ON THE GROUP-LIKE BEHAVIOUR OF THE LE-MURAKAMI-OHTSUKI INVARIANT

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ABSTRACT. We study the effect of Feynman integration and diagrammatic differential operators on the structure of group-like elements in the algebra generated by coloured vertex-oriented uni-trivalent graphs. We provide applications of our results to the study of the LMO invariant, a quantum invariant of manifolds. We also indicate further situations in which our results apply and may prove useful. The enumerative approach that we adopt has a clarity that has enabled us to perceive a number of generalizations.

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1. Introduction

The techniques of Feynman integration and diagrammatic differential operators play an important role in quantum topology. Roughly speaking, these techniques involve "gluing together" formal power series of (coloured uni-trivalent) graphs according to certain recipes arising from the diagrammatic formalism of perturbative Chern-Simons theory.

Feynman diagrams appear in quantum topology as equivalence classes of formal $\mathbb{Q}$-power series of coloured vertex-oriented uni-trivalent graphs. These power series can be equipped with a commutative multiplication (given by the disjoint union) and a coproduct (the sum of all ways of "splitting" diagrams) and can be made into graded Hopf algebras, denoted by $\mathcal{B}$ (see [BN] for details). The primitives of these Hopf algebras are known to be power series of connected elements. Most of the elements of $\mathcal{B}$ which are of interest in quantum topology, such as the values of the Kontsevich or Le-Murakami-Ohtsuki (hereinafter, LMO) invariants, are known to be group-like ([LeMO, Oh2]). A well known property of graded Hopf algebras is that any group-like element may be written as the exponential of a primitive element ([Ab]). This allows one to study the logarithm of quantum invariants rather than the invariants themselves.

In this paper we study the effects that Feynman integration and diagrammatic differential operators have on the structure of group-like elements in $\mathcal{B}$ through the effect on the primitives. We provide applications of our results to the study of the LMO invariant. We also indicate further situations where our results apply and may prove useful. The enumerative approach that we have adopted has a clarity that has enabled us to perceive a number of generalizations.

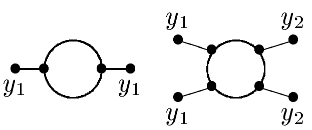
Our approach is to show that these results arise naturally as a generalization of a classical result in algebraic combinatorics.

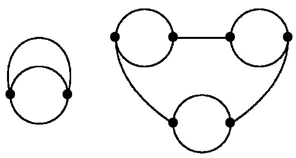
The paper is structured as follows. In Section 2 we describe the problem and our results in a purely combinatorial language. In Section 3 we explain how our results relate to quantum invariants of 3-manifolds and show how to express the values of the primitive LMO invariant in terms of those of the primitive Kontsevich invariant. The enumerative preliminaries are given briefly in Section 4, and the details of the labelling process are given in Section 5. Section 6 deals with the graph-subgraph series for appropriately weighted graphs. The proof of the main theorem appears in Section 7. In Section 8 we explain how our results can be generalized and applied to diagrammatic differential operators. We conclude by applying our results to find closed formulae for the primitive LMO invariant of certain 3-manifolds. These appear in Section 9.

2. The Combinatorial Problem

A $\mathcal{Y}$*-coloured uni-trivalent diagram* is a graph g made of undirected edges with two types of vertices **i)** trivalent vertices equipped with a cyclic ordering of its incident edges and **ii)** univalent vertices with colours assigned from a finite set $\mathcal{Y} = \{y_1, y_2, \ldots\}$, where vertices are to be regarded as mutually distinguishable. If $\mathcal{Y} = \emptyset$ then the graph is *trivalent*.

For example, below are two graphs, g_0 and h_0, each with two components. The graph g_0 contains both trivalent and univalent vertices with colour set $\mathcal{Y} = \{y_1, y_2\}$, while the graph h_0 has only trivalent vertices, so $\mathcal{Y} = \emptyset$.

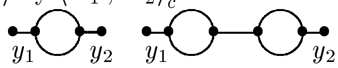
$g_0 =$ 

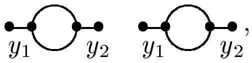
$h_0 =$ 

Note for example, that the two vertices with the colour y_1 on each component of g_0 are distinguishable.

Let $\mathcal{D}(\mathcal{Y})$ be the algebra of formal power series with $\mathcal{Y}$-coloured uni-trivalent graphs as indeterminates and coefficients in $\mathbb{Q}$. The empty graph is allowed, and is denoted by 1 in the algebra. Commutative multiplication is given by disjoint union. Motivated by the Hopf algebra $\mathcal{B}$, we say that an element of $\mathcal{D}(\mathcal{Y})$ is *primitive* if it is a sum of connected graphs and *group-like* if it is the exponential of a primitive. $\mathcal{D}_s(\mathcal{Y})$ is the subalgebra of $\mathcal{D}(\mathcal{Y})$ containing no components of the form  (called *struts*) with the same colour at its vertices.

Let g and h be $\mathcal{Y}$-coloured uni-trivalent graphs. We define a bilinear operator $\langle \cdot\,,\, \cdot\rangle : \mathcal{D}(\mathcal{Y}) \otimes \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\emptyset)$, such that $\langle g\,,\, h\rangle$ is the sum of all ways of identifying each of the y-coloured vertices in g with each of the y-coloured vertices in h, for all colours $y \in \mathcal{Y}$. If the numbers of y-coloured univalent vertices in the two graphs do not match for some $y \in \mathcal{Y}$, the sum is zero. Coloured univalent vertices that are not to be joined under the pairing are indicated by *open vertices* (this will be used in Section 8). We extend this bilinearly to all of $\mathcal{D}(\mathcal{Y}) \otimes \mathcal{D}(\mathcal{Y})$. Once the components of g and h are joined by $\langle \cdot\,,\, \cdot\rangle$, we consider them as *subgraphs* of the resulting trivalent graph. For all $D_1, D_2 \in \mathcal{D}(\mathcal{Y})\otimes\mathcal{D}(\mathcal{Y})$, we denote the primitive part of $\langle D_1\,,\, D_2\rangle$ by $\langle D_1\,,\, D_2\rangle_c$. The following is an example of this operator.

Let $D_1 =$ 

and $D_2 =$ 

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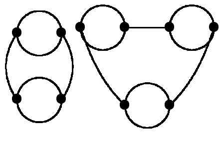
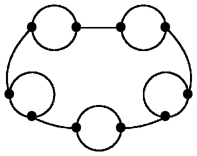
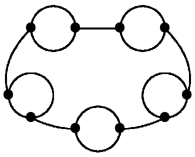
then $\langle D_1\,,\, D_2\rangle = 2$  $+\ 2$  .

The primitive part of this is $\langle D_1\,,\, D_2\rangle_c = 2$  .

Note that it is necessarily linear in the indeterminates. The following is the main theorem of the paper. It determines the primitive structure of

$\langle D_1\,,\,D_2\rangle$ and its relation to the LMO invariant will be discussed in the next section. We will also use the equation to find formulae for the primitive LMO invariant of certain manifolds in Section 9.

Theorem 2.1. *Let $B \in \mathcal{D}(\mathcal{Y})$ and $C \in \mathcal{D}_s(\mathcal{Y})$, be primitive. Then*

$$\langle \exp B\,,\,\exp C\rangle = \exp\,\langle \exp B\,,\,\exp C\rangle_c\,.$$

Before we discuss the relation of this theorem with quantum topology we highlight two important corollaries and a generalization of the theorem. We note that another generalization, which was motivated by the theory of diagrammatic differential operators, will be discussed in section 8.

The first special case of this theorem occurs when B consists entirely of struts. This was the motivating example for this paper and we will see in the following section how it relates to the LMO invariant.

Corollary 2.2. *Let $C \in \mathcal{D}_s(\mathcal{Y})$ be strutless and primitive. Let $\mathcal{Y} = \{y_1, y_2, \cdots\}$, $r_{i,j}(C) \in \mathbb{Q}$ and let $S = \exp\left(\sum_{i\geq j\geq 1} r_{i,j}(C)\, y_i \bullet\!\frown\!\bullet y_j\right)$. Then*

$$\langle S\,,\,\exp C\rangle = \exp\,\langle S\,,\,\exp C\rangle_c\,.$$

A linear operator $\langle\cdot\rangle : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\emptyset)$ also arises in quantum topology (eg. [BNGRT3]). For some $g \in \mathcal{D}(\mathcal{Y})$, $\langle g\rangle$ is defined as the sum of all ways of identifying pairwise all the y-coloured univalent vertices of g for all $y \in \mathcal{Y}$. If g has an odd number of y-coloured univalent vertices for some colour $y \in \mathcal{Y}$, then $\langle g\rangle = 0$. As with our previous operator, let $\langle D\rangle_c$ denote the primitive part of $\langle D\rangle$.

Corollary 2.3. *Let $C \in \mathcal{D}_s(\mathcal{Y})$ be strutless and primitive. Then*

$$\langle \exp C\rangle = \exp\,\langle \exp C\rangle_c\,.$$

Proof. This follows from the observation $\langle D\rangle = \langle \exp(\frac{1}{2}\sum_{y\in\mathcal{Y}} y \bullet\!\frown\!\bullet y\,), D\rangle$ and the theorem. ∎

We note that Garoufalidis made a conjecture of the above form ([Ga]).

The following minor generalization of our theorem will allow us to extend our results to the LMO invariant of links in manifolds. Let $g \in \mathcal{D}_s(\mathcal{Y})$ and

$h \in \mathcal{D}(\mathcal{Y})$. Further suppose that $\mathcal{X} \subset \mathcal{Y}$. Then the definition of $\langle \cdot , \cdot \rangle$ can easily be extended to allow the situation where we only glue together the univalent vertices of g and h whose colours are in $\mathcal{X}$. More precisely, let $\langle \cdot , \cdot \rangle_{\mathcal{X}} : \mathcal{D}(\mathcal{Y}) \otimes \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{Y} - \mathcal{X})$, be the operator such that $\langle g , h \rangle_{\mathcal{X}}$ is the sum of all ways of identifying each of the x-coloured vertices in g with each of the x-coloured vertices in h, for all colours $x \in \mathcal{X}$. If the numbers of x-coloured univalent vertices in the two graphs do not match for some $x \in \mathcal{X}$, the sum is zero. We extend this bilinearly to all of $\mathcal{D}(\mathcal{Y}) \otimes \mathcal{D}(\mathcal{Y})$. Again $\langle \cdot , \cdot \rangle_{\mathcal{X}, c}$ denotes the primitive part of $\langle \cdot , \cdot \rangle_{\mathcal{X}}$. The following generalizes Theorem 2.1.

Theorem 2.4. *Let $\mathcal{X} \subset \mathcal{Y}$, $B \in \mathcal{D}(\mathcal{Y})$ and $C \in \mathcal{D}_s(\mathcal{Y})$, be primitive. Then*

$$\langle \exp B , \exp C \rangle_{\mathcal{X}} = \exp \langle \exp B , \exp C \rangle_{\mathcal{X}, c} .$$

A proof of this statement can be obtained by a simple modification of the proof of Theorems 2.1 and is therefore excluded.

Remark 2.5. The requirement that $C \in \mathcal{D}_s(\mathcal{Y})$ ensures that the calculation of $\langle \exp B , \exp C \rangle$ and its specializations is finite for any given number of vertices.

Remark 2.6. In fact, the proofs of Theorems 2.1, 2.4 and its generalization in Section 8 hold even if we remove the requirement that B and C are uni-trivalent, but since we take our motivation from quantum topology we do not work in this generality here.

3. Motivation from Quantum Topology

Our primary motivation for this study comes from the theory of the LMO invariant, Z^{LMO}, introduced in [LeMO]. This is a universal perturbative invariant of rational homology 3-spheres (see [Oh, BNGRT2, Oh2]), and a universal finite type invariant of integral homology 3-spheres ([Le]). (Recall that for a ring R, a *R-homology sphere* is a 3-manifold M such that $H_*(M; R) = H_*(S^3; R)$.)

The LMO invariant was first derived by considering the behavior of the Kontsevich integral of framed links under the two Kirby moves. A few years later in [BNGRT1, BNGRT2, BNGRT3], the diagrammization of a physical argument led to a reformulation of the LMO invariant. This approach uses the notions of "diagrammatic integration" and the construction is sometimes known as the *Århus integral.* Our motivation comes from this formulation of the LMO invariant. We sketch the construction of this invariant.

Two of the fundamental algebras in quantum topology are the algebras $\mathcal{A}$ of formal $\mathbb{Q}$ power series of uni-trivalent graphs with oriented trivalent vertices and whose uni-valent vertices lie on an oriented compact coloured 1-manifold modulo certain relations, and the algebra $\mathcal{B}$ which is the quotient algebra of $\mathcal{D}(\mathcal{Y})$ generated by some relations. We need not worry about the exact form of these relations here. The (in general non-commutative) multiplication on $\mathcal{A}$ is given by connect summing copies of S^1 and "stacking" copies of the interval so that the colours match (this operation corresponds to the usual composition of tangles). The multiplication on $\mathcal{B}$ is given by disjoint union. In fact, one may make $\mathcal{A}$ and $\mathcal{B}$ into Hopf algebras. We will also need to make use of the coalgebra isomorphism $\chi : \mathcal{B} \to \mathcal{A}$, which is defined to be the average of all ways of placing all of the univalent vertices of an element of $\mathcal{B}$ onto the 1-manifold of $\mathcal{A}$ such that the y-coloured vertices lie on the corresponding component of the 1-manifold, for all $y \in \mathcal{Y}$. This is known as the *Poincaré-Birkhoff-Witt (PBW) isomorphism* as it is the diagrammization of the map from the theory of Lie algebras. Details of these algebras and the theory of finite-type invariants can be found in [BN].

Now let a framed link L represent a rational homology sphere M by surgery. Further suppose that the components of L are in correspondence with a set $\mathcal{Y}$. It is a well known fact that the image of the Kontsevich integral of a framed link under the inverse σ of the PBW isomorphism may be written in the form $\sigma Z(L) = \exp(\sum_{x,y\in\mathcal{Y}} \frac{1}{2} l_{xy}\ x \bullet\!\frown\!\bullet\, y\ + C)$, where $C \in \mathcal{B}_s(Y)$ is primitive and strutless and l_{xy} denotes the linking number. Because of this we can separate the struts and, using the definition of the inner product in Section 2, define $Z_0^{LMO}(L)$ by

$$(1) \qquad Z_0^{LMO}(L) := \left\langle \exp\Big(\sum_{x,y\in\mathcal{Y}} -\frac{1}{2} l^{xy}\ x \bullet\!\frown\!\bullet\, y\ \Big), \exp(C) \right\rangle \in \mathcal{B}(\emptyset),$$

where (l^{xy}) is the inverse of the linking matrix (l_{xy}) (note that since we restrict to rational homology spheres the linking matrix is non-singular). This procedure of gluing together the terms of the Kontsevich integral $\sigma Z(L)$ is known as *formal Gaussian integration* (so called as it is the diagrammization of a perturbed Gaussian integral). The function Z_0^{LMO} we have just defined is invariant under only a handle-slide. To make it invariant under stabilization and therefore into an invariant of 3-manifolds requires the usual trick of normalizing by eigenvalues, and the *LMO invariant* is defined by

$$Z^{LMO}(M) = Z_0^{LMO}(U_+)^{-e_+} Z_0^{LMO}(U_-)^{-e_-} Z_0^{LMO}(L),$$

where $U_\pm$ is the ± 1 framed unknot and $e_\pm$ is the number of $\pm$ve eigenvalues of the linking matrix. One immediately notices that the definition of $\mathring{\mathrm{A}}_0$ is of the form of Corollary 2.2 and we obtain the following.

Proposition 3.1. *The value of the LMO invariant $Z^{LMO}(M)$ is group-like in $\mathcal{B}(\emptyset)$, that is $Z^{LMO}(M) = \exp(C)$, where C is primitive.*

We will discuss this group-like property of the LMO invariant further in Section 9.

The LMO invariant can be easily extended to tangles and links in manifolds. A framed tangle in a rational homology sphere can be represented by a framed tangle $T \subset S^3$ through surgery. Some of the components of this will be distinguished as *surgery components*, *ie.* surgery along these components recovers the original manifold and the remaining components correspond to components of the tangle. Suppose that T is $\mathcal{Y}$-coloured and the surgery components are $\mathcal{X}$-coloured. Then one can construct the LMO invariant of tangles in rational homology spheres in a similar way to the construction outlined above except using the operation $\langle \cdot\,,\,\cdot\rangle_{\mathcal{X}}$ in place of $\langle \cdot\,,\,\cdot\rangle$ and restricting the linking matrix to the surgery components of T. See [BNGRT2, Mof] for details. Theorem 2.4 then gives:

Proposition 3.2. *The value of the LMO invariant of tangles in rational homology spheres is group-like.*

As an example, this property was used in [Mof] to relate the tree part of the LMO invariant of links in integral homology spheres to Milnor's μ-invariants, which are classical link invariants defined through the fundamental group of the link complement.

The group-like structure is a fundamental property of the Kontsevich and LMO invariants. The fact that the Kontsevich integral is group-like is well known. The proof of the group-like property of the LMO invariant using

Le, Murakami and Ohtshuki's construction has a very different flavor than that presented above. It is reduced to the problem of showing that a certain diagram of algebras is commutative. This proof and that for the Kontsevich integral can be found in [LeMO]. One advantage of our proofs for the group-like property of the LMO invariant is that it expresses their values in terms of the values of the primitive Kontsevich integral in a particularly neat way.

We note that Theorem 2.1 is much more general than what is needed for the applications above. In Section 9 we shall need the theorem in its more general form.

Before we return to the combinatorics, we briefly describe one more situation where our formula applies and may prove useful. For brevity, in this paragraph we will assume that the 1-manifold in $\mathcal{A}$ is connected and the colouring set of $\mathcal{B}$ has exactly one element. It was noted earlier that the Poincaré-Birkhoff-Witt isomorphism gives a vectorspace isomorphism $\chi : \mathcal{B} \to \mathcal{A}$. This is not an algebra isomorphism, however the existence of a product on $\mathcal{B}$ which would make this map into an algebra isomorphism is immediate. The multiplication was calculated explicitly in [BNGRT2] (although its existence had been used in several places before) and is given in terms of gluing rooted forests. Explicitly, the multiplication is defined by the formula

$$\sigma(\chi(D_1) \cdot \chi(D_2)) = \langle \exp \Lambda, D_1 \cdot D_2 \rangle,$$

where Λ is the Baker-Campbel-Hausdorf formula (this measures the failure of the identity $e^{x+y} = e^x e^y$ in a Lie algebra) ([Ja]) written as rooted trees. We refer the reader to [BNGRT2] for details. Again notice that Theorem 2.1 applies to this situation and gives a formula for the primitive values.

4. Enumerative Preliminaries

For completeness we include some familiar elementary results. The reader who is familiar with this can pass over this section. A more detailed account is included in [GJ].

4.1. **Labelled structures.** The following notation will be used:

$\mathcal{N}_m = \{1, 2, \ldots, m\}$ and for $p \geq 0$, $\mathcal{N}_{\geq p} = \{p, p+1, \ldots\}$. Let $f(x) = \sum_{n\geq 0} a_n x^n$ be a formal power series with coefficients in $\mathbb{Q}$. For $n \geq 0$ the mapping $[x^n]\colon \mathbb{Q}[[x]] \to \mathbb{Q} : f(x) \mapsto a_n$ is the *coefficient operator*, it is linear.

Let $\mathfrak{A}$ be a set of *combinatorial structures*, and let ε denote the null structure in this set. Let $\omega(A)$ be the *weight* of $\mathfrak{A}$, that is, a function $\omega : \mathfrak{A} \to \mathcal{N}_{\geq 0}$. Let $a(n) = |\{A \in \mathfrak{A} \ : \ \omega(A) = n\}|$. We would like to determine this number for all $n \geq 0$. This is an *enumerative problem* which we denote by $(\mathfrak{A}, \omega)$. A weight function can be *refined* to record more information about a combinatorial structure by tensoring the univariate weight functions for each item of information. So if $\omega_1, \ldots, \omega_r : \mathfrak{A} \to \mathcal{N}_{\geq 0}$ are weight functions, then

$$\omega_1 \otimes \cdots \otimes \omega_r : \mathfrak{A} \to \mathcal{N}_{\geq 0} \times \cdots \times \mathcal{N}_{\geq 0} : A \mapsto (\omega_1(A), \ldots, \omega_r(A)).$$

We shall need labelled structures. If A is a combinatorial structure with generic substructures $\mathbf{s}$ ($\mathbf{s}$-*subobjects*), an $\mathbf{s}$-*labelling* of A is A together with an assignment of the numbers 1 to n to its $\mathbf{s}$-subobjects. For example, in the permutation with 1-line presentation $(2, 1, 3)$, the s-subobjects are the positions onto which labels may be placed. The labels on positions $1, 2, 3$ are $2, 1, 3$. Throughout, the term "label" and "labelling" will be used exclusively in this strict combinatorial sense.

4.2. **Elementary counting lemmas.** The *ordinary generating series* for the enumerative problem$(\mathfrak{A}, \omega)$ in the indeterminate x is $\sum_{A\in\mathfrak{A}} x^{\omega(A)}$ and is denoted by $[(\mathfrak{A}, \omega)]_o$. Thus $a(n)$ is given by $[x^n]$ $[(\mathfrak{A}, \omega)]_o$. The *exponential generating series* for $(\mathfrak{A}, \omega)$ is $\sum_{A\in\mathfrak{A}} x^{\omega(A)}/\omega(A)!$, and is denoted by $[(\mathfrak{A}, \omega)]_e$. Thus $a(n)$ is given by $[x^n/n!][(\mathfrak{A}, \omega)]_e$. This is used when the elements A of $\mathfrak{A}$ are labelled structures. With additional indeterminates, the corresponding generating series is multivariate with a weight function of the form $\omega_1 \otimes \cdots \otimes \omega_r$. Moreover, a multivariate generating series may have some indeterminates that mark some information *ordinarily* and others that mark some information *exponentially.*

We give a brief account of the properties of ordinary and exponential series in terms of elementary operations on sets. These operations are the Cartesian product, the $\star$-product and composition with respect to these. The operations arise very naturally from a combinatorial point of view in the decomposition of sets of structures into their constituents.

With unlabelled structures we use the Cartesian product of sets. With labelled structures, another set product is required, namely one that distributes labels in all possible ways. Let $\mathfrak{A}$ and $\mathfrak{B}$ be sets of labelled combinatorial structures. Let $A \in \mathfrak{A}$ have k $\mathbf{s}$-subobjects and $B \in \mathfrak{B}$ have $n-k$ $\mathbf{t}$-subobjects. Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a set of distinct positive integers. Without loss of generality, $\alpha_1 < \ldots < \alpha_k$. Then $(A)_\alpha$ denotes the structure obtained from A by replacing canonically the label i with α_i for $i = 1, \ldots, k$. Let $\beta = \mathcal{N}_n - \alpha$. Then $((A)_\alpha, (B)_\beta)$ is a labelled structure with n subobjects. Let $\mathfrak{A} \star \mathfrak{B}$ denote the set of all $((A)_\alpha, (B)_\beta)$ for all choices of α as a subset of $\mathcal{N}_n$, for all pairs $(A, B) \in \mathfrak{A} \times \mathfrak{B}$ and where n takes all values greater than or equal to one. It is understood that the contribution of $((A)_\alpha, (B)_\beta)$ is non-null only when $|\alpha|$ and $|\beta|$ are equal to the number of $\mathbf{s}$-subobjects and $\mathbf{t}$-subobjects of A and B, respectively. We have the following familiar counting lemma.

Lemma 4.1 (The $\star$-Product Lemma). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be sets of labelled combinatorial structures with generic $\mathbf{s}$-subobjects and $\mathbf{t}$-subobjects. Let $\omega_{\mathbf{s}}(A)$ be the number of $\mathbf{s}$-subobjects of $A \in \mathfrak{A}$ and $\omega_{\mathbf{t}}(B)$ be the number of $\mathbf{t}$-subobjects in $B \in \mathfrak{B}$. If $\mathbf{r}$ denotes the generic subobject of $\mathfrak{A} \star \mathfrak{B}$, that is either an $\mathbf{s}$-subobject or a $\mathbf{t}$-subobject, then*

$$[(\mathfrak{A} \star \mathfrak{B}, \omega_{\mathbf{r}})]_e = [(\mathfrak{A}, \omega_{\mathbf{s}})]_e \cdot [(\mathfrak{B}, \omega_{\mathbf{t}})]_e.$$

We shall require two auxiliary sets: $\mathfrak{O}(\mathbf{o}) = \{(1, 2, \ldots, k) : k = 0, 1, 2, \ldots\}$ is the set of all *canonical ordered sets* where $\mathbf{o}$ is the generic subobject. Similarly, $\mathfrak{U}(\mathbf{u}) = \{\{1, \ldots, k\} : k = 0, 1, 2, \ldots\}$ is the set of all *canonical unordered sets* where $\mathbf{u}$ is the generic subobject. Trivially, we have the generating series:

$$[(\mathfrak{O}, \omega_{\mathbf{o}})]_e(x) = \frac{1}{1-x} \quad \text{and} \quad [(\mathfrak{U}, \omega_{\mathbf{u}})]_e(x) = \exp x. \tag{2}$$

We denote by $\mathfrak{A} \circledast \mathfrak{B}$ the composition of sets $\mathfrak{A}$ and $\mathfrak{B}$ with respect to the $\star$-product, where each generic subobject of an element $A \in \mathfrak{A}$ is replaced by an element $B \in \mathfrak{B}$ in a unique way. We have immediately the following lemma,

Lemma 4.2 (The Composition Lemma). *Let $\mathfrak{A}(\mathbf{s})$ and $\mathfrak{B}(\mathbf{t})$ be sets of structures with generic $\mathbf{s}$ and $\mathbf{t}$ subobjects, respectively. If $\omega_{\mathbf{s}}$ and $\omega_{\mathbf{t}}$ are the weight functions of $\mathfrak{A}$ and $\mathfrak{B}$, then*

$$[(\mathfrak{A} \circledast \mathfrak{B}, \omega_{\mathbf{t}})]_e = [(\mathfrak{A}, \omega_{\mathbf{s}})]_e \circ [(\mathfrak{B}, \omega_{\mathbf{t}})]_e,$$

where $\circ$ denotes composition.

We illustrate these ideas with the following brief examples.

Example 4.3. Let $\mathfrak{D}$ be the set of all permutations with no fixed points. We give two approaches to determine the number of such permutations on n points: one by an indirect decomposition and the other by a direct decomposition. Let $\mathfrak{P}$ be the set of all permutations and $\mathfrak{I}$ be the set of all the identity permutations. Here the null permutation is regarded as a permutation in each set. Then $\mathfrak{P} \overset{\sim}{\to} \mathfrak{I} \star \mathfrak{D}$, an indirect decomposition for $\mathfrak{D}$. Thus since we have $\mathfrak{P} \overset{\sim}{\to} \mathfrak{O}$, by the $\star$-Product Lemma 4.1 and (2), $(1-x)^{-1} = \exp x \cdot [(\mathfrak{D}, \omega)]_e$, where $\omega(\pi) = n$ if π is a permutation on precisely n symbols. This gives the generating series for $(\mathfrak{D}, \omega)$.

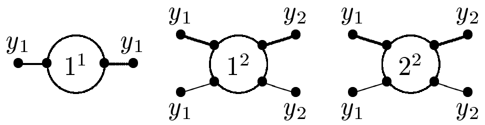
Example 4.4. Alternatively, any permutation can be decomposed into an unordered set of disjoint cycles. Let $\mathfrak{C}$ be the set of all canonical non-null cycles. Then $\mathfrak{P} \overset{\sim}{\to} \mathfrak{U} \circledast \mathfrak{C}$. By the Composition Lemma, $(1-x)^{-1} = \exp[(\mathfrak{C}, \omega)]_e$ where ω is the same as above, giving the generating series for $(\mathfrak{C}, \omega)$. Restricting $\mathfrak{P}$ to $\mathfrak{D}$ we have $\mathfrak{D} \overset{\sim}{\to} \mathfrak{U} \circledast (\mathfrak{C} - \mathfrak{C}_1)$, where $\mathfrak{C}_1$ is the set of all 1-cycles. This is a direct decomposition for $\mathfrak{D}$ and by the Composition Lemma 4.2 and (2) gives $[(\mathfrak{D}, \omega)]_e = (1-x)^{-1} \exp(-x)$.

Example 4.5. A more general instance of Example 4.4 is the following well known result (implicit in the work of Hurwitz) on graph enumeration. Let $\mathfrak{G}$ be the set of simple vertex-labelled graphs and $\mathfrak{G}_c$ be the set of simple connected vertex-labelled graphs, then we have the following decomposition $\mathfrak{G} \overset{\sim}{\to} \mathfrak{U} \circledast \mathfrak{G}_c$, since a vertex-labelled graph is an unordered set of its connected vertex-labelled graphs. Therefore, if ω is a weight function that counts the number of vertices then $[(\mathfrak{G}, \omega)]_e = \exp[(\mathfrak{G}_c, \omega)]_e$ by the Composition Lemma 4.2 and (2). We shall adapt this last result to the graphs in $\mathcal{D}(\mathcal{Y})$.

5. Labelled graphs

5.1. **Labelled uni-trivalent graphs.** When we apply $\langle \cdot\,, \cdot \rangle$ to a power series D in $\mathcal{D}(\mathcal{Y})$, multiple copies of the same trivalent graph can come from one or more graphs in D. These are required to be distinguishable and to do this, we introduce labels and decorations into the graphs.

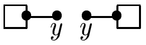
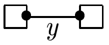
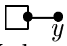
We label the set of all isomorphic components of a $\mathcal{Y}$-coloured unitrivalent diagram ($\mathcal{Y} \neq \emptyset$) by labels from the same set. The set of labels for different isomorphism classes of components are pairwise mutually disjoint. We shall use label sets $\{1^1, 2^1, \ldots\}$, $\{1^2, 2^2, \ldots\}, \ldots$ for this purpose. These labels are applied to components. Moreover, the univalent vertices with the same colour in a component are conveniently distinguished by decorating their univalent edges. To avoid cluttering the diagrams, we have indicated the decoration by using edges of different thicknesses. We call such diagrams *component labelled diagrams*. Let $\mathcal{D}_\ell(\mathcal{Y})$ be the algebra over $\mathbb{Q}$ of formal power series in these graphs as indeterminants. An example of a component labelled diagram is

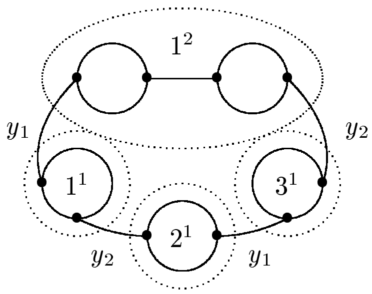


Note that the components labelled 1^1 and 2^1 are isomorphic as graphs, so their labels belong to the same label set. In addition, there are two vertices in the component labelled 1^1 that have the same colour y_1, and these are distinguished by decorating their two incident edges (the two edges have different thicknesses).

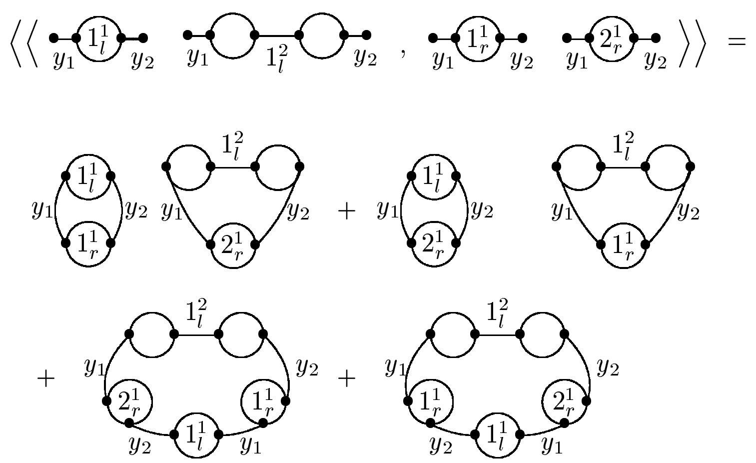
We use g and h to denote unlabelled graphs and u_i, v_j to denote unlabelled connected graphs in $\mathcal{D}(\mathcal{Y})$. To distinguish between unlabelled and labelled structures, we use g and h to denote labelled graphs and u_i, v_j to denote connected labelled graphs in $\mathcal{D}_\ell(\mathcal{Y})$. The graph $\mathsf{u}_1^{\rho_1} \cdots \mathsf{u}_n^{\rho_n}$ has labels $\{1^1, 2^1, \ldots, \rho_1^1\}, \ldots, \{1^n, 2^n, \ldots, \rho_n^n\}$ for its components $\mathsf{u}_1, \ldots, \mathsf{u}_n$. To accommodate the labelling we use the exponential basis of $\mathcal{D}_\ell(\mathcal{Y})$, namely $\{\mathsf{u}_i^j/j! \mid i, j = 0, 1, 2, \ldots\}$. As in $\mathcal{D}(\mathcal{Y})$, there is only one graph in $\mathcal{D}_\ell(\mathcal{Y})$ with ρ_i components of u_i for $1 \leq i \leq n$, this is $\mathsf{u}_1^{\rho_1} \cdots \mathsf{u}_n^{\rho_n}$,. so there is a one-to-one correspondence between graphs in $\mathcal{D}_\ell(\mathcal{Y})$ and $\mathcal{D}(\mathcal{Y})$. In fact, if $\lambda : \mathfrak{D}(\mathcal{Y}) \to \mathfrak{D}_\ell(\mathcal{Y}) : u_i^{\rho_i} \mapsto \mathsf{u}_i^{\rho_i}$ is the operator that labels elements of $\mathfrak{D}(\mathcal{Y})$, then it is an isomorphism. Its inverse λ^{-1} discards labels from the graphs, that is $\lambda^{-1} : \mathfrak{D}_\ell(\mathcal{Y}) \to \mathfrak{D}(\mathcal{Y}) : \mathsf{u}_i^{\rho_i} \mapsto u_i^{\rho_i}$.

5.2. **The $\langle\langle \cdot, \cdot \rangle\rangle$ operator.** We require an operator on $\mathcal{D}_\ell(\mathcal{Y})$ that corresponds to the operator $\langle \cdot, \cdot \rangle$ in $\mathcal{D}(\mathcal{Y})$. Such operator will take graphs

g, h with labelled components and give the sum of all the ways of identifying all of the y-coloured vertices in g with those in h for all colours $y \in \mathcal{Y}$. The identification involves the homeomorphic reduction of the path so formed and the attachment of colour y to the resulting edge. Thus :  $\rightsquigarrow$ , where  denotes a uni-trivalent diagram with a specified y-coloured vertex. We keep decoration in edges when their end vertices are identified (thus, half of an edge may have a different thickness than the other half) . This gives trivalent graphs with some coloured edges, some decorated edges and with the uni-trivalent components as labelled subgraphs. We denote the algebra of power series in such graphs by $\mathcal{D}_\ell(\emptyset)$. The following is an instance of such a graph, with the subgraphs highlighted by dotted closed curves:



We define such operator in the exponential basis of $\mathcal{D}_\ell(\mathcal{Y})$, and extend it bilinearly and denote it by $\langle\langle\cdot\,,\,\cdot\rangle\rangle : \mathcal{D}_\ell(\mathcal{Y}) \otimes \mathcal{D}_\ell(\mathcal{Y}) \to \mathcal{D}_\ell(\emptyset)$. Thus $\langle\langle\cdot\,,\,\cdot\rangle\rangle_c$ is the connected part of $\langle\langle\cdot\,,\,\cdot\rangle\rangle$. For the above combinatorial reason we use label sets $\{1_l^1, 2_l^1, \ldots\}, \{1_l^2, 2_l^2, \ldots\}, \ldots$ for components in the left argument of $\langle\langle\cdot\,,\,\cdot\rangle\rangle$ and $\{1_r^1, 2_r^1, \ldots\}, \{1_r^2, 2_r^2, \ldots\}, \ldots$ for the components in the right argument of $\langle\langle\cdot\,,\,\cdot\rangle\rangle$. What we have gained with the component labelling and decorations is that $\langle\langle\cdot\,,\,\cdot\rangle\rangle$ will give a sum of different trivalent diagrams. Moreover, any rational coefficients in the trivalent diagrams come directly from the original graphs in $\mathcal{D}_\ell(\mathcal{Y})$.



(3)

Note that the first two graphs in the right hand side have different subgraph labelling and the last two are distinguishable by their edge colouring.

As in the case of $\mathcal{D}(\mathcal{Y})$, where $\mathcal{Y} \neq \emptyset$, let G and Γ_i denote a trivalent graph and a trivalent connected graph in $\mathcal{D}(\emptyset)$. Let G and $\mathsf{\Gamma}_i$ denote the labelled and decorated counterparts in $\mathcal{D}_\ell(\emptyset)$. Again, if $\lambda^{-1} : \mathfrak{D}_\ell(\emptyset) \to \mathfrak{D}(\emptyset)$ discards subgraph labels, edge colourings and decorations, then $\lambda^{-1} : \mathsf{\Gamma}_i \mapsto \Gamma_i$. In this case λ^{-1} is not necessarily one-to-one.

6. The graph-subgraph exponential generating series

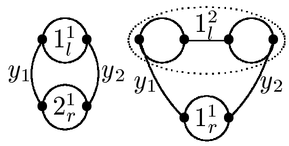
6.1. **Component-Subgraph Decomposition.** To account for the appearance of the external exponential function in the right hand side of Theorem 2.1, we show here that a graph in $\mathcal{D}_\ell(\emptyset)$ can be decomposed into an unordered set of its components and that this decomposition preserves the subgraph labelling. The occurrences of the remaining two internal exponential functions in the right hand side of Theorem 2.1 will be explained very simply in Section 7.

For simplicity, let $\mathsf{G} = \prod_{k=1}^{m} \Gamma_k \in \mathcal{D}_\ell(\emptyset)$ be one of the terms of $\langle\langle \mathsf{u}^\rho\ ,\ \mathsf{v}^\varrho \rangle\rangle$. G has the subgraphs u with all the labels $\{1_l^1, 2_l^1, \ldots, \rho_l^1\}$. Each of the connected components Γ_k has ρ_k of these with the labels $\{\alpha^1_{(k,1)}, \alpha^1_{(k,2)}, \ldots, \alpha^1_{(k,\rho_k)}\}$. Let $\alpha_{(k)} = \{\alpha_{(k,1)}, \alpha_{(k,2)}, \ldots, \alpha_{(k,\rho_k)}\}$, without loss of generality $\alpha_{(k,1)} < \ldots < \alpha_{(k,\rho_k)}$. Similarly, for the subgraph v with labels $\{1_r^1, 2_r^1, \ldots, \varrho_r^1\}$, for each component Γ_k, we get a set $\beta_{(k)} = \{\beta^1_{(k,1)}, \beta^1_{(k,2)}, \ldots, \beta^1_{(k,\varrho_k)}\}$.

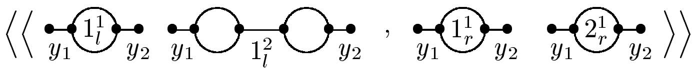
Since $\Gamma_k \in \mathcal{D}_\ell(\emptyset)$ is the corresponding graph by replacing the label sets $\alpha_{(k)}$ and $\beta_{(k)}$ by $\{1, \ldots, \rho_k\}$ and $\{1, \ldots, \varrho_k\}$, respectively, we can denote the Γ_k-component of G using the notation of Section 4.2 as ${}_{\alpha_{(k)}}(\Gamma_k)_{\beta_{(k)}}$. Hence,

$$\prod_{k=1}^{m} \Gamma_k \longmapsto \mathsf{X}_{k=1}^{m}\ {}_{\alpha_{(k)}}(\Gamma_k)_{\beta_{(k)}} \quad \text{where} \quad \dot{\bigcup}_{k=1}^{m} \alpha_{(k)} = \mathcal{N}_\rho \text{ and } \dot{\bigcup}_{k=1}^{m} \beta_{(k)} = \mathcal{N}_\varrho.$$

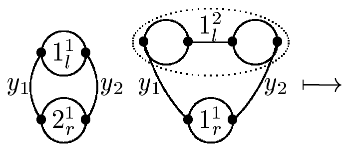
The sets of labels $\alpha_{(k)}$ for $k = 1, \ldots, m$ are mutually disjoint. This also holds for the sets $\beta_{(k)}$. For example, from (3),

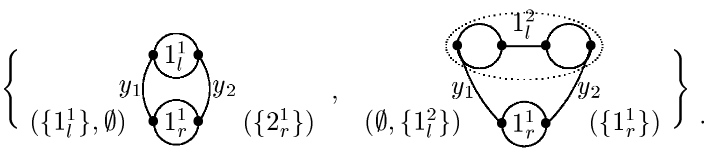


is a term of



and it can be decomposed into





where $\{1_l^1\} \dot\cup \emptyset = \mathcal{N}_1$, $\emptyset \dot\cup \{1_l^2\} = \mathcal{N}_1$ and $\{2_r^1\} \dot\cup \{1_r^1\} = \mathcal{N}_2$. As noted in the previous section, the rational coefficients of the trivalent graphs in $\langle\langle \cdot \, , \cdot \rangle\rangle$ come directly from the graphs in $\mathcal{D}_\ell(\mathcal{Y})$, so we can weight the graph with these rational coefficients. This suggests the following lemma.

Lemma 6.1. *For $\mathsf{D}_1, \mathsf{D}_2 \in \mathcal{D}_\ell(\mathcal{Y})$, let $\mathfrak{G}_c(\mathsf{D}_1, \mathsf{D}_2)$ be the set of subgraph-labelled connected graphs in $\langle\langle \mathsf{D}_1 \, , \mathsf{D}_2 \rangle\rangle$ with rational weights and let $\mathfrak{G}(\mathsf{D}_1, \mathsf{D}_2)$ the set of subgraph-labelled graphs with components from $\mathfrak{G}_c(\mathsf{D}_1, \mathsf{D}_2)$ (their rational weights are obtained by multiplying the weights of the components). Then*

$$\mathfrak{G}(\mathsf{D}_1, \mathsf{D}_2) \xrightarrow{\sim} \mathfrak{U} \circledast \mathfrak{G}_c(\mathsf{D}_1, \mathsf{D}_2).$$

Proof. This is simply an enrichment of Example 4.5. □

6.2. **The weight function for the graphs.** Let $\mathsf{G} = \prod_{k=1}^m \Gamma_k \in \mathcal{D}_\ell(\emptyset)$. Now, for $\Gamma \in \mathcal{D}(\emptyset)$ and $u \in \mathcal{D}(\mathcal{Y})$, we define the following weight functions:

(i) *trivalent component weights:* Let $\omega_\Gamma : \mathcal{D}_\ell(\emptyset) \to \mathcal{N}_{\geq 0}$, where $\omega_\Gamma(\mathsf{G})$ is the number of components of $\lambda^{-1}(\mathsf{G}) = \prod_{k=1}^m \Gamma_k$ that are isomorphic to Γ.
(ii) *subgraph weights:* Let $\omega_u : \mathcal{D}_\ell(\emptyset) \to \mathcal{N}_{\geq 0}$, where $\omega_u(\mathsf{G})$ is the number of appearances of $\lambda(u) = \mathsf{u}$ as a subgraph in G.
(iii) Let $\omega = \left(\bigotimes_{\Gamma \in \mathcal{D}(\emptyset)} \omega_\Gamma\right) \otimes \left(\bigotimes_{u \in \mathcal{D}(\mathcal{Y})} \omega_u\right)$.

Similarly, if $\mathsf{g} = \prod_{i=1}^n \mathsf{u}_i \in \mathcal{D}_\ell(\mathcal{Y})$, we define

(iv) *uni-trivalent component weights:* Let $\theta_u : \mathcal{D}_\ell(\mathcal{Y}) \to \mathcal{N}_{\geq 0}$, where $\theta_u(\mathsf{g})$ is the number of components of $\lambda^{-1}(\mathsf{g}) = \prod_{i=1}^n u_i$ that are isomorphic to u.
(v) Let $\theta = \bigotimes_{u \in \mathcal{D}(\mathcal{Y})} \theta_u$.

6.3. **The linear functions Φ and Φ°.** We use the linear operators Φ and Φ°, respectively, for computing the generating series for labelled trivalent graphs and labelled uni-trivalent graphs. Using $\Gamma \in \mathcal{D}(\emptyset)$, and $u \in \mathcal{D}(\mathcal{Y})$ as indeterminates, Γ marks ordinarily the number of Γ-components, and u marks exponentially the number of u-subgraphs. We therefore work in the ring of formal power series $\mathcal{R} \equiv \mathbb{Q}[\prod_{\Gamma \in \mathcal{D}(\emptyset)} \Gamma]\,[[\prod_{u \in \mathcal{D}(\mathcal{Y})} u]]$ and its subring $\mathcal{R}_{\mathcal{D}(\mathcal{Y})} \equiv \mathbb{Q}[[\prod_{u \in \mathcal{D}(\mathcal{Y})} u]]$. Let ω be as defined in Section 6.2. Then

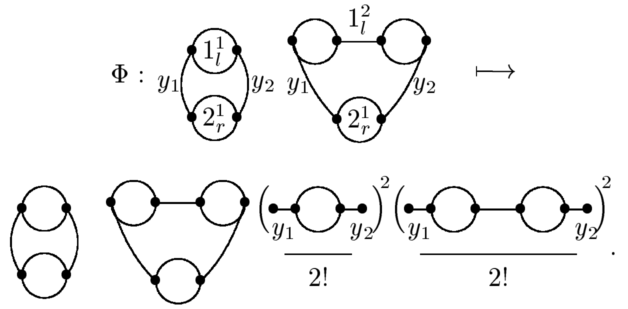
$$\Phi : \mathcal{D}_\ell(\emptyset) \to \mathcal{R} : \mathsf{G} \mapsto [(\mathsf{G}, \omega)]_{(o;e)}\,,$$

where Φ is extended linearly to $\mathcal{D}_\ell(\emptyset)$.

As an example, if $\prod_{k=1}^m \Gamma_k$ is a term in $\langle\langle \mathsf{u}_1^{\rho_1} \cdots \mathsf{u}_{n_1}^{\rho_{n_1}}\,,\, \mathsf{v}_1^{\varrho_1} \cdots \mathsf{v}_{n_2}^{\varrho_{n_2}} \rangle\rangle$, then

$$\Phi(\prod_{k=1}^m \mathsf{\Gamma}_k) = [(\prod_{k=1}^m \mathsf{\Gamma}_k, \omega)]_{(o;e)}(\Gamma, u) = \left(\prod_{k=1}^m \Gamma_k\right) \left(\prod_{j=1}^{n_1} \frac{u_j^{\rho_j}}{\rho_j!} \prod_{j=1}^{n_2} \frac{v_j^{\varrho_j}}{\varrho_j!}\right).$$

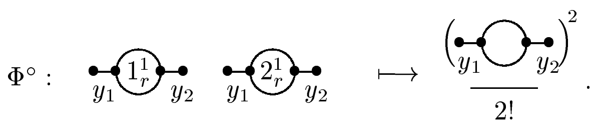
Thus, the generating series encodes the trivalent graph without labels and decorations but with subgraph information. For example, for the first term in the right hand side of (3), we have



We also require combinatorial information from elements in $\mathcal{D}_\ell(\mathcal{Y})$ in the arguments of $\langle\langle\cdot\,,\,\cdot\rangle\rangle$. Let θ be as defined in Section 6.2. Then

$$\Phi^\circ : \mathcal{D}_\ell(\mathcal{Y}) \to \mathcal{R}_{\mathcal{D}(\mathcal{Y})} : \mathsf{g} \mapsto [(\mathsf{g}, \theta)]_{(e)}\,,$$

where Φ° is extended linearly to $\mathcal{D}_\ell(\mathcal{Y})$. As an example, $\Phi^\circ(\prod_{j=1}^n \mathsf{u}_j^{\rho_j}) = \prod_{j=1}^n \mathsf{u}_j^{\rho_j}/\rho_j!$ or for the graph in the right argument of $\langle\langle\cdot\,,\,\cdot\rangle\rangle$ in (3),



If we let $L_a^x f(x) = f(x)\big|_{x=a}$ be the evaluation operator, the following proposition follows.

Proposition 6.2. *Let $\mathcal{Y} = \{y_1, y_2, \ldots\}$, for $\mathsf{D}_1, \mathsf{D}_2 \in \mathcal{D}_\ell(\mathcal{Y})$. Then*

$$\begin{aligned} L_1^{\mathcal{D}(\mathcal{Y})}\Phi(\langle\langle \mathsf{D}_1 , \mathsf{D}_2\rangle\rangle) &= \langle \Phi^\circ(\mathsf{D}_1) , \Phi^\circ(\mathsf{D}_2)\rangle , \\ L_1^{\mathcal{D}(\mathcal{Y})}\Phi(\langle\langle \mathsf{D}_1 , \mathsf{D}_2\rangle\rangle_c) &= \langle \Phi^\circ(\mathsf{D}_1) , \Phi^\circ(\mathsf{D}_2)\rangle_c . \end{aligned}$$

7. Proof of main theorem

Before proving of the main theorem (Theorem 2.1), a prefactory result is first needed.

7.1. A prefactory result. Let $B = \sum_{i=0}^{\infty} w_i$ and $C = \sum_{i=0}^{\infty} z_i$ and where $w_i = r_i u_i$ and $z_i = s_i v_i$ are the scalar products of rational numbers r_i, s_i and connected uni-trivalent graphs $u_i, v_i \in \mathcal{D}(\mathcal{Y})$. Let B and C be the images of B and C in $\mathcal{D}_\ell(\mathcal{Y})$.

Let D_B be the series of all graphs with components in B. By a trivial decomposition into components, $\Phi^\circ(\mathsf{D}_\mathsf{B}) = \exp \Phi^\circ(\mathsf{B})$. Since B is connected then $\Phi^\circ(\mathsf{B}) = B$ and it follows that $\Phi^\circ(\mathsf{D}_\mathsf{B}) = \exp B$. Similarly, if D_C is the series of all graphs with components in C, then $\Phi^\circ(\mathsf{D}_\mathsf{C}) = \exp C$. Let $\mathfrak{G}_c(\mathsf{D}_\mathsf{B}, \mathsf{D}_\mathsf{C})$ and $\mathfrak{G}(\mathsf{D}_\mathsf{B}, \mathsf{D}_\mathsf{C})$ be as in Lemma 6.1. We have the following proposition.

Proposition 7.1. *Let B and C be series of connected graphs in $\mathcal{D}_\ell(\mathcal{Y})$. If D_B and D_C are the series of graphs with components from B and C respectively, then*

$$[(\mathfrak{G}(\mathsf{D}_\mathsf{B}, \mathsf{D}_\mathsf{C}), \omega)]_{(o;e)} = \Phi \langle\langle \mathsf{D}_\mathsf{B} , \mathsf{D}_\mathsf{C}\rangle\rangle .$$

Proof. We shall show that the terms in $\langle\langle \mathsf{D}_\mathsf{B} , \mathsf{D}_\mathsf{C}\rangle\rangle$ are exactly the graphs of $\mathfrak{G}(\mathsf{D}_\mathsf{B}, \mathsf{D}_\mathsf{C})$. Let G_1 and G_2 be terms of $\langle\langle \mathsf{D}_\mathsf{B} , \mathsf{D}_\mathsf{C}\rangle\rangle$. More precisely, they come from $\langle\langle \mathsf{g}_1 , \mathsf{h}_1\rangle\rangle$ and $\langle\langle \mathsf{g}_2 , \mathsf{h}_2\rangle\rangle$ where g_1 and g_2 are graphs with components in B, and h_1 and h_2 are graphs with components in C. For any relabelling of their subgraphs, $\mathsf{G}_1\mathsf{G}_2$ is a term of $\langle\langle \mathsf{g}_1\mathsf{g}_2 , \mathsf{h}_1\mathsf{h}_2\rangle\rangle$, where $\mathsf{g}_1\mathsf{g}_2$ and $\mathsf{h}_1\mathsf{h}_2$ also have components from B and C. Thus $\mathsf{G}_1\mathsf{G}_2$ is a term of $\langle\langle \mathsf{D}_\mathsf{B} , \mathsf{D}_\mathsf{C}\rangle\rangle$. It then follows that all the graphs of $\mathfrak{G}(\mathsf{D}_\mathsf{B}, \mathsf{D}_\mathsf{C})$ are terms of $\langle\langle \mathsf{D}_\mathsf{B} , \mathsf{D}_\mathsf{C}\rangle\rangle$.

On the other hand, let $\mathsf{G}_1\mathsf{G}_2$ be a term of $\langle\langle \mathsf{D}_\mathsf{B} , \mathsf{D}_\mathsf{C}\rangle\rangle$. If it is a term in $\langle\langle \mathsf{g} , \mathsf{h}\rangle\rangle$, we can express $\mathsf{g} = \mathsf{g}_1\mathsf{g}_2$ and $\mathsf{h} = \mathsf{h}_1\mathsf{h}_2$ such that G_1 and G_2 each come from $\langle\langle \mathsf{g}_1 , \mathsf{h}_1\rangle\rangle$ and $\langle\langle \mathsf{g}_2 , \mathsf{h}_2\rangle\rangle$. Since g_1 and g_2 are graphs with components in B, and h_1 and h_2 are graphs with components in C, G_1 and G_2 are terms of $\langle\langle \mathsf{D_B} , \mathsf{D_C}\rangle\rangle$. This implies that the connected components of the terms of $\langle\langle \mathsf{D_B} , \mathsf{D_C}\rangle\rangle$ are all in $\mathfrak{G}_c(\mathsf{D_B}, \mathsf{D_C})$. Thus, the terms of $dbbr\mathsf{D_B}\mathsf{D_C}$ are in $\mathfrak{G}(\mathsf{D_B}, \mathsf{D_C})$. This gives the desired result, since the weight function ω is the same as the one in the definition of Φ. □

7.2. **Proof Theorem 2.1.** We are now in a position to prove the main theorem.

Proof. Let $B = \sum_{i=0}^{\infty} w_i$ and $C = \sum_{i=0}^{\infty} z_i$, where $u_i \in \mathcal{D}(\mathcal{Y})$ and $v_i \in \mathcal{D}_s(\mathcal{Y})$ are the scalar product of rational numbers and connected uni-trivalent graphs. Let $\mathsf{D_B}$ be the series of all graphs with components in B, and $\mathsf{D_C}$ is the series of all graphs with components in C. By Lemma 6.1 applied to $\mathsf{D_B}$ and $\mathsf{D_C}$, we have $\mathfrak{G}(\mathsf{D_B}, \mathsf{D_C}) \longrightarrow \mathfrak{U} \circledast \mathfrak{G}_c(\mathsf{D_B}, \mathsf{D_C})$. Thus, by the Composition Lemma 4.2 $[(\mathfrak{G}(\mathsf{D_B}, \mathsf{D_C}), \omega)]_{(o;e)} = [(\mathfrak{U}, \omega_u)]_e \circ [(\mathfrak{G}_c(\mathsf{D_B}, \mathsf{D_C}), \omega)]_{(o;e)}$. From (2) we have $[(\mathfrak{G}(\mathsf{D_B}, \mathsf{D_C}), \omega)]_{(o;e)} = \exp[(\mathfrak{G}_c(\mathsf{D_B}, \mathsf{D_C}), \omega)]_{(o;e)}$. It is true that $[(\mathfrak{G}_c(\cdot, \cdot), \omega)]_{(o;e)} = \Phi \langle\langle \cdot , \cdot\rangle\rangle_c$, since the graphs of $\mathfrak{G}_c(\cdot, \cdot)$ are exactly the terms of $\langle\langle \cdot , \cdot\rangle\rangle_c$. From this and Proposition 7.1 we have $\Phi \langle\langle \mathsf{D_B} , \mathsf{D_C}\rangle\rangle = \exp \Phi \langle\langle \mathsf{D_B} , \mathsf{D_C}\rangle\rangle_c$. Evaluating $u_i = 1$ for all $u_i \in \mathcal{D}(\mathcal{Y})$ gives
$L_1^{\mathcal{D}(\mathcal{Y})} \Phi \langle\langle \mathsf{D_B} , \mathsf{D_C}\rangle\rangle = \exp L_1^{\mathcal{D}(\mathcal{Y})} \Phi \langle\langle \mathsf{D_B} , \mathsf{D_C}\rangle\rangle_c$. But by Proposition 6.2, this is just $\langle \Phi^\circ(\mathsf{D_B}) , \Phi^\circ(\mathsf{D_C})\rangle = \exp \langle \Phi^\circ(\mathsf{D_B}) , \Phi^\circ(\mathsf{D_C})\rangle_c$. So $\langle \exp B , \exp C\rangle = \exp \langle \exp B , \exp C\rangle_c$, giving the result. □

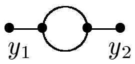
8. A Generalization to Diagrammatic Differential Operators

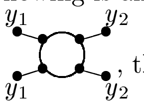
So far we have only discussed "diagrammatic integration". In this penultimate section we show that our results extend to the generality of diagrammatic differential operators ([BLeT, Th]). These are diagrammatic analogues of differential operators and are important in quantum topology. Perhaps the best known use of diagrammatic differential operators comes from the celebrated wheels and wheeling theorems, first proved in [BLeT]. Wheels gives the value of the Kontsevich integral of the unknot Ω (see Section 9), and wheeling states that, in the notation below, $\chi \circ \partial_\Omega : \mathcal{B} \to \mathcal{A}$ is an algebra isomorphism when $\mathcal{B}$ has one colour, the 1-manifold of $\mathcal{A}$ is connected and χ is the PBW isomorphism of vector-spaces. We will not pursue this further and instead move directly to the combinatorial problem.

The bilinear form $\langle \cdot \, , \, \cdot \rangle$ applied to g and h in $\mathcal{D}(Y)$ has the property that whenever the number of y-coloured univalent vertices in g and h do not match for some $y \in \mathcal{Y}$, then $\langle g \, , \, h \rangle = 0$. We can relax this condition and declare it non-zero only if all of the univalent vertices of g are glued to univalent vertices of h. We extend this bilinearly for all $\mathcal{D}(\mathcal{Y}) \otimes \mathcal{D}(\mathcal{Y})$ and denote this new operator (which we call a *diagrammatic differential operator*) by $\partial_g \ (h)$, where

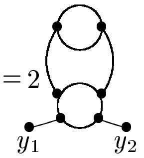
$$\partial_{\cdot} \ (\cdot) : \mathcal{D}(\mathcal{Y}) \otimes \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{Y}).$$

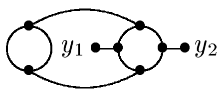
Similarly, $\partial_{\cdot} \ (\cdot)_c$ denotes the primitive part of $\partial_{\cdot} \ (\cdot)$. Note that for $u \in \mathcal{D}(\mathcal{Y})$ we have $\partial_1 \ (u) = u$. The following is an example of this operator.

Let $g_1 =$ 

and $h_1 =$ 

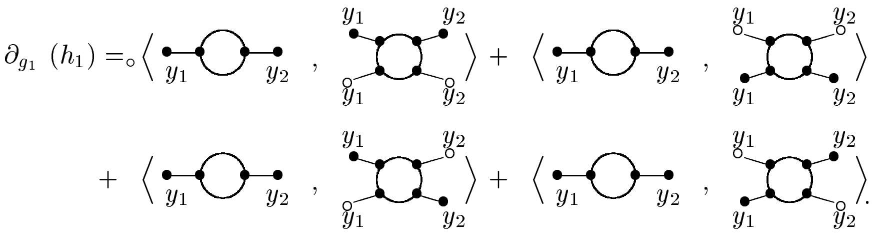
, then

(4) $\partial_{g_1} \ (h_1) = 2$ 

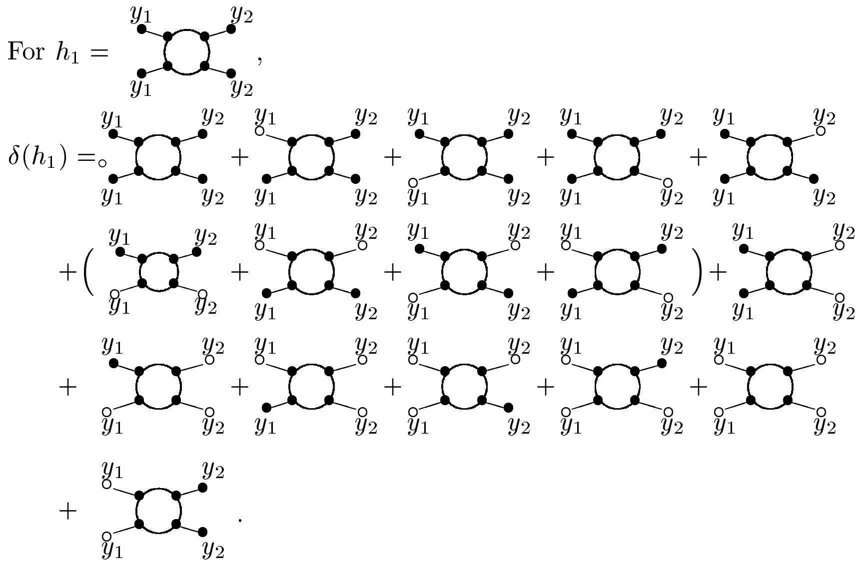
$+ \ 2$ 

.

When calculating $\partial_g \ (h)$, we can specify the coloured univalent vertices of h that will not be identified with the ones in g by marking them as open vertices ($\circ$) and then identifying the remaining ones using $\langle \cdot \, , \, \cdot \rangle$ (by definition, $\langle \cdot \, , \, \cdot \rangle$ treats open vertices as inert). After the identification, all the remaining univalent vertices will be open vertices. By treating these coloured univalent vertices as *filled vertices* ($\bullet$), we can express $\partial_g \ (h)$ as a sum of $\langle g \, , \, \cdot \rangle$. We use $a =_\circ b$ to indicate that $a = b$ where the open vertices of b have been filled. For example from (4),



More concisely, for $h \in \mathcal{D}(\mathcal{Y})$ let $\delta(h) \in \mathcal{D}(\mathcal{Y})$ be the series of uni-trivalent graphs that can be obtained from h by opening univalent vertices in all possible ways. Then $\partial_g(h) =_\circ \langle g, \delta(h)\rangle$. Note that some of the terms of the linear expansion of $\langle g, \delta(h)\rangle$ may be zero. We illustrate this by calculating $\delta(h_1)$ for h_1 in (4),



Note that the terms in parenthesis are the ones that give a nonzero contribution in $\langle g_1, \delta(h_1)\rangle$. Using the above observation we can prove an analogue of the main result.

Corollary 8.1. *Let $B, C \in \mathcal{D}_s(\mathcal{Y})$ be strutless and primitive, then*

$$\partial_{\exp B}\ (\exp C) = \exp\left(\partial_{\exp B}\ (\exp C)_c\right).$$

Proof. Notice that if $C \in \mathcal{D}_s(\mathcal{Y})$ is strutless and primitive, so is $\delta(C)$. It is clear that $\partial_{\exp B}\ (\exp C) =_\circ \langle \exp B\ ,\ \delta(\exp C)\rangle = \langle \exp B\ ,\ \exp \delta(C)\rangle$. But by Theorem 2.1 we have the relation $\langle \exp B\ ,\ \exp \delta(C)\rangle = \exp \langle \exp B\ ,\ \exp \delta(C)\rangle_c$. But $\exp \partial_{\exp B}\ (\exp C)_c\ =_\circ\ \exp \langle \exp B\ ,\ \exp \delta(C)\rangle_c$, completing the proof. □

We hope that Corollary 8.1 may prove useful for finding expressions for values of the Kontsevich invariant in algebras other than $\mathcal{B}$, perhaps through the use of the wheeling theorem mentioned above.

9. Examples

Although the LMO invariant can be computed algorithmically to any finite degree, there are few known examples of the full values of this invariant. Known explicit examples include lens spaces ([LeMO, BNR]) and certain Seifert fiber spaces ([BNR]). As some applications of our results we shall use Theorem 2.1, its corollaries and results of Bar-Natan and Lawrence to determine the logarithm of the LMO invariant of certain manifolds. The logarithm of the LMO invariant is known as the *primitive LMO invariant*, and is denoted by z^{LMO}.

The principle advantage of looking at primitive finite-type and quantum invariants is that their structure and the coefficients of their terms are often more accessible than the original invariant ([Oh2]). Therefore primitive invariants and the corresponding space of primitive diagrams are well studied in knot theory. In addition to this the primitive LMO invariant is known to behave well under the connect sum operations of 3-manifolds and reversal of orientation ([LeMO]). For example, if M and N are two rational homology spheres and $M\#N$ their connected sum, then $z^{LMO}(M\#N) = z^{LMO}(M) + z^{LMO}(N)$ (to see this note that the framed links representing M and N have disjoint colouring so the formal Gaussian integration can be carried out for each set of variables separately). This formula can be applied to the formulae below to obtain expressions for the sums of the manifolds, although we do not include details here.

Through clever use of the wheels and wheeling formulae, Bar-Natan and Lawrence, in [BNR], gave explicit calculations of the Kontsevich integral of integrally framed Hopf links and Hopf chains. Using these calculations they went on to calculate the LMO invariant of lens spaces, which may be presented as integrally framed Hopf chains ([Ro]) and certain Seifert fiber spaces which have a simple "key chain" presentation ([Mon, Sc]). By considering these results, we use our formulae to calculate the primitive LMO invariants of these manifolds.

Let ω_{2n} be the *wheel* of degree $2n$, *ie.* the uni-trivalent graph made from a $2n$-gon with an additional edge coming out from each vertex. We assume ω has x-coloured univalent vertices. Also let $\Omega_x = \exp\left(\sum_{m=1}^{\infty} b_{2m}\, \omega_{2m}\right)$ denote the Kontsevich integral of the unknot and $\Omega_{x/p} = \exp\left(\sum_{m=1}^{\infty} b_{2m}/p^{2m}\, \omega_{2m}\right)$, where the $b_{2m} \in \mathbb{Q}$ are the modified Bernoulli numbers (see *e.g.* [BLeT]). Finally θ denotes the planar trivalent graph with two vertices. All vertex orientations are inherited from the plane.

Bar-Natan and Lawrence show (in the proof of their Proposition 5.1) that the LMO invariant of the (p,q) lens space is given by the formula

$$Z^{\mathrm{LMO}}(L_{p,q}) = \exp\left(\frac{-S(q/p)}{48}\,\theta\right) \langle \Omega_x\, ,\, \Omega_x \rangle^{-1} \left\langle \Omega_x\, ,\, \Omega_{x/p} \right\rangle,$$

where $S(q/p) \in \mathbb{Q}$ is the Dedekind symbol ([KM]), whose definition we do not include here. We may either apply Theorem 2.1 to this formula and obtain

$$Z^{\mathrm{LMO}}(L_{p,q}) = \exp\left(\frac{-S(q/p)}{48}\,\theta\right) \exp\left(\langle \Omega_x\, ,\, \Omega_x \rangle_c\right)^{-1} \exp\left(\left\langle \Omega_x\, ,\, \Omega_{x/p} \right\rangle_c\right),$$

or one can use Corollary 3.5 of [BNR] to write $\langle \Omega_x\, ,\, \Omega_x \rangle^{-1} \left\langle \Omega_x\, ,\, \Omega_{x/p} \right\rangle$ as $\left\langle \Omega_x\, ,\, \Omega_x^{-1}\Omega_{x/p} \right\rangle$, then apply Theorem 2.1 to get

$$Z^{\mathrm{LMO}}(L_{p,q}) = \exp\left(\frac{-S(q/p)}{48}\,\theta\right) \exp\left(\left\langle \Omega_x\, ,\, \Omega_x^{-1}\Omega_{x/p} \right\rangle_c\right).$$

Since $\mathcal{B}$ is a commutative algebra we obtain the following.

Proposition 9.1. *The primitive LMO invariant of a (p,q) lens space is given by*

$$
\begin{aligned}
z^{LMO}(L_{p,q}) &= \left\langle \Omega_x \, , \, \Omega_{x/p} \right\rangle_c - \left\langle \Omega_x \, , \, \Omega_x \right\rangle_c - \frac{S(q/p)}{48}\,\theta \\
&= \left\langle \Omega_x \, , \, \Omega_x^{-1}\Omega_{x/p} \right\rangle_c - \frac{S(q/p)}{48}\,\theta.
\end{aligned}
$$

As our concluding example, if $M = S^3(b, p_1/q_1, \ldots, p_n/q_n)$ is the Seifert fibered space with a spherical base described in Section 5.2 of [BNR], one can use Bar-Natan and Lawrence's formula and Theorem 2.1 in a similar way to calculate its primitive LMO invariant as

$$
\begin{aligned}
z^{\mathrm{LMO}}(M) = &\left\langle \exp\left(\frac{1}{2e_0}\, x \bullet\!\frown\!\bullet\, x\right) \, , \, \Omega_x^{2-n} \prod_i \Omega_{x/p_i} \right\rangle_c - \langle \Omega_x, \Omega_x \rangle_c \\
&+ \frac{1}{4}\left(\lambda_\omega(M) + \frac{1}{12e_0}\left(n - 2 - \sum_i \frac{1}{p_i^2}\right)\right)\theta,
\end{aligned}
$$

where $\lambda_\omega(M)$ denotes the Casson-Walker invariant ([Wa]) of M and $e_0 := b + \sum_i q_i/p_i$.

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