Asymptotic Distribution of the Sum of the Lengths of Ascents or of Descents in Permutations

Lane Clark

Department of Mathematics
Southern Illinois University at Carbondale
Carbondale, IL 62901-4408
lclark@math.siu.edu

Abstract

Let $A_n(\sigma)$ denote the sum of the lengths of ascents of a permutation σ of $\{1, \ldots, n\}$ chosen randomly according to a uniform distribution. We find the exact expectation and variance of A_n and then show that the A_n are asymptotically normally distributed. An identical result holds for the sum of the lengths of descents of a permutation.

Key Words: Ascents, Descents, Asymptotic Normality

A permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$ of $[n] := \{1, \ldots, n\}$ has an **ascent** at $i \in [n-1]$ if and only if $\sigma(i+1) > \sigma(i)$ with **length of ascent** of $\sigma(i+1) - \sigma(i)$ at ascent i. Let $A_n(\sigma)$ denote the sum of the lengths of ascents of σ. For example, $\sigma = (4, 1, 7, 10, 6, 3, 8, 2, 9, 5)$ has ascents of length $6, 3, 5, 7$ at $2, 3, 6, 8$, respectively. Hence, $A_{10}(\sigma) = 6 + 3 + 5 + 7 = 21$.

Let $\mathfrak{S}_n$ denote the set of permutations of $[n]$. We choose each permutation $\sigma \in \mathfrak{S}_n$ randomly according to a uniform distribution so that each σ has probability $\Pr(\sigma) = 1/n!$. As usual, a sequence of random variables S_n on $\mathfrak{S}_n$ is said to be **asymptotically normally distributed** if and only if the sequence of distribution functions of $(S_n - E(S_n))/\sqrt{\mathrm{Var}(S_n)}$ converges weakly to the distribution function of a normal random variable with mean

0 and variance 1, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr\left(\frac{S_n - E(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, dt,$$

where $E(S_n)$ and $\mathrm{Var}(S_n)$ are the expectation and variance of S_n.

For a positive integer m and $\sigma \in \mathfrak{S}_n$, let $S_{n,m}(\sigma)$ denote the number of ascents of σ of length at least m. In [2] it was shown that the $S_{n,1}$ are asymptotically normally distributed on $\mathfrak{S}_n$, and in [3], it was shown that the $S_{n,m}$ are also. Recently, Balcza [1] found the exact expectation and variance for the sum of the lengths of inversions on $\mathfrak{S}_n$. In this paper, we find the exact expectation and variance of A_n and then show that the A_n are asymptotically normally distributed on $\mathfrak{S}_n$. Identical results hold for the sum of the lengths of descents of a permutation.

We denote the nonnegative integers by $\mathbb{N}$; the positive integers by $\mathbb{Z}^+$; the rational numbers by $\mathbb{Q}$; and the real numbers by $\mathbb{R}$. For $k \in \mathbb{Z}^+$ and $r \in \mathbb{R}$, let $(r)_0 = 1$ and $(r)_k = (r)\cdots(r-k+1)$. Notation, terminology and presumed results may be found in [4] and [5].

For $1 \le i \le n-1$, $1 \le r < s \le n$ and $\sigma \in \mathfrak{S}_n$, let

$$X_{(i,r,s)}(\sigma) = \begin{cases} s-r & , \sigma(i) = r,\ \sigma(i+1) = s; \\ 0 & , \text{ otherwise}; \end{cases}$$

and,

$$X = \sum_{\text{all such } (i,r,s)} X_{(i,r,s)},$$

so that $A_n(\sigma) = X(\sigma)$.

Theorem 1. For $\mathfrak{S}_n$,

$$E(A_n) = \frac{n^2-1}{6} \text{ and } \mathrm{Var}(A_n) = \frac{5n^3+5n^2-2n-2}{72}.$$

Proof. For $1 \le i \le n-1$ and $1 \le r < s \le n$,

$$E(X_{(i,r,s)}) = \frac{s-r}{(n)_2},$$

so that,

$$E(X) = \sum_{i=1}^{n-1} \sum_{s=2}^{n} \sum_{r=1}^{s-1} \frac{s-r}{(n)_2} = \frac{n^2-1}{6}.$$

In order to find the variance we consider three sums. First, for $1 \le i \le n-1$ and $1 \le r < s \le n$,

$$E(X^2_{(i,r,s)}) = \frac{(s-r)^2}{(n)_2},$$

so that,

$$E_1 := \sum_{\text{all such } (i,r,s)} E(X^2_{(i,r,s)}) = \sum_{i=1}^{n-1} \sum_{s=2}^{n} \sum_{r=1}^{s-1} \frac{(s-r)^2}{(n)_2} = \frac{n^3-n}{12}\,.$$

Second, for $1 \le i \le n-2$ and $1 \le r < s < t \le n$,

$$E(X_{(i,r,s)}X_{(i+1,s,t)}) = \frac{(s-r)(t-s)}{(n)_3},$$

so that,

$$\begin{aligned}
E_2 &:= \sum_{\text{all such } ((i,r,s),(i+1,s,t))} E(X_{(i,r,s)}X_{(i+1,s,t)}) \\
&= \sum_{i=1}^{n-2} \sum_{t=3}^{n} \sum_{s=2}^{t-1} \sum_{r=1}^{s-1} \frac{(s-r)(t-s)}{(n)_3} \\
&= \frac{1}{2(n)_3} \sum_{i=1}^{n-2} \sum_{t=3}^{n} \sum_{s=1}^{t-1} \{-s^3 + (t+1)s^2 - ts\} \\
&= \frac{1}{24(n)_3} \sum_{i=1}^{n-2} \sum_{t=1}^{n} \{t^4 - 2t^3 - t^2 + 2t\} \\
&= \frac{1}{120(n)_3} \sum_{i=1}^{n-2} \{n^5 - 5n^3 + 4n\} \\
&= \frac{n^3 + n^2 - 4n - 4}{120},
\end{aligned}$$

(where the appropriate summands for $s = 1$ and $t = 1, 2$ are 0, etc.). Third, for $1 \le i \le j-2 \le n-3$, $1 \le r < s \le n$, $1 \le t < u \le n$ with r, s, t, u distinct,

$$E(X_{(i,r,s)}X_{(j,t,u)}) = \frac{(s-r)(u-t)}{(n)_4}\,.$$

For fixed i and j,

$$E_3 := \sum_{\text{all such } ((i,r,s),(j,t,u))} E(X_{(i,r,s)} X_{(j,t,u)})$$

$$= \sum_{s=2}^{n} \sum_{\substack{r=1 \\ r,s,t,u \text{ distinct}}}^{s-1} \sum_{u=2}^{n} \sum_{t=1}^{u-1} \frac{(s-r)(u-t)}{(n)_4}$$

$$= \frac{1}{(n)_4} \sum_{s=2}^{n} \sum_{r=1}^{s-1} (s-r) \left\{ \sum_{u=2}^{n} \sum_{t=1}^{u-1} (u-t) - \sum_{t=1}^{r-1} (r-t) - \sum_{t=1}^{s-1} (s-t) \right.$$

$$\left. - \sum_{u=r+1}^{n} (u-r) - \sum_{u=s+1}^{n} (u-s) + (s-r) \right\}$$

$$= \frac{1}{(n)_4} \sum_{s=2}^{n} \sum_{r=1}^{s-1} (s-r) \left\{ \binom{n+1}{3} + (s-r) - \frac{r^2 - r + s^2 - s + n^2 - (r+s-1)n}{2} \right\}.$$

Now,

$$\sum_{s=2}^{n} \sum_{r=1}^{s-1} (s-r)\{r^2 - r + s^2 - s + n^2 - (r+s-1)n\}$$

$$= \sum_{s=2}^{n} \sum_{r=1}^{s-1} \left\{ -r^3 + (n+1+s)r^2 - (n^2+n+s^2)r + \left[s^3 - (n+1)s^2 + (n^2+n)s \right] \right\}$$

$$= \frac{1}{12} \sum_{s=1}^{n} \left\{ 7s^4 - (8n+14)s^3 + (6n^2+12n+5)s^2 - (6n^2+4n-2)s \right\}$$

$$= \frac{7n^5 - 15n^3 + 8n}{60}.$$

Then,

$$E_3 = \frac{1}{(n)_4} \left\{ \binom{n+1}{3}^2 + \frac{n^4 - n^2}{12} - \frac{7n^5 - 15n^3 + 8n}{120} \right\}$$

$$= \frac{10n^5 - 21n^4 + 10n^3 + 45n^2 - 20n - 24}{360(n-1)_3}.$$

Finally, for $1 \le i \le j-2 \le n-3$, $1 \le r < s \le n$, $1 \le t < u \le n$ with r, s, t, u distinct,

$$E_4 := \sum_{\text{all such }((i,r,s),(j,t,u))} E(X_{(i,r,s)}, X_{(j,t,u)}) = \binom{n-2}{2} E_3$$
$$= \frac{10n^4 - 11n^3 - n^2 + 44n + 24}{720}.$$

Hence,

$$E(X^2) = E_1 + 2E_2 + 2E_4 = \frac{10n^4 + 25n^3 + 5n^2 - 10n}{360},$$

so that,

$$\mathrm{Var}(X) = \frac{5n^3 + 5n^2 - 2n - 2}{72}. \qquad \blacksquare$$

We say $\tau = (\tau(1), \ldots, \tau(n)) \in \mathfrak{S}_n$ has an **excedance** at $i \in [n]$ if and only if $\tau(i) > i$ with **length of excedance** of $\tau(i) - i$ at excedance i. Let $X_n(\tau)$ denote the sum $\sum_{i=1}^n \max\{0, \tau(i) - i\}$ of the lengths of excedances of τ. In [7], an explicit bijection $f\ \mathfrak{S}_n \to \mathfrak{S}_n$ was given so that the number of ascents of σ equals the number of excedances of $f(\sigma)$ for all $\sigma \in \mathfrak{S}_n$. We next give a different bijection f that also satisfies $A_n(\sigma) = X_n(f(\sigma))$ for all $\sigma \in \mathfrak{S}_n$.

Suppose $\sigma = (\sigma(1), \ldots, \sigma(n)) \in \mathfrak{S}_n$ has ascents at $1 \leq i_1 < \cdots < i_\ell \leq n-1$. Order $\sigma(i_1), \ldots, \sigma(i_\ell)$ as $1 \leq \sigma(j_1) < \cdots < \sigma(j_\ell) \leq n$ and $[n] - \{\sigma(j_k + 1)\ 1 \leq k \leq \ell\}$ as $1 \leq t_1 < \cdots < t_{n-\ell} \leq n$. Now construct $\tau \in \mathfrak{S}_n$ as follows. Place $\sigma(j_k + 1)$ at coordinate $\sigma(j_k)$ of τ $(1 \leq k \leq \ell)$. Necessarily, $t_1 = 1$ (as all $\sigma(j_k + 1) \geq \sigma(j_k) + 1 \geq 2$) which we place in the left-most unused coordinate s_1 of τ. Having placed $t_1, \ldots, t_q$ in (the left-most unused) coordinates $1 \leq s_1 < \cdots < s_q$, we place t_{q+1} in the left-most unused coordinate s_{q+1} ($> s_q$, necessarily) of τ $(1 \leq q \leq n - \ell - 1)$. Clearly, $\tau \in \mathfrak{S}_n$.

Assume that all of $1, \ldots, t_q$ have appeared in coordinates $1, \ldots, s_q$ of τ where $1 \leq q \leq n - \ell - 1$. Let $s_{q+1} = s_q + a$, $t_{q+1} = t_q + b$ and $s_q = t_q + c$ with $a, b \in \mathbb{Z}^+$ and $c \in \mathbb{N}$. Suppose that $t_{q+1} \geq s_{q+1} + 1$. For $1 \leq x \leq a + c$, $t_q + 1 \leq t_q + x \leq t_q + a + c = s_{q+1} \leq t_{q+1} - 1$. Then, each $t_q + x = \sigma(j_k + 1)$ is at coordindate $\sigma(j_k)$ with $\sigma(j_k) \leq t_q + x - 1 \leq t_q + a + c - 1 = s_{q+1} - 1$. Hence, all of $t_q + 1, \ldots, t_q + a + c = s_{q+1}$ appear in coordinates $1, \ldots, s_{q+1} - 1$. Consequently, all of $1, \ldots, s_{q+1}$ appear in coordinates $1, \ldots, s_{q+1} - 1$, which is a contradiction. Then $t_{q+1} \leq s_{q+1}$ and, as above, all of $t_q+1,\ldots,t_{q+1}$ appear in coordinates $1,\ldots,s_{q+1}$. Hence, all of $1,\ldots,t_{q+1}$ appear in coordinates $1,\ldots,s_{q+1}$.

Consequently, $s_q \geq t_q$ $(1 \leq q \leq n-\ell)$, so that the number of ascents of σ equals the number of excedances of τ and $A_n(\sigma) = X_n(\tau)$. As it is immediately seen that $f\,\mathfrak{S}_n \to \mathfrak{S}_n$ by $f\,\sigma \mapsto \tau$ is a bijection, we have proved the following result.

Lemma 2. With the above notation, $f\ \mathfrak{S}_n \to \mathfrak{S}_n$ is a bijection where the number of ascents of σ equals the number of excedances of $f(\sigma)$ and $A_n(\sigma) = X_n(f(\sigma))$ for all $\sigma \in \mathfrak{S}_n$.

We use the following result of Hoeffding [6; Theorem 3]. Given $c_n\,[n]^2 \to \mathbb{R}$, let $d_n\,[n]^2 \to \mathbb{R}$ by

$$d_n(i,j) = c_n(i,j) - \frac{1}{n}\sum_{g=1}^{n} c_n(g,j) - \frac{1}{n}\sum_{h=1}^{n} c_n(i,h) + \frac{1}{n^2}\sum_{g=1}^{n}\sum_{h=1}^{n} c_n(g,h).$$

Theorem (Hoeffding [6]). For $\tau \in \mathfrak{S}_n$, $S_n = S_n(\tau) = \sum_{i=1}^{n} c_n(i,\tau(i))$ is asymptotically normally distributed if

$$\lim_{n\to\infty} \frac{\max\limits_{1\leq i,j\leq n} d_n^2(i,j)}{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j)} = 0.$$

We let $c_n\,[n]^2 \to \mathbb{N}$ by $c_n(i,j) = \max\{0, j-i\}$ so that $d_n\,[n]^2 \to \mathbb{Q}$ by

$$d_n(i,j) = \begin{cases} j - i - \frac{1}{n}\binom{j}{2} - \frac{1}{n}\binom{n-i+1}{2} + \frac{\binom{n+1}{3}}{n^2} & ,1 \leq i < j \leq n; \\ -\dfrac{1}{n}\dbinom{j}{2} - \dfrac{1}{n}\dbinom{n-i+1}{2} + \dfrac{\binom{n+1}{3}}{n^2} & ,1 \leq j \leq i \leq n. \end{cases}$$

It is readily seen that all $d_n(i,j) \leq d_n(1,n) = (n^2-1)/6n < n/6$. For $1 \leq i \leq \lfloor n/3 \rfloor + 1$, $\lceil 2n/3 \rceil \leq j \leq n$ and $n \geq 3$, $d_n(i,j) \geq n/20$ so that $\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j)/n \geq n^3/3600$ and, hence,

$$\frac{\max\limits_{1\leq i,j\leq n} d_n^2(i,j)}{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j)} = O(n^{-1}) \quad \text{as } n \to \infty.$$

Since Lemma 2 implies A_n and S_n have the same distribution on $\mathfrak{S}_n$, we have proved our main result.

Theorem 3. For $\mathfrak{S}_n$ and $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr\left(\frac{A_n - E(A_n)}{\sqrt{\mathrm{Var}(A_n)}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, dt,$$

where $E(A_n) = (n^2-1)/6$ and $\mathrm{Var}(A_n) = (5n^3+5n^2-2n-2)/72$.

A permutation $\sigma = (\sigma(1), \ldots, \sigma(n)) \in \mathfrak{S}_n$ has a **descent** at $i \in [n-1]$ if and only if $\sigma(i) > \sigma(i+1)$ with **length of descent** of $\sigma(i) - \sigma(i+1)$ at descent i. Let $D_n(\sigma)$ denote the sum of the lengths of descents of σ. It is easily seen that $f\ \mathfrak{S}_n \to \mathfrak{S}_n$ by $f(\sigma)(i) = n+1-\sigma(i)$ is a bijection where σ has ascent of length $\sigma(i+1) - \sigma(i)$ at i if and only if $f(\sigma)$ has descent of length $\sigma(i+i) - \sigma(i)$ at i. Hence, A_n, D_n or X_n have the same distribution (they are not pair-wise independent, however) on $\mathfrak{S}_n$ and Theorem 3 remains valid with A_n replaced by D_n or X_n.

References

1. Balcza, L., "Sum of lengths of inversions in permuations," Discrete Math. 111, 41–48, 1993.

2. Carlitz, L., Kurtz, D.C., Scoville, R., and Stackelberg, O.P., "Asymptotic properties of eulerian numbers," Zeit. Wahrsch. Verw. Gebiete 23, 47–54, 1972.

3. Clark, L., "Asymptotic normality of the generalized eulerian numbers," Ars Comb. 48, 213–218, 1998.

4. Comtet, L., Advanced combinatorics, D. Reidel, Boston, 1974.

5. Durrett, R., Probability: theory and examples, Wadsworth, Belmont, CA, 1991.

6. Hoeffding, W., "A combinatorial central limit theorem," Ann. Math. Statist. 22, 558–566, 1951.

7. Stanley, R., Enumerative combinatorics V. 1, Wadsworth, Belmont, CA, 1986.