GRAD AND CLASSES WITH BOUNDED EXPANSION III. RESTRICTED GRAPH HOMOMORPHISM DUALITIES

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Abstract. We study restricted homomorphism dualities in the context of classes with bounded expansion. This presents a generalization of restricted dualities obtained earlier for bounded degree graphs and also for proper minor closed classes. This is related to distance coloring of graphs and to the "approximative version" of Hadwiger conjecture.

1. Introduction

We motivate this paper by the following two examples.

Example1. Celebrated Grötzsch's theorem (see e.g. [2]) says that every planar graph is 3-colourable. In the language of homomorphisms this says that for every triangle free planar graph G there is a homomorphism of G into K_3. Here a *homorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ which preserves adjacency: $\{f(x), f(y)\} \in E(H)$ whenever $\{x,y\} \in E(G)$. $G \xrightarrow{f} H$ or $f : G \longrightarrow H$ denotes that f is a homomorphism from G to H. The existence of a homomorphism from G to H is noted $G \longrightarrow H$, while the non-existence of such a homomorphism is noted $G \not\longrightarrow H$. It is also clear that the relation $G \leq H$ defined as $G \longrightarrow H$ is a quasiorder on the class of all finite graphs. This quasiorder becames partial order if we restrict it to the class of all minimal retracts (i.e. *cores*). This partial order is called *homomorphism order*. See [5] for a recent introduction to graphs and homomorphisms.

Using the partial order terminology the Grötzsch's theorem says that K_3 is an upper bound (in the homomorphism order) for the class $\mathcal{P}_3$ of all planar triangle free graphs. As obviously $K_3 \notin \mathcal{P}_3$ a natural question (first formulated in [9]) suggests: Is there yet a smaller bound? The answer, which may be viewed as a strengthening of Grötzsch's theorem, is positive. Thus there exists a triangle free 3-colorable graph H such that $G \longrightarrow H$

for every graph $G \in \mathcal{P}_3$. This has been proved in [15, 12] in a stronger version for minor closed classes. The case of planar graphs and triangle is interesting in its own and it has been related to the Seymour conjecture and Guenin's theorem [3], see [7] and seems to found a proper setting in the context of TT-continuous mappings, see [16]. Restricted duality results have been generalized since to proper minor closed classes of graphs and to other forbidded subgraphs. In fact to any finite set of connected graphs, see [15]. This then implies that Grötzsch's theorem can be strengthened by a sequence of ever stronger bounds and that the supremum of the class of all triangle free planar graphs does not exist, see [11].

Example2. Let us consider all sub-cubic graphs (i.e. graph with maximum degree ≤ 3). By Brooks theorem (see e.g. [2]) all these graphs are 3-colorable with the single connected exception K_4. What about the class of all sub-cubic *triangle free* graphs? Does there exists a triangle free 3-colorable bound? The positive answer to this question is given in [20] and [4]. In fact for every finite set $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ of connected graphs there exists a graph H with the following properties:

- H is 3-chromatic;

- $G \longrightarrow H$ for every subcubic graph $G \in \mathrm{Forb_h}(\mathcal{F})$.

(Here $\mathrm{Forb_h}(\mathcal{F})$ is the class of all graphs G which satisfy $F_i \not\longrightarrow G$ for every $i = 1, 2, \ldots, t$.) In this case we briefly say that the class of all sub-cubic graphs has all *restricted dualities*. (We shall motivate this terminology below.)

It is interesting to note that while sub-cubic graphs have restricted dualities (and, more generally, this also holds for the classes of bounded degree graphs) for the classes of degenerated graphs a similar statement is not true (in fact, with a few trivial exceptions, it is never true), see [9, 11].

Where lies the boundary for validity of restricted dualities? This is the central question of this paper. We give a very general sufficient condition for a class to have restricted dualities. But first we introduce another source for restricted dualities. Chronologically this is also the original cotext.

The following is a partial order formulation of an important homomorphism (or coloring) problem:

Definition 1.1. A pair F, D of graphs is called *dual pair* if for every graph G holds:

(1) $$F \not\longrightarrow G \iff G \longrightarrow D.$$

We also say that F and D form a duality, D is called *dual of F*. Dual pairs of graphs and even of relational structures were characterized in [17], the notion itself goes back to [8]. Equivalently, one can describe a dual pair F, D by saying that for the class $\mathrm{Forb_h}(F)$ the graph D is the maximum graph (in the homomorphism order).

It appears (and this is the main result of [17]) that (up to the homomorphism equivalence) all the dualities are of the form (T, D_T) where T is a finite (relational) tree. Every dual D_T is uniquelly determined by the tree T (but its structure is by far more difficult to describe, see e.g. [18, 19]). These results imply in most cases infinitely many examples. But a much richer spectrum (and in fact a surprising richness of results) is obtained by restricting the validity of (1) to a particular class of graphs $\mathcal{K}$:

Definition 1.2. A class $\mathcal{K}$ admits a *restricted duality* if, for any finite set of connected graphs $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$, there exists a finite graph $D_{\mathcal{F}}^{\mathcal{K}}$ such that $F_i \not\longrightarrow D_F^{\mathcal{K}}$ for $i = 1, \ldots, t$ and such that for all $G \in \mathcal{K}$ holds:

$$(F_i \not\longrightarrow G), i = 1, 2, \ldots, t, \iff (G \longrightarrow D_F^{\mathcal{K}})$$

.

It is easy to see that using the homomorphism order we can reformulate this definition as follows: A class $\mathcal{K}$ has restricted duality if for any finite set of connected graphs $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ the class $\mathrm{Forb_h}(\mathcal{F}) \cap \mathcal{K}$ is bounded in the class $\mathrm{Forb_h}(\mathcal{F})$.

In our companion papers [13, 14] we defined the notion of grad and bounded expansion class. For the benefit of the reader we recall these definitions in Section 2. The following is then the main result of this paper:

Theorem 1.1. *Any class of graphs with bounded expansion has all restricted dualities.*

As both proper minor closed classes and bounded degree graphs form classes of bounded expansion this result generalizes both Examples 1. and 2. In fact the seeming incomparability of bounded degree graphs and minor closed classes led us to the definition of bounded expansion classes.

This paper is organized as follows. In Section 2 we recall basic definitions and results of [13] which will be needed. In Section 3 we reformulate the restricted dualities in terms of local homomorphism properties and introduce the basic construction. In Section 4 we prove Theorem 1.1 and in Section 5 we list several corollaries. Among them is a surprising result that exact odd powers of graphs in any given bounded expansion class have bounded chromatic number.

2. Bounded expansion classes.

In [15], we introduced the *tree-depth* $\mathrm{td}(G)$ of a graph G as follows:

A *rooted forest* is a disjoint union of rooted trees. The *height* of a vertex x in a rooted forest F is the number of vertices of a path from the root (of the tree to which x belongs to) to x and is noted $\mathrm{height}(x, F)$. The *height* of F is the maximum height of the vertices of F. Let x, y be vertices of F. The vertex x is an *ancestor* of y in F if x belongs to the path linking y and the root of the tree of F to which y belongs to. The *closure* $\mathrm{clos}(F)$ of a rooted forest F is the graph with vertex set $V(F)$ and edge set $\{\{x, y\} : x \text{ is an ancestor of } y \text{ in } F, x \neq y\}$. A rooted forest F defines a partial order on its set of vertices: $x \leq_F y$ if x is an ancestor of y in F. The comparability graph of this partial order is obviously $\mathrm{clos}(F)$. The *tree-depth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest F such that $G \subseteq \mathrm{clos}(F)$. As a consequence, we have an algorithmic definition of the tree depth :

Lemma 2.1 ([15]). *Let G be a graph and let $G_1, \ldots, G_p$ be its connected components. Then:*

$$\mathrm{td}(G) = \begin{cases} 1, & \text{if } |V(G)| = 1; \\ 1 + \min_{v \in V(G)} \mathrm{td}(G - v), & \text{if } p = 1 \text{ and } |V(G)| > 1; \\ \max_{i=1}^{p} \mathrm{td}(G_i), & \text{otherwise.} \end{cases}$$

We say that a class $\mathcal{C}$ has a *low tree-depth coloring* if, for any integer $p \geq 1$, there exists an interger $N(p)$ such that any graph $G \in \mathcal{C}$ may be vertex-colored using $N(p)$ colors so that each of the connected components of the subgraph induced by any $i \leq p$ parts has tree-depth at most i. As it obviously holds $\mathrm{td}(G) \geq \mathrm{tw}(G) - 1$ any class having a low-tree depth coloring has also low tree-width coloring (in the sense of [1].

The existence of low-tree depth colorings is related to the ntion of p-centered coloring, which have also been introduced in [15]: A *p-centered coloring* of a graph G is a vertex coloring such that, for any connected subgraph H, either some color $c(H)$ appears exactly once in H, or H gets at least p colors. For the sake of completeness we recall some results of [15]. These statements establish the relationship of centered colorings and low tree/depth colorings. They are easy to prove (with the exception of Theorem 2.4 which is the central result of [15]):

Lemma 2.2 ([15]). *Let G, H be graphs, let $p = \mathrm{td}(H)$, let c be a q-centered coloring of G where $q \geq p$. Then any subgraph H' of G isomorphic to H gets at least p colors in the coloring c of G.* □

¿From this lemma follows that p-centered colorings induce low tree-depth colorings:

Corollary 2.3. *Let p be an integer, let G be a graph and let c be a p-centered coloring of G.*

Then $i < p$ parts induce a subgraph of tree-depth at most i

Proof. Let G' be any subgraph of G induced by $i < p$ parts. Assume $\mathrm{td}(G') > i$. According to Lemma 2.1, the deletion of one vertex decreases the tree-depth by at most one. Hence there exists an induced subgraph H of G' such that $\mathrm{td}(H) = i + 1 \leq p$. According to lemma 2.2, H gets at least p colors, a contradiction. □

Theorem 2.4. *Any graph G has p-centered coloring for any $p \leq td(G)$.*

The following was established in [15] for the case of proper minor closed classes of graphs. We prove it here in full generality.

Theorem 2.5. *Let $\mathcal{C}$ be a class of graphs having low tree-depth colorings and let p be an integer. Then there exists integer $X(p)$, such that every graph in $\mathcal{C}$ has a p-centered coloring using $X(p)$ colors.*

Proof. Let $G \in \mathcal{C}$. According to the assumption, there exists a vertex partition into $C(p)$ parts, such that any p parts form a graph of tree-depth at most p. This partition will be defined as a coloring $\bar{c} : V(G) \longrightarrow \{1, 2, \ldots, C(p)\}$. For any set P of p parts let G_P be the graph induced by all the parts in P. It is easy to see that any graph G with $\mathrm{td}(G) \leq p$ has a p-centered coloring c by $\mathrm{td}(G($ colors: we simply assign to any connected subgraph H of G the minimal level of a vertex of H in the tree F satisfying $H \subset \mathrm{clos}(F)$ (see the definition of tree depth at the beginning of this section. Consider the following ("product") coloring c defined as

$$c(v) = (\bar{c}(v), (c_P(v); |P| = p, P \subset \{1, 2, \ldots, C(p)\})).$$

Take the product of the coloring of G by $C(p)$ colors and of the colorings of the G_P as a new coloring of G (with $X(p) = C(p)N(p, p-1)^{\binom{C(p)}{p}}$ colors). Let H be a connected subgraph of G. Then, either H gets at least $p+1$ colors, or $V(H)$ is included in some subgraph G_P of G induced by p parts. In the later case, some color appears exactly once in H. □

Recall that the *maximum average degree* $\mathrm{mad}(G)$ of a graph G is the maximum over all subgraphs H of G of the average degree of H, that is $\mathrm{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$. The *distance* $d(x, y)$ between two vertices x

and y of a graph is the minimum length of a pth linking x and y, or ∞ if x and y do not belong to same connected component.

We introduce several notations:

- The *radius* $\rho(G)$ of a connected graph G is:

$$\rho(G) = \min_{r \in V(G)} \max_{x \in V(G)} \mathrm{d}(r, x)$$

- A *center* of G is a vertex r such that $\max_{x \in V(G)} \mathrm{d}(r, x) = \rho(G)$.

Definition 2.1. Let G be a graph. A *ball* of G is a subset of vertices inducing a connected subgraph. The set of all the families of pairwise disjoint balls of G is noted $\mathfrak{B}(G)$.

Let $\mathcal{P} = \{V_1, \dots, V_p\}$ be a family of pairwise disjoint balls of G.

- The *radius* $\rho(\mathcal{P})$ of $\mathcal{P}$ is $\rho(\mathcal{P}) = \max_{X \in \mathcal{P}} \rho(G[X])$
- The *quotient* $G/\mathcal{P}$ of G by $\mathcal{P}$ is a graph with vertex set $\{1, \dots, p\}$ and edge set $E(G/\mathcal{P}) = \{\{i, j\} : (V_i \times V_j) \cap E(G) \neq \emptyset \text{ or } V_i \cap V_j \neq \emptyset\}$.

We introduce several invariants that generalize the one of maximum average degree:

Definition 2.2. The *greatest reduced average density (grad)*of G with rank r is

$$\nabla_r(G) = \max_{\substack{\mathcal{P} \in \mathfrak{B}(G) \\ \rho(\mathcal{P}) \leq r}} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}$$

For the sake of simplicity, we also define:

The *grad of* G:

$$\nabla(G) = \max_r \nabla_r(G) = \max_{H \preceq G} \frac{|E(H)|}{|V(H)|}$$

Notice the two following well known facts (usually expressed by mean of the maximum average degree):

Lemma 2.6. *Let G be a graph. Then G has an orientation such that the maximum indegree of G is at most k if and only if $k \geq \nabla_0(G)$.*

Lemma 2.7. *Let G be a graph. Then G is $\lfloor 2\nabla_0(G) \rfloor$-degenerated, hence $\lfloor 2\nabla_0(G) + 1 \rfloor$-colorable.*

The following is our key definition:

Definition 2.3. A class of graphs $\mathcal{C}$ has *bounded expansion* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathcal{C}$ and every r holds

$$\nabla_r(G) \leq f(r). \tag{2}$$

f is called the expansion function.

The following is a special case of the main result of [13]

Theorem 2.8. *For a class $\mathcal{C}$ of graphs are the following statements equivalent*

- *$\mathcal{C}$ has bounded expansion,*
- *$\mathcal{C}$ has low tree-depth colorings,*
- *$\mathcal{C}$ has p-centered coloring for every $p \geq 1$.*

3. A construction

Definition 3.1. Let G, H be graphs and let $\mathcal{P}$ be a system of subsets of $V(G)$. We say that G is *$\mathcal{P}$-locally homomorphic to H* and we denote $G \overset{\mathcal{P}}{\dashrightarrow} H$ if for every subset $A \in \mathcal{P}$:

$$G[A] \longrightarrow H$$

.

We shall deal mostly with the following systems: If $\phi : V(G) \longrightarrow X$ is a function and p a positive integer then we can consider the system $\mathcal{P} = \{A \subset V(G); |\phi(A)| \leq p\}$. This system will be denoted by $\mathcal{P}_{(\phi,p)}$. In this case we also say that G is (ϕ, p)-locally homomorphic to H (instead of $\mathcal{P}_{(\phi,p)}$-locally homomorphic). This is also denoted by $G \overset{(\alpha,p)}{\dashrightarrow} H$.

Example 1. For $H = K_2$ and ϕ an identical map $V(G) \longrightarrow V(G)$ a graph G is (ϕ, p)-locally homomorphic to H iff the odd-girth of G is $> p$.

The following is a modification of a construction introduced in [12]:

Definition 3.2. Let G, H be finite graphs and let $1 \leq p < |V(H)|$ be an integer. For $v \in V(H)$, define $\mathcal{A}_v = \{(I, v); I \in \binom{V(H)}{p} : v \in I\}$, where $\binom{V(H)}{p}$ stands for the subsets of $V(H)$ with cardinality p. Define the sets $V_v = V(G)^{\mathcal{A}_v}, W = \bigcup_{v \in V(H)} V_v$ and the function $\alpha : W \to V(H)$ by $\alpha(z) = v$ if $z \in V_v$.

The *p-truncated H-power* $G^{\Uparrow_p^H}$ of G is the graph with vertex set W and with the edge set F defined as follows: $\{z, z'\} \in F$ iff $z \in V_v, z' \in V_{v'}$ and for every $I \in \mathcal{A}_v \cap \mathcal{A}_{v'}$ holds $\{z_{(I,v)}, z'_{(I,v')}\} \in E(G)$.

Remark 3.1. $K_1^{\Uparrow_1^H}$ is isomorphic to H.

Remark 3.2. The order of $G^{\Uparrow_p^H}$ is $|V(H)|.|V(G)|^{\binom{|V(H)|-1}{p-1}}$.

The function $\alpha : V(G^{\Uparrow^H_p}) \to V(H)$ is called the *color projection* of $G^{\Uparrow^H_p}$. This is justified by

Lemma 3.3. *The color projection α of $G^{\Uparrow^H_p}$ is a homomorphism from $G^{\Uparrow^H_p}$ to H:*

$$\alpha : G^{\Uparrow^H_p} \longrightarrow H$$

Proof. By definition, for any edge $\{z, z'\}$ of $G^{\Uparrow^H_p}$, there exists an edge $\{v, v'\}$ of H such that $z \in \mathcal{A}_v$ and $z' \in \mathcal{A}_{v'}$, i.e. $v = \alpha(z)$ and $v' = \alpha(z')$. $\square$

Lemma 3.4. *Let α be the color projection of $G^{\Uparrow^H_p}$. Then the graph $G^{\Uparrow^H_p}$ is (α, p)-locally homomorphic to G:*

$$\begin{tikzcd} G^{\Uparrow^H_p} \arrow[r, dotted, "(\alpha,p)"] & G \end{tikzcd}$$

.

Proof. Let A be a subset of $V(G^{\Uparrow^H_p})$ such that $|\alpha(A)| \leq p$. Let I be any subset of $V(H)$ of cardinality p such that $\alpha(A) \subset I$. According to the definition of $G^{\Uparrow^H_p}$, $\{z_{(I,\alpha(z))}, z'_{(I,\alpha(z'))}\} \in E(G)$ for any $\{z, z'\} \in E(G^{\Uparrow^H_p}[A])$. It follows that the mapping $z \mapsto z_{(I,\alpha(z))}$ is a homomorphism from $G^{\Uparrow^H_p}[A]$ to G. $\square$

Lemma 3.5. *Let G, H, U be finite graphs, let p be an integer. Assume that $\gamma : G \longrightarrow H$ is a homomorphims and that G is (γ, p)-locally homomorphic to U. Schematically:*

Assume
$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[d, dotted, "(\gamma,p)"'] & H \\ U & \end{tikzcd}$$

Then there exists a homomorphism $f : G \longrightarrow U^{\Uparrow^H_p}$ such that $\gamma = f \circ \alpha$. Moreover, $U^{\Uparrow^H_p}$ is (α, p)-locally homomorphic to U.

Schematically, this can be expressed by the following scheme:

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[d, dotted, "(\gamma,p)"'] \arrow[dr, "f"] & H \\ U & U^{\Uparrow^H_p} \arrow[l, dotted, "(\alpha,p)"] \arrow[u, "\alpha"'] \end{tikzcd}$$

Proof. For $I \in \binom{V(H)}{p}$ put $G_I = G[\gamma^{-1}(I)]$ and let g_I be a homomorphism from G_I to U. Define f as follows: Given $x \in V(G)$ we define $f(x) \in gamma(x)$ by the following formula $f(x)(I, \gamma(x)) = g_I(x)$ (see the above definition of $\Uparrow_p^H$). Obviously $f \circ \alpha = \gamma$. We prove that f is a homomorphism.

Let $\{x, y\}$ be any edge. It is $\{\gamma(x), \gamma(y)\} \in E(H)$ as $\gamma : G \longrightarrow H$. For any I which contains both $\gamma(x)$ and $\gamma(y)$ holds $\{f(x)_{(I,\gamma(x))}, f(y)_{(I,\gamma(y)} = \{g_I(x), g_I(y)\} \in E(U)$. It follows that $\{f(x), f(y)\}$ is an edge of $U^{\Uparrow_p^H}$ and thus f is a homomorphism. □

This lemma highlights a fundamental property of $G^{\Uparrow_p^H}$ which we will state as follows:

Lemma 3.6. *Let G, H, U be finite graphs and let p be an integer. Then: there is a homomorphism $G \longrightarrow U^{\Uparrow_p^H}$ iff there exists a homomorphism $\gamma : G \longrightarrow H$ and G is (γ, p)-locally homomorphic to U. Schematically, this may be depicted as follows:*

$$G \longrightarrow U^{\Uparrow_p^H} \qquad \Longleftrightarrow \qquad \begin{tikzcd} G \arrow[r, "\gamma"] \arrow[d, dotted, "{(\gamma,p)}"'] & H \\ U \end{tikzcd}$$

Proof. First, assume $f : G \longrightarrow U^{\Uparrow_p^H}$. Let α be the color projection of $U^{\Uparrow_p^H}$ to H. Put $\gamma = \alpha \circ f$. We have $\gamma : G \longrightarrow H$. Let $A \subseteq V(G)$. The condition $|\gamma(A)| \leq p$ is equivalent to the condition $|\alpha(f(A))| \leq p$. Hence the homomorphism $f : G \longrightarrow U^{\Uparrow_p^H}$ together with the (α, p)-local homomorphism of $\Uparrow_p^H$ to U implies (γ, p)-local homomorphism of $\Uparrow_p^H$ to U. The reverse implications follows from the previous lemma. □

It is interesting to note that if we consider $G = U$ we get:

Corollary 3.7. $G \longrightarrow G^{\Uparrow_p^H} \iff G \longrightarrow H$. *In particular, G is homomorphism-equivalent to $G^{\Uparrow_p^G}$.*

Theorem 3.8. *A class of graphs $\mathcal{C}$ has restricted dualities iff for any finite set $\mathcal{F}$ of graphs there exist a graph H and a graph $U \in \mathrm{Forb_h}(\mathcal{F})$ such that for every $G \in \mathcal{C} \cap \mathrm{Forb_h}(\mathcal{F})$ there exists a homomorphism $\gamma : G \longrightarrow H$ for which G is (γ, p)-locally homomorphic to U, where $p = \max\{|V(F)|; F \in \mathcal{F}\}$.*

Proof. If $\mathcal{C}$ has restricted dualities and a set $\mathcal{F}$ of graphs has a dual $D_{\mathcal{F}}^{\mathcal{C}}$ then we may put $U = H = D_{\mathcal{F}}^{\mathcal{C}}$.

Now assume that graphs U, H and a homomorphism γ exist. Put $p = \max\{|V(F)|; F \in \mathcal{F}\}$. In this situation we prove that $U^{\Uparrow_p^H}$ is a dual of $\mathcal{F}$. Then for every $F \in \mathcal{F}$ holds $F \not\longrightarrow U^{\Uparrow_p^H}$. (Suppose contrary, let $F \xrightarrow{g} U^{\Uparrow_p^H}$. By Lemma 3.5 $U^{\Uparrow_p^H}$ is (p,p)-locally homomorphic to U and this together with $|g(V(F))| \leq |V(F)| \leq p$ would imply $F \longrightarrow U$. If $F \not\longrightarrow G$ for every $F \in \mathcal{F}$ then $G \longrightarrow U^{\Uparrow_p^H}$ according to Lemma 3.6. If $G \longrightarrow U^{\Uparrow_p^H}$ then $F \not\longrightarrow G$ as $F \longrightarrow G$ would imply $F \longrightarrow U^{\Uparrow_p^H}$. $\square$

4. Restricted dualities

We shall need one more ("finitness") result proved in [15], Corollary 3.3:

Lemma 4.1. *For any positive integer p there exists a number $F(G))$ any graph G with $\mathrm{td}(G) \leq p$ is hom-equivalent to one of its induced subgraph of order at most $F(\mathrm{td}(G))$.*

Theorem 4.2. *Let $\mathcal{F}$ be a finite set of finite connected graphs. Then, for any class of graph $\mathcal{K}$ with bounded expansion there exists a finite graph $U(\mathcal{K},\mathcal{F}) \in \mathrm{Forb_h}(\mathcal{F})$ such that any graph of $\mathcal{K} \cap \mathrm{Forb_h}(\mathcal{F})$ has a homomorphism to $U(\mathcal{K},\mathcal{F})$.*

Proof. Let $p = \max_{F \in \mathcal{F}} |V(F)| + 1$. There exists an integer N, such that any graph $G \in \mathcal{K}$ has a proper N-coloring in which any p colors induce a graph of tree depth at most p. According Lemma 4.1, there exists a finite set $\hat{\mathcal{D}}_k$ of graphs with tree depth at most k, so that any graph with tree-depth at most k is hom-equivalent to one graph in the set. Let $U(\mathcal{D}_k, F)$ be the disjoint union of the graphs in $\hat{\mathcal{D}}_k \cap \mathrm{Forb_h}(F)$. In this situation we can use Theorem 3.8 and put $U(\mathcal{K},\mathcal{F}) = U(\mathcal{D}_k, F)^{\Uparrow(p+1)\cdots\Uparrow N}$. $\square$

5. Concluding remarks

1. On Hadwiger conjecture

Let us list the following corollary of Theorem 4.2

Corollary 5.1. *Let $\mathcal{K}$ be a proper minor closed class of graphs. Let $\mathcal{F}$ be a finite set of finite connected graphs. Then there exists a finite graph $U(\mathcal{K},\mathcal{F}) \in \mathrm{Forb_h}(\mathcal{F})$ such that any graph of $\mathcal{K} \cap \mathrm{Forb_h}(\mathcal{F})$ has a homomorphism to $U(\mathcal{K},\mathcal{F})$.*

It is well known ([5]) that one can reformulate the Hadwiger conjecture as the existence of a maximum (in the homomorphism order) for every proper minor closed class. Let $h = h(\mathcal{K})$ be the Hadwiger number of the class $\mathcal{K}$.

Then $K_{h+1} \notin \mathcal{K}$ and Corollary 5.1 gives at least a K_{h+1}-free bound of the class $\mathcal{K}$. In fact we can get a bound with any set of the same local properties as the class $\mathcal{K}$ itself.

2. On bounded expansion classes

Let $\mathcal{K}$ be the class of all graphs G which have bounded expansion with the expansion function f. Formally, $\mathcal{K} = \{G; \nabla_r(G) \leq f(r), r = 1, 2, \ldots\}$. Assume that p is minimal with $K_p \notin \mathcal{K}$. Then $\nabla_0(G) \leq p - 1$ for every $G \in \mathcal{K}$. Thus every $G \in \mathcal{K}$ is $p-1$-degenerated, If f is monotonne then also $K_p \in \mathcal{K}$ and thus $\mathcal{K}$ has maximum. Thus Hadwiger conjecture holds for bounded expansion classes determined by a monotonne expansion function.

Note also that for constant expansion functions the bounded expansion classes are proper just minor closed classes. This may be seen as follows:

Assume that $\mathcal{K}$ is bounded expansion class bounded by a constant function C. Explicitely, we assume that for every $G \in \mathcal{K}$ holds $\nabla_r(G) \leq C$ and thus also $\nabla(G) \leq C$. Let H be a minor of G. Then obviously $\nabla(H) \leq \nabla(G) \leq C$ and thus $\mathcal{K}$ is minor closed. It follows that K_{2C+1} is a forbidden minor of $\mathcal{K}$.

3. On distal colorings - exact powers

We now explain a particular consequence of our main result in a greater detail. Let G be a graph, p a positive integer. Denote by $G^{\natural p}$ the graph $(V, E^{\natural p})$ where $\{x, y\}$ is an edge of $E^{\natural p}$ iff there exists a path P in G from x to y of length p. The graph $G^{\natural p}$ could be called *exact p-power* of G. Clearly graphs $G^{\natural 2}$ and all graphs $G^{\natural p}, p$ even, may have unbounded chromatic number even for the case of trees (consider subdivision stars), and the only (obvious) bound is $\chi(G^{\natural p}) \leq \Delta(G)^p + 1$. Similarly, for any odd p there are 3-colorable graphs G for which is the chromatic number $\chi(G^{\natural p})$ may be arbitrarily large. However for p odd and arbitrary proper minor closed class and even class with bounded expansion we have the following (perhaps surprising):

Theorem 5.2. *For any class $\mathcal{K}$ with bounded expansion and for every odd integer $p \geq 1$, there exists an integer $N = N(\mathcal{K}, p)$ such that all the graphs $G^{\natural p}, G \in \mathcal{K}$ and* $\mathrm{odd-girth}(G) > p$ *have chromatic number $\leq N$: For any $G \in \mathcal{K}$,*

$$\mathrm{odd-girth}(G) > p \Longrightarrow \chi(G^{\natural p}) \leq N$$

Theorem 5.2 follows immediately from Theorem 4.2. It suffices to consider $\mathcal{F} = \{C_p\}$. In this case any graph $U(\mathcal{K}, \mathcal{F}) \in \mathrm{Forb_h}(\mathcal{F})$ and any homomorphism $c : G \longrightarrow U(\mathcal{K}, \mathcal{F})$ gives a desired coloring by $N = |(|V(U(\mathcal{K}, \mathcal{F}))$ colors.

With a little more care one can prove the following result which we state without proof (as it may be generalized in yet another direction, see [10]. We take time for a definition of *exact distance* graph: Let G be a graph, p a positive integer. Denote by $G^{[\natural p]}$ the graph $(V, E^{[\natural p]})$ where $\{x, y\}$ is an edge of $E^{[\natural p]}$ iff the distance of x and y in G is p.

Theorem 5.3. *For any class $\mathcal{K}$ with bounded expansion and for every odd integer $p \geq 1$, there exists an integer $N' = N(\mathcal{K}, p)$ such that all the graphs $G^{[\natural p]}, G \in \mathcal{K}$ have chromatic number $\leq N$.*

Note that in both Theorems 5.2,5.3 we cannot replace the conditions in the definition of powers $G^{\natural p}, G^{[\natural p]}$ by the existence of a path (or even induced path) of length p. See [10] for a more detailed disscussion).

4. On universality of posets. It follows from [6] that the class $\mathcal{SPG}$ of all finite series paralel graphs is universal partial ordered class. What this means (using non-trivial result that the homomorphism order of all finite graphs is universal, [6, 5]) is that to every finite graph G we can associate a series parallel graph $\Phi(G)$ such that for any two graphs G, H

$$G \leq H \iff \Phi(G) \leq \Phi(H).$$

Thus bounded tree width graphs form a homomorphism universal class. Note that for the tree depth such a statement is deeply not true as we cannot even find an infinite antichain. In fact up to homomorphism equivalence the class of all graphs with a bounded tree depth is a finite class (see Lemma 4.1). Also this indicate that low tree-depth partitions are much more restrictive than low tree width partitions.

5. Regular partitions.

Implicit in our proof of restricted dualities is the following partition result. By Lemma 4.1 there exists a finite set $\hat{\mathcal{D}}_p$ of graphs with tree depth p, so that any graph with tree-depth p is hom-equivalent to one graph in the set. This implies:

Theorem 5.4. *For every class $\mathcal{K}$ with bounded expansion and for every positive integer p there exists a positive integer $N = N(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ there exists a coloring $V(G) = V_1 \cup \ldots \cup V_N$ such that the subgraph G_J of G induced by any $j = |J|$ classes has the following property:*

- *each component of G_J is homomorphism aequivalent to one of the graphs in the finite set $\hat{\mathcal{D}}_p$.*

This stronger decomposition theorem may be used for an alternative proof of Theorem 4.2. Finally note that our results may be regarded as (very) regular partitions of graphs with bounded expansion. While the celebrated Szemeredi regularity lemma [21] applies to dense graphs the classes of bounded expansion are on the other side of spectrum: their edge densities are (hereditarily) small. For these classes one can the achieve a very regular partitions, essentially describing all components which may occur in graphs induced by a few color classes.

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