ENCODING POINTED MAPS BY DOUBLE OCCURRENCE WORDS

PATRICE OSSONA DE MENDEZ AND PIERRE ROSENSTIEHL

Hommage à Antoine Panayatopoulos,
talentueux combinatoriste chez Pascal comme chez Pythagore.

Abstract. We show that pointed maps with m edges are in bijection with standard double occurrence words with $(m+1)$ symbols.

1. Introduction

1.1. **Graphs, Maps and Rotation Schemes.** Recall that a *graph* consists of a set V (of *vertices*) and a set E (of *edges*), each edge having two *ends* (said *opposite*), each incident to a vertex. A *loop* is an edge whose ends are both incident to a same vertex.

A crossing-free drawing of a graph on a surface divides the surface into *regions*. If each region is homeomorphic to an open disk, the graph is said to be *cellularly embedded* on the surface [9].

A *map* is a graph cellularly embedded in a surface (up to topological equivalence). A map is *pointed* if some end of some edge of the underlying graph has been distinguished (see Fig. 1). This will be displayed by a *mark* on the corresponding edge end.

It is common and convenient to use a polygonal representation of surfaces. The standard polygonal representation of the double torus is an octagon whose sides are pairwise identified according to some labels and orientations put on each side. The use of such a polygonal representation allows a crossing free representation of any map in the plane (see Fig. 2).

A *rotation scheme* of a graph is a circular order of the incident ends of edges around each vertex of the graph. Obviously, a map on an orientable surface typologically defines a rotation scheme of the embedded graph. Less obviously, the converse is true: any rotation scheme of a connected graph G induces a unique cellular embedding of G on an orientable surface (up to

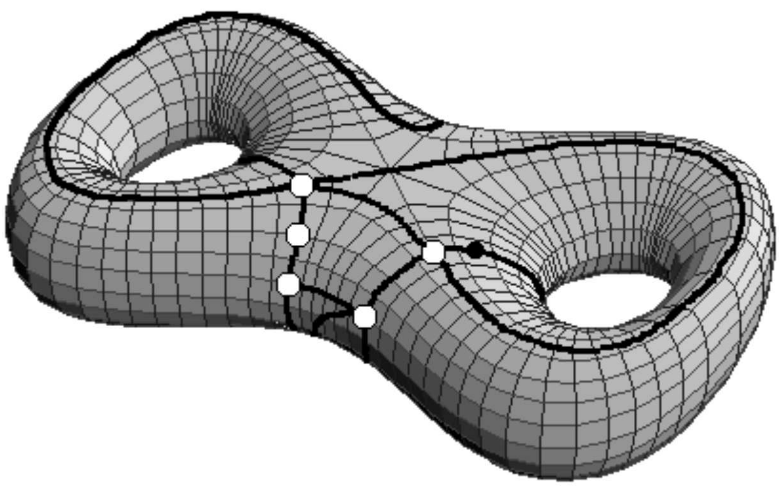


FIGURE 1. A pointed map embedded on the double torus $\mathbb{S}_2$

orientation preserving topological equivalence). This equivalence of maps and rotation scheme, which is present in a dual form in the work of Heffter [10] and which has been extensively used by Ringel in the 50s, have been independently expressed in the primal form by Edmonds [7], popularized by Youngs [14] and further generalized by Gross and Alpert [8]. According to this equivalence, a map may be represented as a *normal drawing* in the plane (that is: a drawing where two edges may cross at most once) where the rotation scheme is preserved (see Fig. 3). Notice that the equivalence of the maps of Fig. 1, Fig. 2 and Fig. 3 may be asserted by checking that they define the same rotation schemes.

1.2. **Double Occurrence Words.** A *word* over an *alphabet* Σ (which is some set of *symbols*) is a sequence of symbols belonging to Σ. This sequence is usually noted multiplicatively ($a\ a\ b$ is a word over $\{a, b, c\}$, for instance). The *concatenation* of two words is the word formed by the sequence of symbols of the first word followed by the sequence of symbols of the second

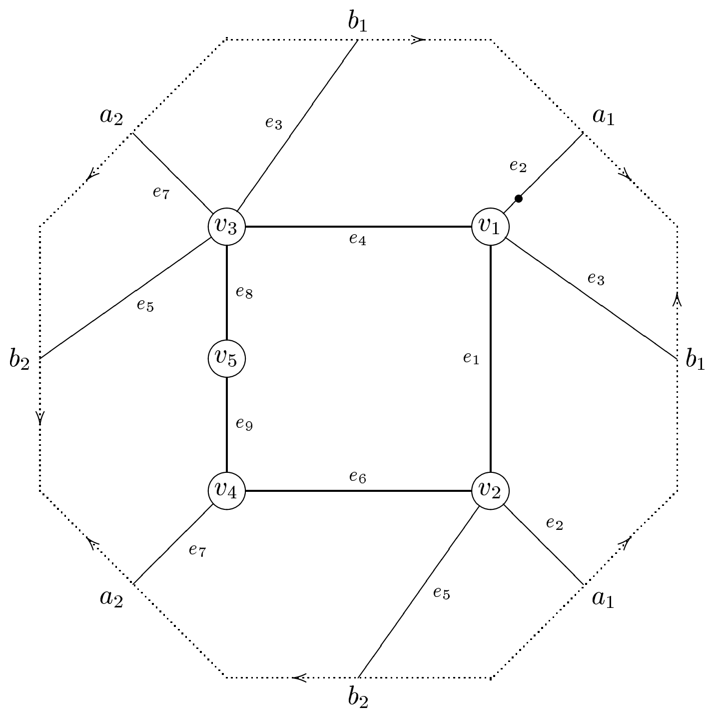


FIGURE 2. The map of Fig. 1 drawn in a standard polygonal representation of the double torus

one. This concatenation is also noted multiplicatively, as it is nothing more than the mere operation of putting the words "the one after the other" as we do for symbols when we form a word. A word w over an alphabet Σ is *equivalent* to a word w' over an alphabet Σ' if their exists a one-to-one mapping which sends Σ to Σ' and w to w'. Notice that a word w using k symbols is equivalent to a unique word over a totally ordered alphabet of size k where the symbols appear (for the first time) in the word in increasing order. Such a word is said *standard*.

A *double occurrence word* is a word in which any symbol appears exactly twice. For instance, the word $a\ b\ c\ c\ a\ b$ is a double occurrence word. A double occurrence word w that may not be factorized as $w = w_1\ w_2$, where w_1 and w_2 are both non-empty double occurrence words is said to

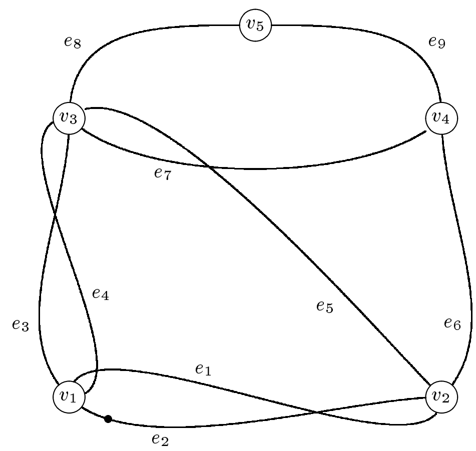


FIGURE 3. A normal drawing of the pointed map of Fig. 1 in the plane

be *connected*. For instance, $a\ b\ c\ a\ c\ b$ is connected, but $a\ b\ b\ a\ c\ d\ c\ d$ is not, as it is the concatenation of $a\ b\ b\ a$ and $c\ d\ c\ d$, which are both double occurrence words.

1.3. **Pointed Maps and Connected Double Occurrence Words.** Although it was known for years, in particular in quantum physics, that the sequence of the numbers of pointed maps with m edges is the same as the sequence of the *indecomposable involutions* (which are equivalent to standard connected double occurrence words) [6] [1], no bijective proof of this numerical equivalence was known.

Our purpose is restricted to a display of our encoding and decoding algorithms enlightened by pictures. So, what may be the obviousness of each step of these algorithms we refer for a complete proof to the expository of our more general result (may be a more obscure one) on encoding hypermaps [12][13]. As a first course, Table 1 displays the codes for the 10 pointed maps with two edges.

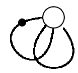
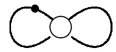
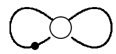
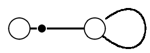
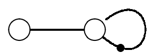
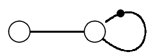
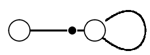
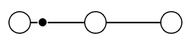
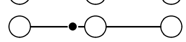
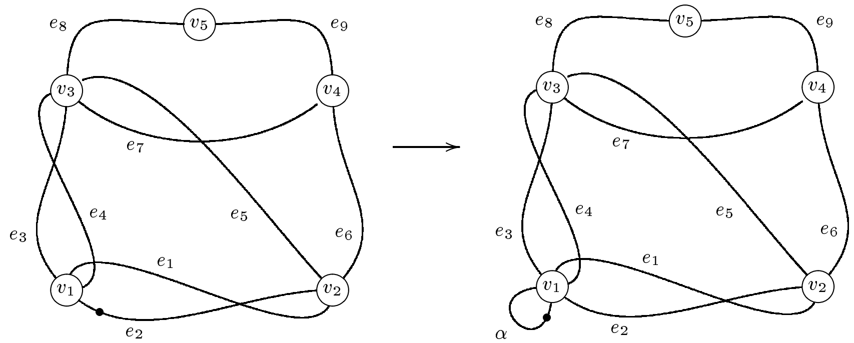
α	a	b	a	b	α	
α	a	a	b	b	α	
α	a	b	b	a	α	
α	a	b	α	b	a	
α	a	α	b	b	a	
α	a	a	b	α	b	
α	a	b	a	α	b	
α	a	b	b	α	a	
α	a	α	b	a	b	
α	a	b	α	a	b	

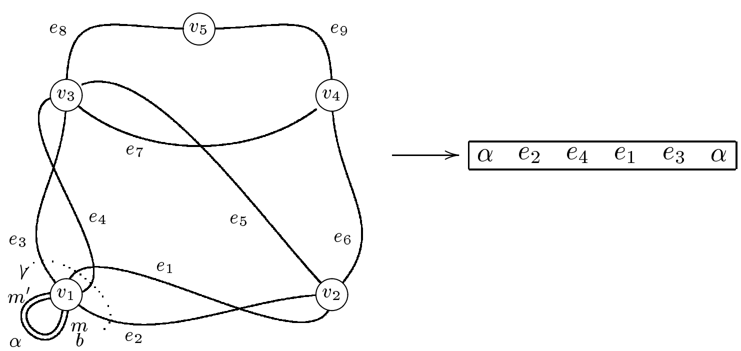
Table 1. Codes for the maps with two edges (using the alphabet $\{\alpha, a, b\}$ ordered by $\alpha < a < b$)

2. Encoding a pointed map

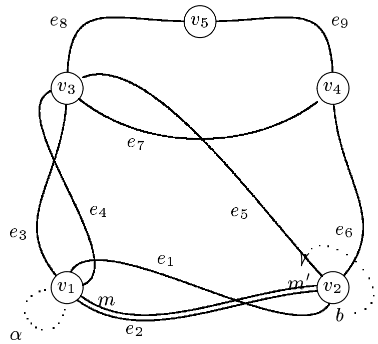
First, we add a loop "α" just before the pointed incidence and move the mark to the loop:



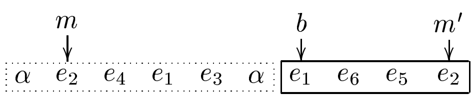
We call m the edge end where the mark is, m' the opposite end of the same edge, and b the edge end next to m' in the rotation order. Then we list all the edge labels encountered while traveling from b to m' around the vertex m' is incident to. So is formed a list of *visited* ends.

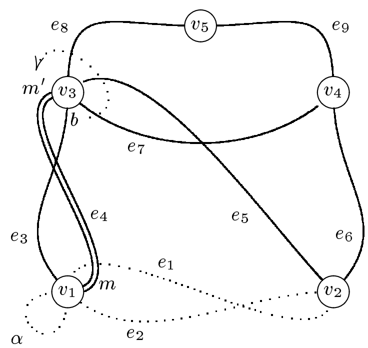


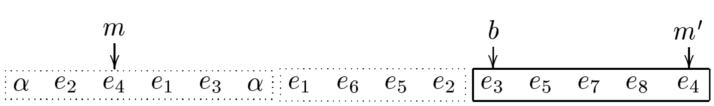
Then, m is moved to the next end of the list of visited ends, whose opposite end has not yet been visited (here the end of e_2 incident to v_1):

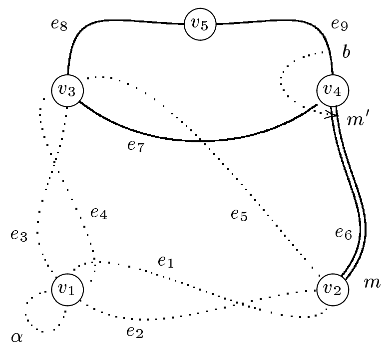

e8
v5
e9
v3
v4
e7
e4
e5
e3
e6
e1
v1
m
m′
v2
b
α
e2

We iterate:

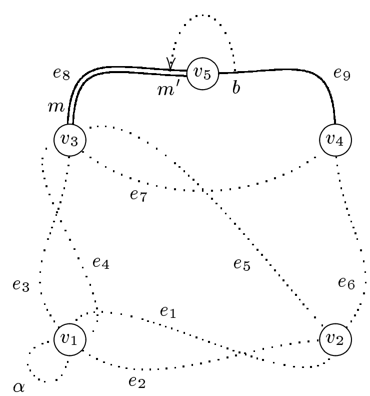

m
b
m′
α e2 e4 e1 e3 α
e1 e6 e5 e2


e8
v5
e9
m′
v3
v4
b
e7
e4
e5
e3
e6
e1
v1
m
v2
α
e2


m
b
m′
α e2 e4 e1 e3 α
e1 e6 e5 e2
e3 e5 e7 e8 e4



							m								b		m'
α	e_2	e_4	e_1	e_3	α	e_1	e_6	e_5	e_2	e_3	e_5	e_7	e_8	e_4	e_9	e_7	e_6



													m					b	m'
α	e_2	e_4	e_1	e_3	α	e_1	e_6	e_5	e_2	e_3	e_5	e_7	e_8	e_4	e_9	e_7	e_6	e_9	e_8

This code can be put in standard form using a linearly ordered alphabet:

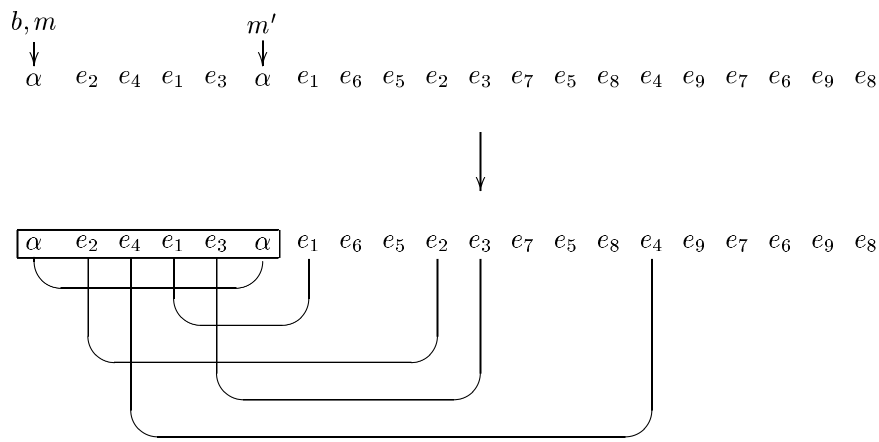
0 1 2 3 4 0 3 5 6 1 4 6 7 8 2 9 7 5 9 8

3. Decoding a connected double occurrence word

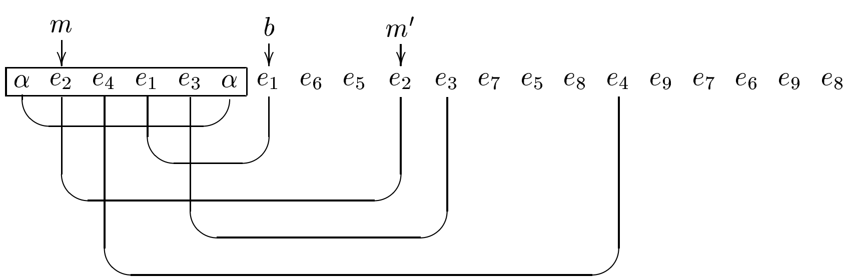
Starting from the double occurrence word, we will connect the two occurrences of each symbol and group symbols into boxes.

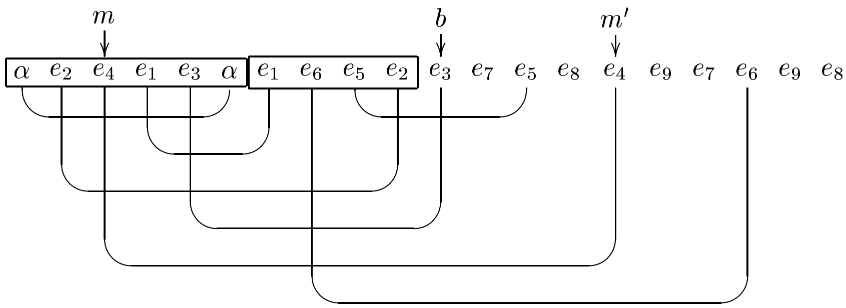
Let b be the leftmost symbol not in a box (here b is the first occurrence of α) and let m be the leftmost first occurrence symbol whose second occurrence m' is not in a box (here m and m' are the first and second occurrences of α)

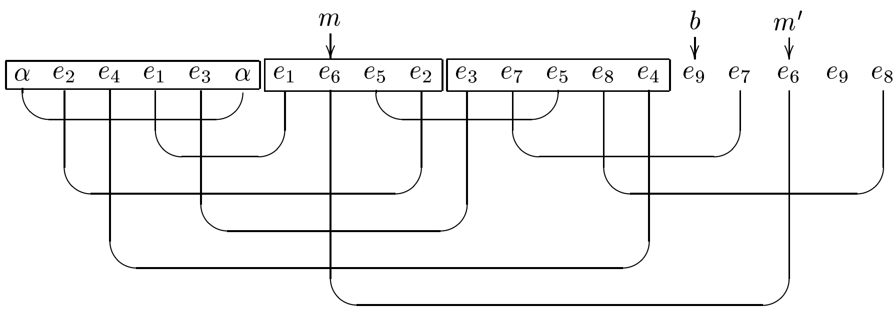
Create a box that contains all symbols from b to m' and then connect each first occurrence symbol in the box to its second occurrence (wherever it is):

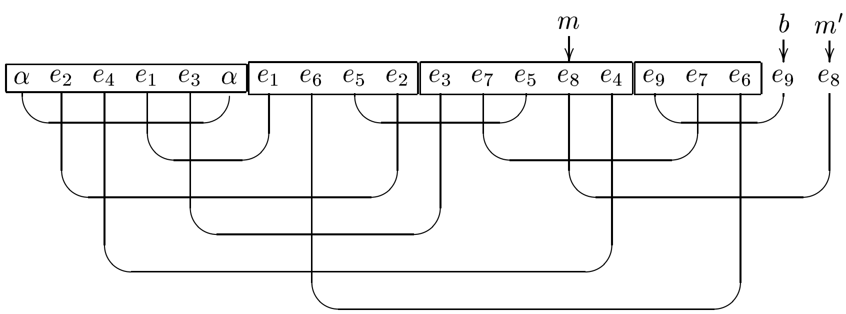


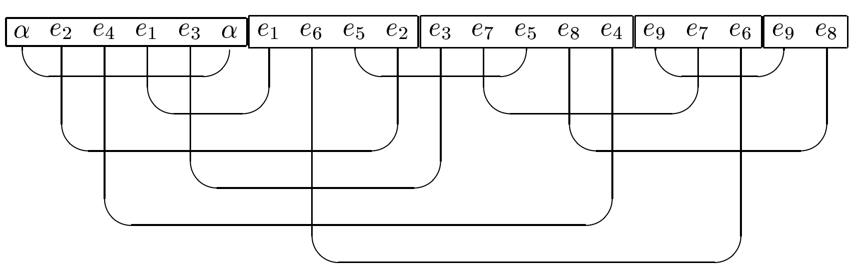
Now, b is the second occurrence of e_1, m and m' are the first and second occurrences of e_2 and we iterate:



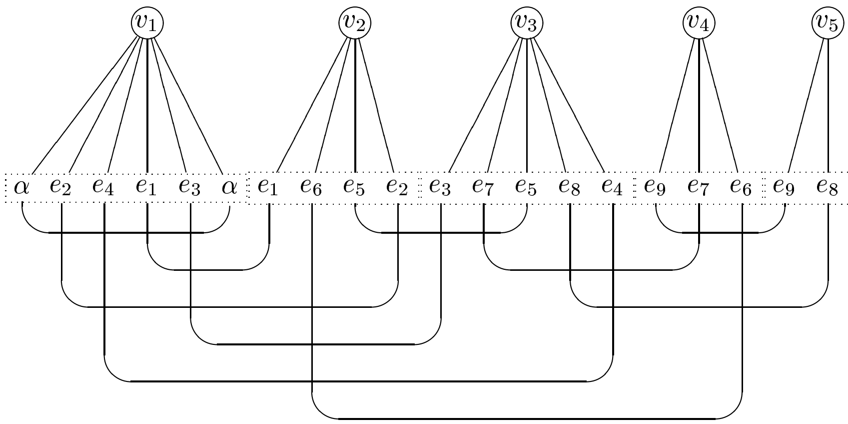

m
b
m′
α e2 e4 e1 e3 α
e1 e6 e5 e2
e3 e7 e5 e8 e4 e9 e7 e6 e9 e8


m
b
m′
α e2 e4 e1 e3 α
e1 e6 e5 e2
e3 e7 e5 e8 e4
e9 e7 e6 e9 e8

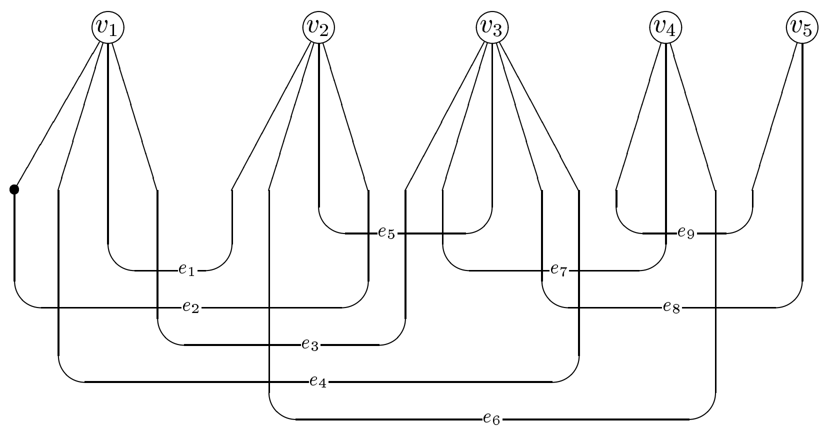

m
b
m′
α e2 e4 e1 e3 α
e1 e6 e5 e2
e3 e7 e5 e8 e4
e9 e7 e6
e9 e8


α e2 e4 e1 e3 α
e1 e6 e5 e2
e3 e7 e5 e8 e4
e9 e7 e6
e9 e8

Now, create a vertex per box and connect it to the edges symbols in the box according to their order of appearance:



Last, remove the α loop, point at the first symbol and forget the boxes:



4. Conclusion

We shall notice that, in the decoding process, the first symbol (α) plays a special "bootstrapping" rule necessary to initiate the inductive construction of the pointed map.

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Centre d'Analyse et de Mathématiques Sociales, CNRS, UMR 8557, 54 Bd Raspail, 75006 Paris, France

E-mail address: pom@ehess.fr

Centre d'Analyse et de Mathématiques Sociales, CNRS, UMR 8557, 54 Bd Raspail, 75006 Paris, France

E-mail address: pr@ehess.fr