1(\boldsymbol{\lambda},\mu)$, M_2, respectively. We will calculate the description of the set $\mathcal{S}_{B^1B^2}$ from Remark 3. Since $4 \in B^1$, we will proceed along the second paragraph of Remark 3.

First, we will deal with the set $\mathcal{S}_{B^1}$. From Lemma 2 we have $\mathcal{S}_{B^1} = \mathcal{S}_{(2,4)} = \mathbb{R}^3 \setminus \big(\mathcal{P}_{(2)} \cup -\mathcal{P}_{(2)}\big)$. The convex polyhedral set

$$\{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{A}_{B^1}\mathbf{x} = \mathbf{b}_{B^1},\ \mathbf{A}_{N^1}\mathbf{x} \le \mathbf{b}_{N^1}\} =$$
$$\{\mathbf{x} \in \mathbb{R}^2 \mid x_2 = 5,\ -x_1 \le 0,\ x_1 - 2x_2 \le 0\}$$

contains two vertices $\mathbf{x}^1 = (0,5)^T$, $\mathbf{x}^2 = (10,5)^T$. Hence we obtain

$$\begin{aligned}
\mathcal{P}_{(2)} &= \{(\boldsymbol{\lambda},\mu) \in \mathbb{R}^3 \mid 5\lambda_2 > \mu,\ 10\lambda_1 + 5\lambda_2 > \mu\},\\
\mathcal{S}_{(2,4)} &= \{(\boldsymbol{\lambda},\mu) \in \mathbb{R}^3 \mid 5\lambda_2 \ge \mu,\ 10\lambda_1 + 5\lambda_2 \le \mu\} \cup\\
&\quad\ \{(\boldsymbol{\lambda},\mu) \in \mathbb{R}^3 \mid 5\lambda_2 \le \mu,\ 10\lambda_1 + 5\lambda_2 \ge \mu\}.
\end{aligned}$$

The description of the set $\mathcal{S}^{\mathcal{Q}}_{B^1B^2}$ follows from the guide of Remark 3. The convex polyhedral cone described as

$$\begin{pmatrix} 0 & 1 & 5 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & -1 & -8 & 1 \end{pmatrix}\mathbf{y} \le \mathbf{0}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}\mathbf{y} \ge 0$$

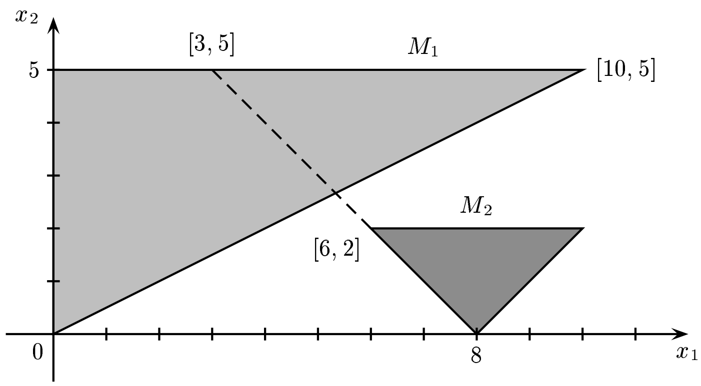

Figure 3: Illustration to Example 3.

has edges in directions of vector $\mathbf{g}_1 = (2, -1, 0, 1)^T$, $\mathbf{h}_1 = (6, 2, -1, 0)^T$, $\mathbf{h}_2 = (-3, -5, 1, 0)^T$, $\mathbf{h}_3 = (1, 0, 0, 0)^T$. Hence

$$\begin{aligned} \mathcal{S}^{\mathcal{Q}}_{B^1B^2} &= \{(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^3 \mid 2\lambda_1 - \lambda_2 + 1 > 0,\ 6\lambda_1 + 2\lambda_2 - \mu \geq 0, \\ &\qquad -3\lambda_1 - 5\lambda_2 + \mu \geq 0,\ \lambda_1 \geq 0\} \\ &= \{(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^3 \mid 6\lambda_1 + 2\lambda_2 - \mu \geq 0,\ -3\lambda_1 - 5\lambda_2 + \mu \geq 0,\ \lambda_1 \geq 0\}. \end{aligned}$$

The set in question has the desription

$$\begin{aligned} \mathcal{S}_{B^1B^2} &= \mathcal{S}_{B^1} \cap \mathcal{S}^{\mathcal{Q}}_{B^1B^2} \\ &= \{(\boldsymbol{\lambda}, \mu) \in \mathbb{R}^3 \mid \quad 10\lambda_1 + 5\lambda_2 - \mu \geq 0,\ 6\lambda_1 + 2\lambda_2 - \mu \geq 0, \\ &\qquad -3\lambda_1 - 5\lambda_2 + \mu \geq 0\}. \end{aligned}$$

7 Conclusion

In this article, we were concerned with the separation properties of two convex polyhedral sets $\mathcal{M}_1$, $\mathcal{M}_2$, which depended on parameters. Parameters were situated in one row of the constraint matrix from the description of the one of these convex polyhedral sets. The situation, when there are parameters in the right-hand side of inequalities was dealt with in [6] and the situation, when there are parameters in one column of the constraint matrix was dealt with in [7]. We defined so called solution set (a set of parameters for which $\mathcal{M}_1$, $\mathcal{M}_2$ are strongly separable) and stability sets (a set of parameters for which separability of $\mathcal{M}_1$, $\mathcal{M}_2$ has the same characteristics). On stability sets, there could be applied various kinds of postoptimality analyses (parametric analysis, sensitivity analysis or tolerance analysis – see e.g. [2]), but it was not the subject of this paper. We produced also a lot of examples, which were carried out on a computer.

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