

The Grad of a Graph and Classes with Bounded Expansion

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Abstract

We introduce classes of graphs with *bounded expansion* as a generalization of both proper minor closed classes and degree bounded classes. Such classes are based on a new invariant, the *greatest reduced average density (grad) of G with rank r* , $\nabla_r(G)$. We generalize to these classes some results proved for proper minor closed classes and bounded degree graphs, such as the existence of low tree-width colorings and homomorphism dualities.

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1 Tree-width, Tree-depth and Coloring

A k -tree is a graph which is either a single vertex, or is obtained from a smaller k -tree by adding a vertex joined to the vertices of a clique of size at most k . The *tree-width* $\text{tw}(G)$ of a graph G is the smallest k such that G is a subgraph of a k -tree. Tree-width has been proved to be a fundamental parameter, especially in the study of minor closed classes of graphs.

In a paper motivated by a question of R. Thomas [5], DeVos, Ding, Oporowski, Sanders, Reed, Seymour and Vertigan [1] proved that for every graph K and integer $j \geq 1$ there is an integer $i_V = i_V(K, j)$, such that every graph with no K -minor has a vertex partition into i_V graphs, so that any $j' \leq j$ parts form a graph with tree-width at most $j' - 1$. This proof relies on an important result of Robertson and Seymour on the structure of graphs without a particular graph as a minor [?].

In a previous work [3], we introduced the *tree-depth* $\text{td}(G)$ of graph G as the minimum height of a tree which closure contains G as a subgraph. This minor monotone invariant $\text{td}(G)$ is obviously at least $\text{tw}(G) + 1$ but $\text{td}(G)$ fails to be bounded by a function of $\text{tw}(G)$ (a path of length n has tree-width 1 and tree-depth $\lceil \log_2(n + 1) \rceil$).

Using the above mentioned result [1], we proved in [3] that for any proper minor closed class of graphs \mathcal{C} (that is: any minor closed class of graphs excluding at least one minor) and integer $j \geq 1$ there is an integer $F_j(\mathcal{C})$, such that every graph in \mathcal{C} has a vertex partition into $F_j(\mathcal{C})$ graphs, so that any $j' \leq j$ parts form

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a graph with tree-depth at most j' , and that the tree-depth is the maximum graph invariant for which such a statement holds.

For graph G and integer $j \geq 1$, define $\chi_j(G)$ has the smallest integer N , such that G may be N -colored in such a way that, for any $H \subseteq G$, H gets at least $\min(j, \text{td}(H))$ colors. Then the previous statement may be restated as:

Theorem 1.1 *For any proper minor closed class of graphs \mathcal{C} and any integer $j \geq 1$, $\chi_j(G)$ is bounded on \mathcal{C} .*

Notice that for any graph G of order n , $\chi_1(G) = 1$, $\chi_2(G)$ is the usual chromatic number $\chi(G)$. We have then

$$\chi_1(G) \leq \chi_2(G) \leq \dots \leq \chi_n(G) = \text{td}(G).$$

2 Homomorphism Duality

The previous result has then been applied to prove that any proper minor closed class \mathcal{C} of graphs has a *restricted homomorphism duality* for any connected graph F , that is: For any connected graph F , there exists a graph $D_F^{\mathcal{C}}$ so that:

- F has no homomorphism to $D_F^{\mathcal{C}}$:

$$F \not\rightarrow D_F^{\mathcal{C}}$$

- any graph $G \in \mathcal{C}$ has no homomorphism **from** F if and only if it has a homomorphism **to** $D_F^{\mathcal{C}}$:

$$\forall G \in \mathcal{C}, \quad (F \not\rightarrow G) \iff (G \rightarrow D_F^{\mathcal{C}})$$

Such a restricted duality exists for proper minor closed classes of graphs [3] and for classes of bounded degree graphs [4,2].

3 Grad of a Graph and Classes with Bounded Expansion

The *greatest reduced average density (grad)* of G with rank r : $\nabla_r(G)$ is related to the maximum average degree of a minor obtained by contracting a family of disjoint connected subgraphs, each having radius bounded by r . Precisely, a *connected partition* of G is a partition $\mathcal{P} = (V_1, \dots, V_p)$ of G such that each subgraph $G[V_i]$ of G induced by V_i is connected. The *radius* $\rho(\mathcal{P})$ of a connected partition \mathcal{P} is

$$\rho(\mathcal{P}) = \max_{X \in \mathcal{P}} \min_{x \in X} \max_{y \in X} \text{dist}_{G[X]}(x, y)$$

and the *grad of rank r* of G is

$$\nabla_r(G) = \max_{\rho(\mathcal{P}) \leq r} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}$$

where the maximum is taken over all the connected partitions \mathcal{P} of G having radius at most r and where G/\mathcal{P} stands for the **simple** graph obtained by contracting each of the $G[V_i], V_i \in \mathcal{P}$ to a single vertex.

For any graph G of order n , $\nabla_0(G)$ is half of the maximum average degree (mad) of G and

$$\frac{\text{mad}(G)}{2} = \nabla_0(G) \leq \nabla_1(G) \leq \dots \leq \nabla_n(G) \leq \text{td}(G) - 1$$

The last inequality is straightforward: any minor of order n of a graph with tree-depth at most k has tree-depth at most k hence size at most $(k-1)n$. Moreover, $\nabla_r(G)$ is obviously bounded by a constant for any proper minor closed class of graphs and by $O(\Delta^r)$ for any graph with maximum degree Δ .

A class \mathcal{C} has a *bounded expansion* if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\forall r \in \mathbb{N}, \forall G \in \mathcal{C}, \quad \nabla_r(G) \leq f(r).$$

4 Transitive Fraternal Orientation and Augmentation

It is well known that a graph G is a k -tree (for some k) if and only if it admits an *acyclic fraternal orientation*, that is an orientation of its edges such that when (x, z) and (y, z) are both arcs of G then either (x, y) or (y, x) is an arc of G .

We will relax the properties of acyclic fraternal orientations to build some locally fraternal orientations. In the following, directed graphs may have, for distinct vertices x and y , one arc (at most) from x to y **and** one arc (at most) from y to x .

Let \mathbf{G} be a directed graph. A *1-transitive fraternal augmentation* of \mathbf{G} is a directed graph \mathbf{H} with the same vertex set, including all the arcs of \mathbf{G} and such that, for any vertices x, y, z ,

- if (x, z) and (z, y) are arcs of \mathbf{G} then (x, y) is an arc of \mathbf{H} (*transitivity*),
- if (x, z) and (y, z) are arcs of \mathbf{G} then (x, y) or (y, x) or both are arcs of \mathbf{H} (*fraternity*).

A *transitive fraternal augmentation* of a directed graph \mathbf{G} is a sequence $\mathbf{G} = \mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \dots \subseteq \mathbf{G}_i \subseteq \mathbf{G}_{i+1} \subseteq \dots$, such that \mathbf{G}_{i+1} is a 1-transitive fraternal augmentation of \mathbf{G}_i for any $i \geq 1$.

Theorem 4.1 *Let \mathcal{C} be a class with bounded expansion. There exists a function $F : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that for any integer Δ^- , there exists a sequence $\mathcal{C} = \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \dots \subseteq \mathcal{C}_i \subseteq \dots$ of classes with bounded expansion, so that any orientation \mathbf{G} of a graph*

$G \in \mathcal{C}$ with $\Delta^-(\mathbf{G}) \leq \Delta^-$ has a transitive fraternal augmentation $\mathbf{G} = \mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \dots \subseteq \mathbf{G}_i \subseteq \dots$ where \mathbf{G}_i is an orientation with maximum indegree $\Delta^-(\mathbf{G}_i) \leq F(\Delta^-, i)$ of some $G_i \in \mathcal{C}_i$.

5 Main Results

Our first main result stands in a characterization of classes with bounded expansion, while the second one expresses the existence of restricted dualities for such classes.

Theorem 5.1 *Let \mathcal{C} be a class of graphs. The following conditions are equivalent:*

- \mathcal{C} has bounded expansion,
- for any integer c , the class $\mathcal{C}[K_c] = \{G[K_c] : G \in \mathcal{C}\}$ has bounded expansion, where $G[K_c]$ denotes the lexicographic product of G and K_c ;
- there exists a function $F : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that any orientation \mathbf{G} of a graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\mathbf{G} = \mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \dots \subseteq \mathbf{G}_i \subseteq \dots$ where $\Delta^-(\mathbf{G}_i) \leq F(\Delta^-(\mathbf{G}), i)$;
- for any integer i , $\chi_i(G)$ is bounded on \mathcal{C} .

From a computational point of view, it has to be noticed that, for any fixed class \mathcal{C} with bounded expansion and any fixed integer j , the proof of the theorems yields a linear time algorithm to compute, for any graph $G \in \mathcal{C}$, a coloring using at most $N(\mathcal{C}, j)$ colors so that any $j' \leq j$ parts not only induce a subgraph of tree-depth at most j' , but actually induce a subgraph which connected components have the property that some color appears exactly once in them, which is a stronger statement.

From the previous theorem, we further prove that classes with bounded expansion also admits restricted dualities:

Theorem 5.2 *Any class of graphs with bounded expansion has a restricted duality for any connected graph: for every connected graph F , there exists a graph $D_F^{\mathcal{C}}$ so that $F \not\rightarrow D_F^{\mathcal{C}}$ and*

$$\forall G \in \mathcal{C}, \quad (F \not\rightarrow G) \iff (G \rightarrow D_F^{\mathcal{C}})$$

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