

On the computational complexity of the $L_{(2,1)}$ -labeling problem for regular graphs ^{*}

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Abstract. An $L_{(2,1)}$ -labeling of a graph of span t is an assignment of integer labels from $\{0, 1, \dots, t\}$ to its vertices such that the labels of adjacent vertices differ by at least two, while vertices at distance two are assigned distinct labels.

We show that for all $k \geq 3$, the decision problem whether a k -regular graph admits an $L_{(2,1)}$ -labeling of span $k + 2$ is NP-complete. This answers an open problem of R. Laskar.

1 Introduction

Motivated by models of channel assignment in wireless communication [7, 6], generalized graph coloring and in particular the concept of $L_{(2,1)}$ -labeling have drawn significant attention in the graph-theory community in the past decade [1].

Besides the practical aspects, also purely theoretical questions became very interesting. Among other we shall highlight a long-lasting conjecture of Griggs and Yeh that the span of any optimal $L_{(2,1)}$ -labeling is upperbounded by $\Delta(G)^2$, where $\Delta(G)$ is the maximum degree of the given graph G [6]. So far this conjecture is still open, though it has been verified for various classes of graphs (e.g., for chordal graphs [11, 8] or for graphs of diameter at most two [6]).

We focus our attention on the computational complexity of the decision problem whether a given graph G allows an $L_{(2,1)}$ -labeling of span at most λ .

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If λ is a part of the input, the problem becomes NP-complete by a reduction from the Hamiltonian path problem [6]. If λ is a fixed constant (i.e., the parameter of the problem), the computational complexity was settled in [3], by constructing a polynomial time algorithm for $\lambda \leq 3$ and by showing that the problem is NP-complete otherwise. The core argument of the NP-hardness proof is based on the fact that vertices of high degree may allow only extremal labels (i.e., 0 or λ) of the given spectrum.

In response to this fact, R. Laskar asked at the DIMACS/DIMATIA/-Rényi Workshop on Graph Colorings and their Generalizations (Rutgers University, 2003) what is the computational complexity of the $L_{(2,1)}$ -labeling problem when restricted to regular graphs, hoping that the restriction might provide new ideas for a general proof of hardness results on distance constrained labelings. In this note we settle the computational complexity of the $L_{(2,1)}$ -labeling problem on regular graphs in the following sense:

Theorem 1. *For every integer $k \geq 3$, it is NP-complete to decide whether a k -regular graph admits an $L_{(2,1)}$ -labeling of span (at most) $\lambda = k + 2$.*

The result is the best possible in terms of the span, since no k -regular graph (for $k \geq 2$) allows an $L_{(2,1)}$ -labeling of span $k + 1$ (see e.g. a paper by Georges and Mauro [5] on labelings of regular graphs). Though our result is not totally unexpected, the reduction (namely the garbage collection) is surprisingly uneasy to design. It utilizes so called multicovers introduced in [10].

The paper is organized as follows: The next section provides necessary definitions and facts used latter. In Section 3 we prepare tools used in the construction and discuss their properties. The main result is then proven in Section 4.

2 Preliminaries

All graphs considered in this paper are finite and simple, i.e., with a finite vertex set and without loops or multiple edges. A graph G is denoted as a pair (V_G, E_G) , where V_G stands for a finite set of vertices and E_G is a set of edges, i.e. unordered pairs of vertices. The distance $\text{dist}_G(u, v)$ between two vertices u and v of a graph G is the length (the number of edges) of a shortest path connecting u and v . If two vertices belong to different components, we let their distance be unspecified.

The set of vertices adjacent to a vertex u is called *the neighborhood* of u and it is denoted by $N_G(u)$. The *degree* of a vertex u is the cardinality of its neighborhood, i.e., $\deg(u) = |N_G(u)|$. A graph is called *k-regular* if all its vertices are of degree k .

A vertex labeling by nonnegative integers $f : V_G \rightarrow \mathbb{Z}_0^+$ is called an $L_{(2,1)}$ -labeling of G if $|f(u) - f(v)| \geq 2$ holds for any pair of adjacent vertices u and v , and the labels $f(u)$ and $f(v)$ are distinct whenever $\text{dist}(u, v) = 2$.

The *span* of an $L_{(2,1)}$ -labeling is the difference between the largest and the smallest label used. The parameter $\lambda_{(2,1)}(G)$ is the minimum possible span of an $L_{(2,1)}$ -labeling of G . Such a labeling will be called *optimal*, and we may assume that it uses labels from the discrete interval $[0, \dots, \lambda_{(2,1)}(G)]$.

With an optimal labeling f we associate its *symmetric* labeling f' , defined by $f'(u) = \lambda_{(2,1)}(G) - f(u)$. Clearly the symmetric labeling is also optimal.

$L_{(2,1)}$ -labelings are closely related to graph covers: A *full covering projection* from a graph G to a graph H is a graph homomorphism $h : V_G \rightarrow V_H$ such that the neighborhood $N_G(u)$ of any vertex $u \in V_G$ is mapped bijectively on the neighborhood $N_H(h(u))$ of $h(u)$.

Similarly, if the mapping is locally injective, i.e., if $N_G(u)$ is mapped injectively into $N_H(h(u))$, we call h a *partial covering projection*. Obviously every full covering projection is also a partial covering projection.

The relationship between $L_{(2,1)}$ -labelings and (partial) covering projections was discussed in [2]:

Proposition 1. *Every $L_{(2,1)}$ -labeling of a graph G of span λ corresponds to a partial covering projection $G \rightarrow \overline{P_{\lambda+1}}$, and vice versa.*

In particular, $\overline{C_{\lambda+1}} \subset \overline{P_{\lambda+1}}$, hence every partial covering projection to $\overline{C_{\lambda+1}}$ is also an $L_{(2,1)}$ -labeling of span at most λ .

Kratochvíl, Proskurowski and Telle [10] gave an explicit construction of a special multicover graph allowing many extensions to full covering projections. We will use it in our gadgets.

Proposition 2 ([10]). *For any regular graph F , there exists a graph H (called a multicover of F) with a distinguished vertex $u \in V_H$ such that any locally injective homomorphism $h' : N_H(u) \cup u \rightarrow F$ can be extended to a locally bijective homomorphism $h : H \rightarrow F$.*

3 Gadgets

3.1 Polarity gadget

Let $k \geq 3$ be a positive integer. Consider the graph F_p on $k + 5$ vertices $v_1, \dots, v_{k-1}, u_1, \dots, u_4, x, y$, with edges defined as follows:

$$\begin{aligned}
 E(F_p) = & \{(v_i, v_j) \mid 1 \leq i, j \leq k-1, |i-j| \geq 2\} \\
 & \cup \{(v_i, u_j) \mid 1 \leq i \leq k-1, j = 1, 2, 3\} \\
 & \cup \{(v_i, u_4) \mid 2 \leq i \leq k-2, \} \\
 & \cup \{(u_1, u_2), (u_3, u_4), (u_4, x), (u_4, y)\}
 \end{aligned}$$

See Fig. 1 for an example of such a graph. Observe also that each vertex except x and y is of degree k .

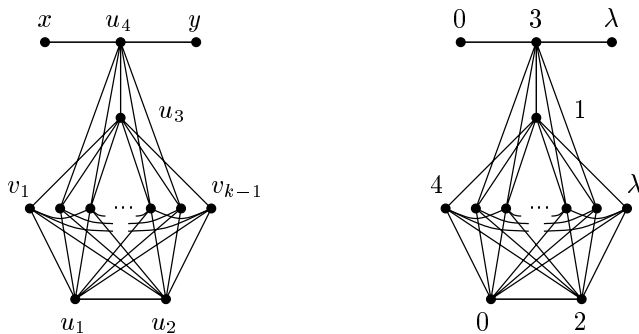


Fig. 1. The polarity gadget F_p and its $L_{(2,1)}$ -labeling.

Lemma 1. *In the graph F_p , the pair of vertices x and y are labeled by 0 and λ (or vice versa) under any $L_{(2,1)}$ -labeling f of span $\lambda = k + 2$.*

Proof. The edge (u_1, u_2) participates in $k - 1$ triangles. If both u_1 and u_2 were labeled by labels different from 0 and λ , then at most $\lambda - 4 < k - 1$ labels would remain for v_1, \dots, v_{k-1} , which is insufficient. So without loss of generality we may assume $f(u_1) = 0$, and then u_2 may get only two possible labels: 2 or λ . The latter would, however, exclude all possible choices for u_3 .

Now up to a symmetry of labelings we have $f(u_1) = 0$, $f(u_2) = 2$ and for the vertices v_1, \dots, v_{k-1} remain the labels $4, 5, \dots, \lambda$. Since F_p restricted onto these vertices is the complement of a path on $k - 1$ vertices, only two possible labelings exist: either $f(v_i) = i + 3$ or $f(v_i) = \lambda + 1 - i$. In both cases (they are equivalent under an automorphism of F_p) only one possible label remains for the vertex u_3 , namely $f(u_3) = 1$. We deduce by similar arguments that $f(u_4) = 3$.

Finally, since u_4 is adjacent to vertices labeled by $1, 5, \dots, \lambda - 1$, its remaining neighbors x and y must be labeled either one by 0 and the other one by λ as claimed.

3.2 Swallowing gadget

In our construction we involve multicovers allowing two different $L_{(2,1)}$ -labelings as follows. Let H, u be a multicover of the k -regular graph \underline{C}_{k+3} . For the swallowing gadget we take two copies H_1, H_2 of the graph H (with the notation that the copy of vertex v in H_i is denoted by v_i , $i = 1, 2$), insert two new vertices x, y and modify the edge set as follows:

$$E(F_s) = (E(H_1) \cup E(H_2) \setminus \{(u_1, v_1), (u_2, v_2)\}) \cup \{(x, v_1), (y, v_2), (u_1, u_2)\})$$

where v is an arbitrary neighbor of u in H . Observe that again all vertices except x and y are of degree k . The construction of the swallowing gadget is illustrated in Fig. 2.

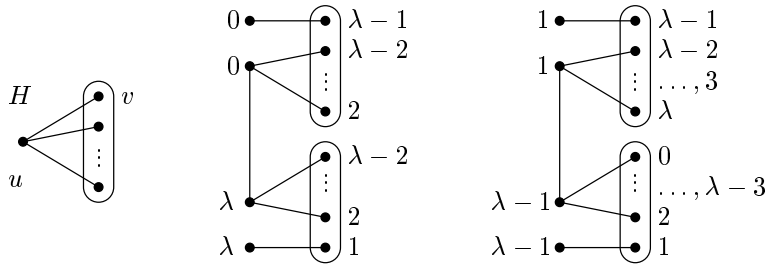


Fig. 2. The swallowing gadget F_s and the two of its $L_{(2,1)}$ -labelings.

Lemma 2. *The graph F_s allows two $L_{(2,1)}$ -labelings f and f' , such that $f(v_1) = f'(v_1) = \lambda - 1$, $f(v_2) = f'(v_2) = 1$, while $f(x) = 0$, $f'(x) = 1$ and $f(y) = \lambda$, $f'(y) = \lambda - 1$.*

Proof. We label the vertices of $\overline{C_{k+2}}$ by integers $[0, \lambda]$, in a usual sequential way. For the construction of f we choose the covering projection from H_1 to $[0, \lambda]$ where u is mapped on 0 and v on $\lambda - 1$. The remaining neighbors of u are mapped onto $2, 3, \dots, \lambda - 1$. On H_2 we use the symmetric labeling and get a valid $L_{(2,1)}$ -labeling of F_s , since the ‘‘central’’ vertices u_1 and u_2 got labels 0 and λ which are sufficiently separated for the desired $L_{(2,1)}$ -labeling f .

Similarly f' can be obtained in a similar way from an $L_{(2,1)}$ -labeling of H where u is mapped on 1, the vertex v on $\lambda - 1$ and the remaining neighbors of u are mapped on the set $3, 4, \dots, \lambda - 2, \lambda$.

Both labelings are schematically depicted in Fig. 2.

3.3 Coupling gadget

Let a be an integer in the range $[1, \lambda - 1]$. Set $T = \{1, 2, \dots, a - 1, a + 1, \dots, k\}$. We construct the following graph (called the *coupling gadget*) F_c^a on $k^2 + 2k + 1$ vertices $\{v_i^t : i = 0, 1, \dots, \lambda, t \in T\} \cup \{u_1, u_2, x, y\}$ by setting the edges as follows:

$$\begin{aligned} E(F_c^a) = & \{(v_i^t, v_j^t) \mid 0 \leq i, j \leq \lambda, |i - j| \geq 2, t \in T\} \\ & \setminus \{(v_t^t, v_\lambda^t), (v_{\lambda-t}^t, v_0^t) \mid t \in T\} \\ & \cup \{(u_1, v_t^t), (u_2, v_{\lambda-t}^t) \mid t \in T\} \\ & \cup \{(u_1, x), (u_2, y)\} \end{aligned}$$

An example of the coupling gadget for a particular choice $\lambda = 5, a = 2$ is depicted in Fig 3. Observe that similarly as above all vertices except x and y are of degree k .

Lemma 3. *The graph F_c^a allows an $L_{(2,1)}$ -labeling f of span λ such that $f(x) = a$, $f(y) = \lambda - a$, $f(u_1) = \lambda$ and $f(u_2) = 0$.*

Proof. Set $f(v_i^t) = i$ for $i = 0, 1, \dots, \lambda, t \in T$. (See Fig. 3 for an example.) An easy check shows that it is a valid $L_{(2,1)}$ -labeling.

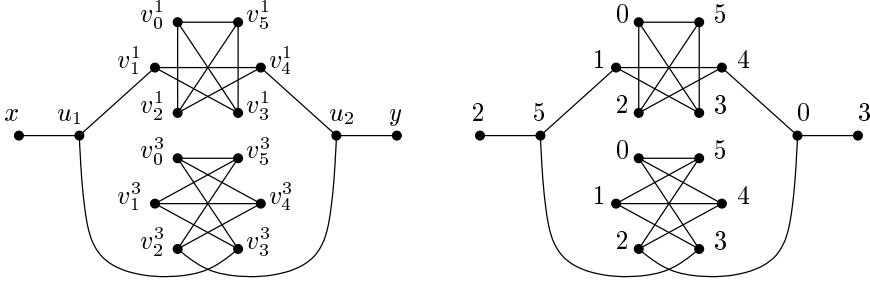


Fig. 3. The coupling gadget F_c^2 for $\lambda = 5$ and its $L_{(2,1)}$ -labeling.

4 Main result

Theorem 2. *For every integer $k \geq 3$, it is NP-complete to decide whether a k -regular graph admits an $L_{(2,1)}$ -labeling of span (at most) $\lambda = k + 2$.*

Proof. The problem is clearly in NP. Moreover, no k -regular graph admits an $L_{(2,1)}$ -labeling of span less than $k + 2$ (as long as $k \geq 1$), so we can restrict our attention only to labelings of span exactly $k + 2$.

We prove the theorem by a reduction from the NOT-ALL-EQUAL 3-SATISFIABILITY problem. The input of this problem is a Boolean formula Φ in conjunctive normal form with exactly three literals in each clause and the question is whether it is NAE-satisfiable, i.e, if a truth assignment exists such that each clause contains at least positively and at least one negatively valued literal. Determining whether Φ is NAE-satisfiable has been shown NP-complete by Schaefer [12] (see also [4, Problem LO3]).

Without loss of generality we may assume that with each clause C the formula Φ contains also its complementary clause C' consisting of the complements of all literals in C . In particular, each variable has then the same number of positive and negative occurrences.

From such a formula Φ we construct a k -regular graph G such that G allows an $L_{(2,1)}$ -labeling f of span $k + 2$ if and only if Φ is NAE-satisfiable. The graph G is constructed by local replacements of variables and clauses by variable and clause gadgets described below. (Consult Fig. 4 for details of the construction.)

Variable gadgets: Assume first that $k \neq 4$. For each variable with t positive and t negative occurrences, we insert in G $2t$ copies of the polarity

gadget F_p , arranged in a circular manner, i.e., the vertex y_i of the i -th gadget will be identified with the vertex x_{i+1} of the consequent gadget. (The last and the first gadgets are joined accordingly as well.)

The vertices x_i with odd indices will represent positive occurrences of the associated variable, while even indices will be used as negated occurrences of the variable. For $k \geq 5$, we conclude the construction of each variable gadget by inserting $t(k+1)$ new vertices v_i^s , $i = 1, \dots, k+1$, $s = 1, \dots, t$, and t triples of coupling gadgets F_c^1, F_c^3 and F_c^3 linked by the following edges:

- (v_i^s, v_j^s) if $|i - j| \geq 2$, i.e., each $(k+1)$ -tuple with the same upper index induces the complement of a path on $k+1$ vertices
- $(v_i^s, x_{2s-1}), (v_i^s, x_{2s})$ if $i \neq 1, 3, \lambda - 3, \lambda - 1$

Moreover, for each $s = 1, \dots, t$ the vertices v_1^s and $v_{\lambda-1}^s$ are identified with the x, y vertices of its uniquely associated coupling gadget F_c^1 , and similarly v_3^s and $v_{\lambda-1}^s$ are merged with the x, y of a pair of F_c^3 s.

When $k = 4$, we join polarity gadgets in a similar way: Use t copies of the polarity gadget, the x -vertices represent positive occurrences and the y -vertices negations. Now with a help of the $t(k+1)$ new vertices v_1^1, \dots, v_{k+1}^t , we define the remaining edges as:

- (v_i^s, v_j^s) if $|i - j| \geq 2$, i.e., each $(k+1)$ -tuple with the same upper index induces the complement of a path on $k+1$ vertices
- $(v_i^s, y_s), (v_i^s, x_{s+1})$ for $i = 2, 4$

As above, two coupling gadgets F_c^1, F_c^3 are joined to vertices v_1^s and v_3^s (gadget F_c^1) and to v_3^s (gadget F_c^3); both connections terminate in v_3^s . See Fig 4 (right) for a detail of this construction.

Observe that at this moment vertices of variable gadgets are of degree $k-1$ (the x_i 's) or of degree k (all others).

Clause gadgets: Each clause gadget consists of $k+3$ vertices $z_1, z_2, z_3, w_1, \dots, w_k$, and of the following edges:

- (w_i, w_j) if $|i - j| \geq 2$, inducing the complement of a path on k vertices
- (z_i, w_j) if $i = 1, 2, 3$ and $2 \leq j \leq k-2$

Clause gadgets of complementary clauses C and C' are joined by use of

- three swallowing gadgets F_s where for each $i = 1, 2, 3$, the vertices x, y are identified with z_i and z'_i . (Both z vertices must represent the same variable — one a positive occurrence, the other one a negated occurrence),
- two coupling gadgets F_c^k where both x 's are merged with w_k and both y 's with w'_k ,
- a coupling gadget F_c^1 between w_1 and w'_1 ,
- the edge (w_1, w'_1) .

Completing the construction: Finally, all variable and clause gadgets are composed together as follows: The x -vertices of variable gadgets are linked in one-by-one manner to the z -vertices of clause gadgets such that edges between gadgets represent the variable-clause incidence relation in Φ between the associated variables and clauses. As was already noted above, vertices x_i with odd i indicate positive occurrences of the associated variable, while those with an even i represent negations. (Formally, if a variable v occurs positively in a clause c , we pick a unique x_i , i odd, of the gadget representing v and a unique z_j of the clause gadget representing c and insert into G the edge (x_i, z_j) . Similarly for a negated variable we choose x_i with an even index i .) This concludes the construction of the graph G , see Fig. 4 for an illustration.

Clearly G is k -regular. It remains to be shown that G admits an $L_{(2,1)}$ -labeling of span $k + 2$ if and only if Φ is NAE-satisfiable. In particular, we prove that the NAE-satisfying Boolean assignments ϕ are in one-to-one correspondence with valid labelings f of G via the equivalence:

$$(*) \quad \phi(v) = \text{TRUE} \Leftrightarrow f(x_1) = 0 \text{ for } x_1 \text{ of the variable gadget representing } v.$$

Assume first that Φ is NAE-satisfied by an assignment ϕ . Then the partial of x_1 's can be extended to all variable gadgets such that each x_i is inside the gadget incident with vertices $2, 3, \dots, \lambda - 2$. (Just set $f(v_i^s) = i$ and extend it to the polarity and coupling gadgets.) Consider a clause C and its gadget. Without loss of generality we may assume that C contains one positively and two negatively valued literals, i.e., (up to a permutation of indices) z_1 is adjacent to a variable vertex labeled by λ , and z_2, z_3 to vertices labeled by 0. We extend f onto the clause gadget by letting $f(z_1) = 0$, $f(z_2) = \lambda - 1$, $f(z_3) = \lambda$ and $f(w_i) = i$. For the complementary clause C' , we label its gadget symmetrically and extend f to the remaining swallowing and coupling gadgets to get a valid labeling of the entire graph G .

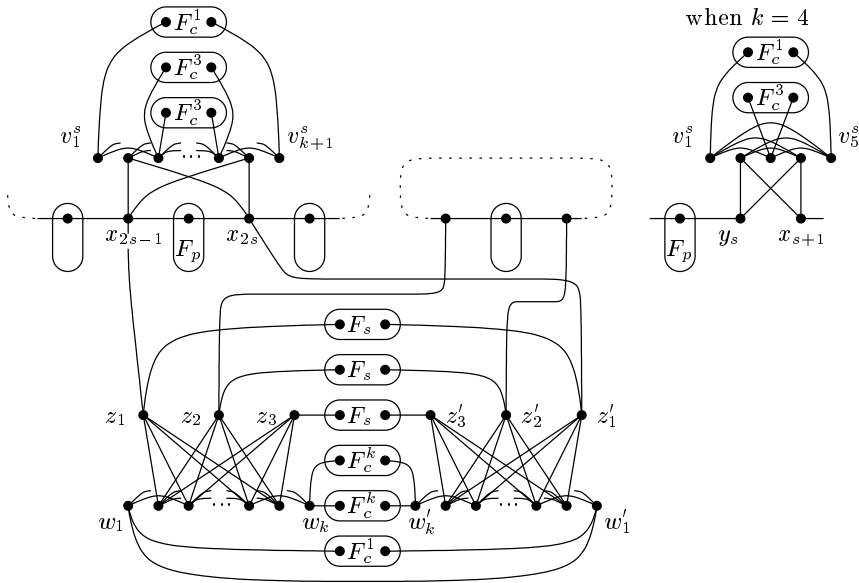


Fig. 4. Construction of G , variable gadgets in the upper part, two complementary clause gadgets at the bottom. The different construction of the variable gadget for $k = 4$ shown on the right side.

In the opposite direction, it is easy to observe that in a valid $L_{(2,1)}$ -labeling f of G of span $k + 2$ the following arguments hold:

- Up to symmetry the polarity gadgets allow only one possible labeling, where in each variable gadget $f(x_i) \neq f(x_{i+1})$, $f(x_i) \in \{0, \lambda\}$.
- For $k \geq 5$, the $k - 3$ common neighbors $\{v_2^s, v_4^s, v_5^s, \dots, v_{\lambda-4}^s, v_{\lambda-2}^s\}$ of x_{2s-1} and x_{2s} must be labeled by the set $\{2, 4, 5, \dots, \lambda - 4, \lambda - 5\}$ regardless the labeling of x_{2s-1} and x_{2s} . (Note that each x_i gets neighbors labeled 3 and $\lambda - 3$ inside the polarity gadgets.) Similarly for $k = 4$, it holds that $f(y_s) \neq f(x_{s+1})$ and $f(z_2^s), f(z_4^s) \in \{2, 4\}$.
- If x_i is labeled 0, then its neighbor z_j is labeled either λ or $\lambda - 1$ and symmetrically if $f(x_1) = \lambda$ then $f(z_j) \in \{0, 1\}$.
- In each clause gadget the labels of z_1, z_2 and z_3 must be distinct since they share a common neighbor (e.g., the vertex w_1). Then from both sets $\{0, 1\}$ and $\{\lambda - 1, \lambda\}$ at least one label is used on $\{z_1, z_2, z_3\}$.

Then ϕ defined by (*) is a NAE-satisfying assignment for Φ , i.e., each clause contains some positively as well as also some negatively valued literals. This concludes the proof of NP-hardness of the problem. Membership in NP is obvious.

5 Conclusion

We have shown NP-hardness of determining the minimum span of $L(2,1)$ -labelings of regular graphs by proving that for every $k \geq 3$, the decision problem whether $\lambda_{(2,1)}(G) \leq k + 2$ is NP-complete for k -regular graphs G . Note that the bound $k + 2$ is the minimum possible, no k -regular graph allows an $L(2,1)$ -labeling of span less than $k + 2$.

We conjecture that for every $k \geq 3$, there exists a constant c_k (depending on k) such that the decision problem $\lambda_{(2,1)}(G) \leq \lambda$ restricted to k -regular graphs is NP-complete for every fixed $\lambda \in \{k + 2, k + 3, \dots, c_k\}$ and polynomially solvable for all other values of λ . The latter is certainly true for small λ (i.e., $\lambda \leq k + 1$). The upper bound is more interesting. In particular, if our conjecture is true, it still remains a question how far is the c_k from $\lambda_k = \max\{c : \exists k\text{-regular } G \text{ s.t. } \lambda_{(2,1)}(G) > c\}$. We conjecture that $c_k \neq \lambda_k$, i.e., that in the upper part of the spectrum there will be space for nontrivial polynomial time algorithms. Note finally that $\lambda_k \leq k^2 + k - 2$ follows from [9], and that $\lambda_k \leq k^2 - 1$ if the conjecture of Griggs and Yeh is true.

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