Asymptotic Distribution of the Sum of the Lengths of Ascents or of Descents in Permutations

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Abstract

Let $A_n(\sigma)$ denote the sum of the lengths of ascents of a permutation σ of $\{1,\ldots,n\}$ chosen randomly according to a uniform distribution. We find the exact expectation and variance of A_n and then show that the A_n are asymptotically normally distributed. An identical result holds for the sum of the lengths of descents of a permutation.

Key Words: Ascents, Descents, Asymptotic Normality

A permutation $\sigma = (\sigma(1),\ldots,\sigma(n))$ of $[n] := \{1,\ldots,n\}$ has an **ascent** at $i \in [n-1]$ if and only if $\sigma(i+1) > \sigma(i)$ with **length of ascent** of $\sigma(i+1) - \sigma(i)$ at ascent i. Let $A_n(\sigma)$ denote the sum of the lengths of ascents of σ. For example, $\sigma = (4,1,7,10,6,3,8,2,9,5)$ has ascents of length $6,3,5,7$ at $2,3,6,8$, respectively. Hence, $A_{10}(\sigma) = 6+3+5+7 = 21$.

Let $\mathfrak{S}_n$ denote the set of permutations of $[n]$. We choose each permutation $\sigma \in \mathfrak{S}_n$ randomly according to a uniform distribution so that each σ has probability $\Pr(\sigma) = 1/n!$. As usual, a sequence of random variables S_n on $\mathfrak{S}_n$ is said to be **asymptotically normally distributed** if and only if the sequence of distribution functions of $(S_n - E(S_n))/\sqrt{\mathrm{Var}(S_n)}$ converges weakly to the distribution function of a normal random variable with mean

0 and variance 1, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr\left(\frac{S_n - E(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, dt,$$

where $E(S_n)$ and $\mathrm{Var}(S_n)$ are the expectation and variance of S_n.

For a positive integer m and $\sigma \in \mathfrak{S}_n$, let $S_{n,m}(\sigma)$ denote the number of ascents of σ of length at least m. In [2] it was shown that the $S_{n,1}$ are asymptotically normally distributed on $\mathfrak{S}_n$, and in [3], it was shown that the $S_{n,m}$ are also. Recently, Balcza [1] found the exact expectation and variance for the sum of the lengths of inversions on $\mathfrak{S}_n$. In this paper, we find the exact expectation and variance of A_n and then show that the A_n are asymptotically normally distributed on $\mathfrak{S}_n$. Identical results hold for the sum of the lengths of descents of a permutation.

We denote the nonnegative integers by $\mathbb{N}$; the positive integers by $\mathbb{Z}^+$; the rational numbers by $\mathbb{Q}$; and the real numbers by $\mathbb{R}$. For $k \in \mathbb{Z}^+$ and $r \in \mathbb{R}$, let $(r)_0 = 1$ and $(r)_k = (r)\cdots(r-k+1)$. Notation, terminology and presumed results may be found in [4] and [5].

For $1 \le i \le n-1$, $1 \le r < s \le n$ and $\sigma \in \mathfrak{S}_n$, let

$$X_{(i,r,s)}(\sigma) = \begin{cases} s-r & , \sigma(i) = r,\ \sigma(i+1) = s; \\ 0 & , \text{ otherwise;} \end{cases}$$

and,

$$X = \sum_{\text{all such } (i,r,s)} X_{(i,r,s)},$$

so that $A_n(\sigma) = X(\sigma)$.

Theorem 1. For $\mathfrak{S}_n$,

$$E(A_n) = \frac{n^2-1}{6} \text{ and } \mathrm{Var}(A_n) = \frac{5n^3+5n^2-2n-2}{72}.$$

Proof. For $1 \le i \le n-1$ and $1 \le r < s \le n$,

$$E(X_{(i,r,s)}) = \frac{s-r}{(n)_2},$$

so that,

$$E(X) = \sum_{i=1}^{n-1} \sum_{s=2}^{n} \sum_{r=1}^{s-1} \frac{s-r}{(n)_2} = \frac{n^2-1}{6}.$$

In order to find the variance we consider three sums. First, for $1 \leq i \leq n-1$ and $1 \leq r < s \leq n$,

$$E(X^2_{(i,r,s)}) = \frac{(s-r)^2}{(n)_2},$$

so that,

$$E_1 := \sum_{\text{all such } (i,r,s)} E(X^2_{(i,r,s)}) = \sum_{i=1}^{n-1} \sum_{s=2}^{n} \sum_{r=1}^{s-1} \frac{(s-r)^2}{(n)_2} = \frac{n^3-n}{12}\,.$$

Second, for $1 \leq i \leq n-2$ and $1 \leq r < s < t \leq n$,

$$E(X_{(i,r,s)}X_{(i+1,s,t)}) = \frac{(s-r)(t-s)}{(n)_3},$$

so that,

$$\begin{aligned}
E_2 &:= \sum_{\text{all such } ((i,r,s),(i+1,s,t))} E(X_{(i,r,s)}X_{(i+1,s,t)}) \\
&= \sum_{i=1}^{n-2} \sum_{t=3}^{n} \sum_{s=2}^{t-1} \sum_{r=1}^{s-1} \frac{(s-r)(t-s)}{(n)_3} \\
&= \frac{1}{2(n)_3} \sum_{i=1}^{n-2} \sum_{t=3}^{n} \sum_{s=1}^{t-1} \left\{-s^3 + (t+1)s^2 - ts\right\} \\
&= \frac{1}{24(n)_3} \sum_{i=1}^{n-2} \sum_{t=1}^{n} \left\{t^4 - 2t^3 - t^2 + 2t\right\} \\
&= \frac{1}{120(n)_3} \sum_{i=1}^{n-2} \left\{n^5 - 5n^3 + 4n\right\} \\
&= \frac{n^3 + n^2 - 4n - 4}{120},
\end{aligned}$$

(where the appropriate summands for $s = 1$ and $t = 1, 2$ are 0, etc.). Third, for $1 \leq i \leq j-2 \leq n-3$, $1 \leq r < s \leq n$, $1 \leq t < u \leq n$ with r, s, t, u distinct,

$$E(X_{(i,r,s)}X_{(j,t,u)}) = \frac{(s-r)(u-t)}{(n)_4}\,.$$

For fixed i and j,

$$
\begin{aligned}
E_3 &:= \sum_{\text{all such } ((i,r,s),(j,t,u))} E(X_{(i,r,s)}X_{(j,t,u)}) \\
&= \sum_{s=2}^{n} \sum_{\substack{r=1 \\ r,s,t,u \text{ distinct}}}^{s-1} \sum_{u=2}^{n} \sum_{t=1}^{u-1} \frac{(s-r)(u-t)}{(n)_4} \\
&= \frac{1}{(n)_4} \sum_{s=2}^{n} \sum_{r=1}^{s-1} (s-r) \left\{ \sum_{u=2}^{n} \sum_{t=1}^{u-1} (u-t) - \sum_{t=1}^{r-1} (r-t) - \sum_{t=1}^{s-1} (s-t) \right. \\
&\qquad \left. - \sum_{u=r+1}^{n} (u-r) - \sum_{u=s+1}^{n} (u-s) + (s-r) \right\} \\
&= \frac{1}{(n)_4} \sum_{s=2}^{n} \sum_{r=1}^{s-1} (s-r) \left\{ \binom{n+1}{3} + (s-r) - \frac{r^2-r+s^2-s+n^2-(r+s-1)n}{2} \right\}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\sum_{s=2}^{n} \sum_{r=1}^{s-1} (s-r)\{r^2 - r + s^2 - s + n^2 - (r+s-1)n\} \\
=& \sum_{s=2}^{n} \sum_{r=1}^{s-1} \left\{ -r^3 + (n+1+s)r^2 - (n^2+n+s^2)r + \left[s^3 - (n+1)s^2 + (n^2+n)s \right] \right\} \\
=& \frac{1}{12} \sum_{s=1}^{n} \left\{ 7s^4 - (8n+14)s^3 + (6n^2+12n+5)s^2 - (6n^2+4n-2)s \right\} \\
=& \frac{7n^5 - 15n^3 + 8n}{60}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
E_3 &= \frac{1}{(n)_4} \left\{ \binom{n+1}{3}^2 + \frac{n^4 - n^2}{12} - \frac{7n^5 - 15n^3 + 8n}{120} \right\} \\
&= \frac{10n^5 - 21n^4 + 10n^3 + 45n^2 - 20n - 24}{360(n-1)_3}.
\end{aligned}
$$

Finally, for $1 \le i \le j-2 \le n-3$, $1 \le r < s \le n$, $1 \le t < u \le n$ with r, s, t, u distinct,

$$E_4 := \sum_{\text{all such } ((i,r,s),(j,t,u))} E(X_{(i,r,s)}, X_{(j,t,u)}) = \binom{n-2}{2} E_3$$
$$= \frac{10n^4 - 11n^3 - n^2 + 44n + 24}{720}.$$

Hence,

$$E(X^2) = E_1 + 2E_2 + 2E_4 = \frac{10n^4 + 25n^3 + 5n^2 - 10n}{360},$$

so that,

$$\mathrm{Var}(X) = \frac{5n^3 + 5n^2 - 2n - 2}{72}. \qquad \blacksquare$$

We say $\tau = (\tau(1), \ldots, \tau(n)) \in \mathfrak{S}_n$ has an **excedance** at $i \in [n]$ if and only if $\tau(i) > i$ with **length of excedance** of $\tau(i) - i$ at excedance i. Let $X_n(\tau)$ denote the sum $\sum_{i=1}^{n} \max\{0, \tau(i) - i\}$ of the lengths of excedances of τ. In [7], an explicit bijection $f\ \mathfrak{S}_n \to \mathfrak{S}_n$ was given so that the number of ascents of σ equals the number of excedances of $f(\sigma)$ for all $\sigma \in \mathfrak{S}_n$. We next give a different bijection f that also satisfies $A_n(\sigma) = X_n(f(\sigma))$ for all $\sigma \in \mathfrak{S}_n$.

Suppose $\sigma = (\sigma(1), \ldots, \sigma(n)) \in \mathfrak{S}_n$ has ascents at $1 \leq i_1 < \cdots < i_\ell \leq n-1$. Order $\sigma(i_1), \ldots, \sigma(i_\ell)$ as $1 \leq \sigma(j_1) < \cdots < \sigma(j_\ell) \leq n$ and $[n] - \{\sigma(j_k + 1)\ 1 \leq k \leq \ell\}$ as $1 \leq t_1 < \cdots < t_{n-\ell} \leq n$. Now construct $\tau \in \mathfrak{S}_n$ as follows. Place $\sigma(j_k + 1)$ at coordinate $\sigma(j_k)$ of τ $(1 \leq k \leq \ell)$. Necessarily, $t_1 = 1$ (as all $\sigma(j_k + 1) \geq \sigma(j_k) + 1 \geq 2$) which we place in the left-most unused coordinate s_1 of τ. Having placed $t_1, \ldots, t_q$ in (the left-most unused) coordinates $1 \leq s_1 < \cdots < s_q$, we place t_{q+1} in the left-most unused coordinate s_{q+1} ($> s_q$, necessarily) of τ $(1 \leq q \leq n - \ell - 1)$. Clearly, $\tau \in \mathfrak{S}_n$.

Assume that all of $1, \ldots, t_q$ have appeared in coordinates $1, \ldots, s_q$ of τ where $1 \leq q \leq n - \ell - 1$. Let $s_{q+1} = s_q + a$, $t_{q+1} = t_q + b$ and $s_q = t_q + c$ with $a, b \in \mathbb{Z}^+$ and $c \in \mathbb{N}$. Suppose that $t_{q+1} \geq s_{q+1} + 1$. For $1 \leq x \leq a + c$, $t_q + 1 \leq t_q + x \leq t_q + a + c = s_{q+1} \leq t_{q+1} - 1$. Then, each $t_q + x = \sigma(j_k + 1)$ is at coordindate $\sigma(j_k)$ with $\sigma(j_k) \leq t_q + x - 1 \leq t_q + a + c - 1 = s_{q+1} - 1$. Hence, all of $t_q + 1, \ldots, t_q + a + c = s_{q+1}$ appear in coordinates $1, \ldots, s_{q+1} - 1$. Consequently, all of $1, \ldots, s_{q+1}$ appear in coordinates $1, \ldots, s_{q+1} - 1$, which is a contradiction. Then $t_{q+1} \leq s_{q+1}$ and, as above, all of $t_q + 1, \ldots, t_{q+1}$ appear in coordinates $1, \ldots, s_{q+1}$. Hence, all of $1, \ldots, t_{q+1}$ appear in coordinates $1, \ldots, s_{q+1}$.

Consequently, $s_q \geq t_q$ $(1 \leq q \leq n - \ell)$, so that the number of ascents of σ equals the number of excedances of τ and $A_n(\sigma) = X_n(\tau)$. As it is immediately seen that $f\,\mathfrak{S}_n \to \mathfrak{S}_n$ by $f\,\sigma \mapsto \tau$ is a bijection, we have proved the following result.

Lemma 2. With the above notation, $f\,\mathfrak{S}_n \to \mathfrak{S}_n$ is a bijection where the number of ascents of σ equals the number of excedances of $f(\sigma)$ and $A_n(\sigma) = X_n(f(\sigma))$ for all $\sigma \in \mathfrak{S}_n$.

We use the following result of Hoeffding [6; Theorem 3]. Given $c_n\,[n]^2 \to \mathbb{R}$, let $d_n\,[n]^2 \to \mathbb{R}$ by

$$d_n(i,j) = c_n(i,j) - \frac{1}{n}\sum_{g=1}^{n} c_n(g,j) - \frac{1}{n}\sum_{h=1}^{n} c_n(i,h) + \frac{1}{n^2}\sum_{g=1}^{n}\sum_{h=1}^{n} c_n(g,h).$$

Theorem (Hoeffding [6]). For $\tau \in \mathfrak{S}_n$, $S_n = S_n(\tau) = \sum_{i=1}^{n} c_n(i, \tau(i))$ is asymptotically normally distributed if

$$\lim_{n\to\infty} \frac{\displaystyle\max_{1\leq i,j\leq n} d_n^2(i,j)}{\displaystyle\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j)} = 0.$$

We let $c_n\,[n]^2 \to \mathbb{N}$ by $c_n(i,j) = \max\{0, j-i\}$ so that $d_n\,[n]^2 \to \mathbb{Q}$ by

$$d_n(i,j) = \begin{cases} j - i - \frac{1}{n}\binom{j}{2} - \frac{1}{n}\binom{n-i+1}{2} + \frac{\binom{n+1}{3}}{n^2} & ,1 \leq i < j \leq n; \\ -\dfrac{1}{n}\dbinom{j}{2} - \dfrac{1}{n}\dbinom{n-i+1}{2} + \dfrac{\binom{n+1}{3}}{n^2} & ,1 \leq j \leq i \leq n. \end{cases}$$

It is readily seen that all $d_n(i,j) \leq d_n(1,n) = (n^2-1)/6n < n/6$. For $1 \leq i \leq \lfloor n/3 \rfloor + 1$, $\lceil 2n/3 \rceil \leq j \leq n$ and $n \geq 3$, $d_n(i,j) \geq n/20$ so that $\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j)/n \geq n^3/3600$ and, hence,

$$\frac{\displaystyle\max_{1\leq i,j\leq n} d_n^2(i,j)}{\displaystyle\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j)} = O(n^{-1}) \quad \text{as } n \to \infty.$$

Since Lemma 2 implies A_n and S_n have the same distribution on $\mathfrak{S}_n$, we have proved our main result.

Theorem 3. For $\mathfrak{S}_n$ and $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr\left(\frac{A_n - E(A_n)}{\sqrt{\mathrm{Var}(A_n)}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, dt,$$

where $E(A_n) = (n^2-1)/6$ and $\mathrm{Var}(A_n) = (5n^3 + 5n^2 - 2n - 2)/72$.

A permutation $\sigma = (\sigma(1), \ldots, \sigma(n)) \in \mathfrak{S}_n$ has a **descent** at $i \in [n-1]$ if and only if $\sigma(i) > \sigma(i+1)$ with **length of descent** of $\sigma(i) - \sigma(i+1)$ at descent i. Let $D_n(\sigma)$ denote the sum of the lengths of descents of σ. It is easily seen that $f\ \mathfrak{S}_n \to \mathfrak{S}_n$ by $f(\sigma)(i) = n + 1 - \sigma(i)$ is a bijection where σ has ascent of length $\sigma(i+1) - \sigma(i)$ at i if and only if $f(\sigma)$ has descent of length $\sigma(i+i) - \sigma(i)$ at i. Hence, A_n, D_n or X_n have the same distribution (they are not pair-wise independent, however) on $\mathfrak{S}_n$ and Theorem 3 remains valid with A_n replaced by D_n or X_n.

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